Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 310679, 11 pages
http://dx.doi.org/10.1155/2013/310679

Research Article

N-Dimensional Fractional Lagrange's Inversion Theorem

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Received 10 November 2012; Revised 14 January 2013; Accepted 20 January 2013

Academic Editor: Ciprian A. Tudor

Using Riemann-Liouville fractional differential operator, a fractional extension of the Lagrange inversion theorem and related formulas are developed. The required basic definitions, lemmas, and theorems in the fractional calculus are presented. A fractional form of Lagrange's expansion for one implicitly defined independent variable is obtained. Then, a fractional version of Lagrange's expansion in more than one unknown function is generalized. For extending the treatment in higher dimensions, some relevant vectors and tensors definitions and notations are presented. A fractional Taylor expansion of a function of N-dimensional polyadics is derived. A fractional N-dimensional Lagrange inversion theorem is proved.

1. Introduction

The fractional calculus (FC) may be considered as an old and yet novel topic. It dates back to the end of the seventeenth century through the pioneering works of Leibniz, Euler, Lagrange, Abel, Liouville, and many others. In a letter to L'Hospital in 1695, Leibniz raised the possibility of generalizing the operation of differentiation to noninteger orders, and L'Hospital asked what would be the result of half-differentiating x. Leibniz replied: It leads to a paradox, from which one day useful consequences will be drawn. The paradoxical aspects are due to the fact that there are several different ways of generalizing the differentiation operator to non-integer powers, leading to inequivalent results.

The fractional calculus (FC) generalizes the ordinary differentiation and integration so as to include any arbitrary real or even complex order instead of being only the positive integers (see, e.g., Samko et al. [1], Kilbas, et al. [2], Magin [3], and Podlubny [4]).

During the second half of the twentieth century till now, FC gained considerable popularity and importance. Many authors have explored the world of FC giving new insight into many areas of scientific research in physics, mechanics, and mathematics. Miller and Ross [5] pointed out that there is hardly a field of science or engineering that has remained untouched by the new concepts of FC.

Fractional derivatives provide an excellent as well as very powerful tool for the description and modeling of many phenomena in nature. There are many applications where the fractional calculus can be widely used, for example, viscoelasticity, electrochemistry, diffusion processes, control theory, heat conduction, electricity, mechanics, chaos and fractals, turbulence, fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and so forth, see for details [2–12] and the references therein.

In a very good book by Baleanu et al. [13], readers were given the possibility of finding very important mathematical tools for working with fractional models and solving fractional differential equations, such as a generalization of Stirling numbers in the framework of fractional calculus and a set of efficient numerical methods. Moreover, they introduced some applied topics, in particular, fractional variational methods which are used in physics, engineering, or economics. They also discussed the relationship between semi-Markov continuous-time random walks and the space-time fractional diffusion equation, which generalized the usual theory relating random walks to the diffusion equation.

Debbouche and Baleanu [14] introduced a new concept called implicit evolution system to establish the existence results of mild and strong solutions of a class of fractional nonlocal nonlinear integrodifferential system; then they proved the exact null controllability result of a class of fractional evolution nonlocal integrodifferential control systems in Banach space. As an application that illustrates their abstract results, they provided two examples.

Babakhani and Baleanu [15] considered a class of nonlinear fractional order differential equations involving Caputo fractional derivative with lower terminal at 0 in order to study the existence solution satisfying the boundary conditions or satisfying the initial conditions. They derived unique solution under Lipschitz condition. In order to illustrate their results they presented several examples.

Finally and roughly speaking, the fractional calculus may improve the smoothness properties of functions rather than the calculus with integer orders. The development of the FC theory is due to the contributions of many mathematicians such as Euler, Liouville, Riemann, and Letnikov. Several definitions of a fractional derivative have been proposed. These definitions include Riemann-Liouville, Grunwald-Letnikov,Weyl, Caputo, Marchaud, and Riesz fractional derivatives, see Miller and Ross [5] and Riewe [16]. Riemann-Liouville derivative is the most used generalization of the derivatives. It is based on the direct generalization of Cauchy's formula for calculating an n-fold or repeated integral, see Oldham and Spanier [17].

In 1770, Lagrange (1736–1813) published his power series solution of the implicit equation. However, his solution used cumbersome series expansions of logarithms. [18, 19]. This expansion was generalized by Bürmann [20–22]. There is a straightforward derivation using complex analysis and contour integration; the complex formal power series version is clearly a consequence of knowing the formula for polynomials; so the theory of analytic functions may be applied. Actually, the machinery from analytic function theory enters only in a formal way in this proof. In 1780, Laplace (1749–1827) published a simpler proof of the theorem, based on the relations between partial derivatives with respect to the variable and the parameter, see [23, 24], Hermite (1822–1901) presented the most straightforward proof of the theorem by using contour integration [25–27].

In mathematical analysis, this series expansions is known as Lagrange inversion theorem, also known as the Lagrange-Bürmann formula, giving the Taylor series expansion of the inverse function. Suppose that $z = f(w)$, where f is analytic function at a point a and $f(a) \neq 0$. Then, it is possible to invert or solve the equation for w such that $w = g(z)$ on a neighborhood of $f(a)$, where g is analytic at the point $f(a)$. This is also called reversion of series. The series expansion of g is given by

$$\begin{aligned} w &= g(z) \\ &= a + \sum_{n=1}^{\infty} \left(\lim_{w \to a} \left(\frac{(z - f(a))^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w-a}{f(w) - f(a)} \right)^n \right) \right). \end{aligned} \tag{1}$$

In this work, we will apply the concepts of fractional calculus to obtain a fractional form of the Lagrange expansion and some generalizations.

2. Basic Definitions and Theorems

Definition 1. By D, we denote the operator that maps a differentiable function onto its integer derivative; that is, $Df(x) = f'$; by J_a, we denote the integer integration operator that maps a function f, assumed to be (Riemann) integrable on the compact interval $[a, b]$, onto its primitive centered at a; that is, $J_a f(x) = \int_a^x f(t)dt$ for all $a \le x \le b$.

Definition 2. By D^n and $J_a^n, n \in \mathbb{N}$, we denote the n-fold iterates of D and J_a, respectively. Note that D^n is the left inverse of J_a^n in a suitable space of functions.

Lemma 3. *Let f be Riemann integrable on $[a, b]$. Then, for $a \le x \le b$ and $n \in \mathbb{N}$, one has*

$$J_a^n\ f(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt, \quad n \in \mathbb{N}. \tag{2}$$

Definition 4. The operator $\mathcal{J}_a^\alpha$, defined on Lebesgue space $L_1[a, b]$, denotes the Riemann-Liouville fractional operator of order α. That is,

$$\mathcal{J}_a^\alpha\ f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad a \le x \le b,\ \alpha \in \mathbb{R}. \tag{3}$$

Remark 5. It is evident that $\mathcal{J}_a^\alpha \equiv J_a^n$, for all $\alpha \in \mathbb{N}$, except for the fact that we have extended the domain from Riemann integrable functions to Lebesgue integrable functions (which will not lead to any problems in our development). Moreover, in the case $\alpha \ge 1$, it is obvious that the integral $\mathcal{J}_a^\alpha f(x)$ exists for every $x \in [a, b]$ because the integrand is the product of an integrable function f and the continuous function $(x - \bullet)^{\alpha-1}$. One important property of integer-order integral operators, is preserved by this generalization. That is,

$$\begin{aligned} \mathcal{J}_a^\alpha \left(\mathcal{J}_a^\beta f(x) \right) &= \mathcal{J}_a^\beta \left(\mathcal{J}_a^\alpha\ f(x) \right) \\ &= \mathcal{J}_a^{\alpha+\beta} f(x), \quad \alpha, \beta > 0,\ f(x) \in L_1[a, b]. \end{aligned} \tag{4}$$

Definition 6. Let $\alpha \in \mathbb{R}^+$ and let $m = \lceil \alpha \rceil$, The Riemann-Liouville fractional differential operator of order α is defined as such that. Then, $\mathcal{D}_a^\alpha = D_a^m \mathcal{J}_a^{m-\alpha}$. That is,

$$\begin{aligned} \mathcal{D}_a^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx} \right)^m \int_a^x (x-t)^{m-\alpha-1} \\ \times f(t)\, dt, \quad a \le x \le b,\ \alpha \in \mathbb{R}. \end{aligned} \tag{5}$$

Lemma 7. *Let $\alpha \in \mathbb{R}^+$ and let $m \in \mathbb{N}$ such that $> \alpha$. Then, $\mathcal{D}_a^\alpha = D_a^m \mathcal{J}_a^{m-\alpha}$.*

Proof. Since $m > \alpha$ yields $m \ge \lceil \alpha \rceil$.

Thus,

$$\begin{aligned} D^m \mathcal{J}_a^{m-\alpha} &= D^{\lceil \alpha \rceil} D^{m-\lceil \alpha \rceil} J^{m-\lceil \alpha \rceil} J^{\lceil \alpha \rceil - \alpha} \\ &= D^{\lceil \alpha \rceil + m - \lceil \alpha \rceil - m + \lceil \alpha \rceil - \lceil \alpha \rceil + \alpha} = D^\alpha = \mathcal{D}_a^\alpha. \end{aligned} \tag{6}$$

□

Theorem 8. *Let $(\alpha \geq 0) \in \mathbb{R}^+$. Then, for every $f(x) \in L_1[a,b]$, $\mathscr{D}_a^\alpha \mathscr{I}_a^\alpha f(x) = f(x)$.*

Proof. For $\alpha = 0$, both operator, are the identity. For $\alpha > 0$, let $m \geq \lceil \alpha \rceil$; then,

$$\begin{aligned} \mathscr{D}_a^\alpha \mathscr{I}_a^\alpha f(x) &= D_a^m \mathscr{I}_a^{m-\alpha} \mathscr{I}_a^\alpha f(x) = D_a^m \mathscr{I}_a^m f(x) \\ &= D_a^m J_a^m f(x) = f(x). \end{aligned} \tag{7}$$

□

Corollary 9. *Let f be analytic in $(a-h,\ a+h)$ for some $h > 0$, and let $(\alpha \geq 0) \in \mathbb{R}^+$.*

Then,

$$\begin{aligned} &(I\text{-}1) \quad \mathscr{I}_a^\alpha f(x) = \sum_{k=0}^{\infty} \frac{(-1)^m (x-a)^{k+\alpha}}{k!\,(\alpha+k)\,\Gamma(\alpha)} D_a^k f(x), \\ &\qquad\qquad \forall a \leq x < a + \frac{h}{2}, \\ &(I\text{-}2)\ \mathscr{I}_a^\alpha f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{k+\alpha}}{\Gamma(k+1+\alpha)} D_a^k f(a), \\ &\qquad\qquad \forall a \leq x < a + h, \\ &(D\text{-}1)\ \mathscr{D}_a^\alpha f(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(x-a)^{k-\alpha}}{(k+1-\alpha)} D_a^k f(x), \\ &\qquad\qquad \forall a \leq x < a + \frac{h}{2}, \\ &(D\text{-}2)\ \mathscr{D}_a^\alpha f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{k-\alpha}}{(k+1-\alpha)} D_a^k f(a), \\ &\qquad\qquad \forall a \leq x < a + h. \end{aligned} \tag{8}$$

The binomial coefficients for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ are defined as

$$\begin{aligned} \binom{\alpha}{k} &= \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} \\ &= \frac{\alpha!}{k!\,(\alpha-k)!}. \end{aligned} \tag{9}$$

Proof. For the first two statements $(I\text{-}1)$, $(I\text{-}2)$ and we use the definition of the Riemann-Liouville integral operator $\mathscr{I}_a^\alpha$ and expand $f(t)$ into a power series about x. Since $x \in [a, a+h/2)$, the power series converges in the entire interval of integration and exchanges summation and integration. Then, we use the explicit representation for the fractional integral of the power function:

$$\mathscr{I}_a^\alpha (x-a)^k = \frac{\Gamma(k+1)}{(\alpha+k+1)} (x-a)^{k+\alpha}. \tag{10}$$

$(I\text{-}1)$ follows immediately. For the second statement, we proceed in a similar way; but we now expand the power series at a and not at x. This allows us again to conclude the convergence of the series in the required interval. The analyticity of $\mathscr{I}_a^\alpha$ follows immediately from the second statement.

To prove $(D\text{-}1)$ we use the relation

$$\begin{aligned} &\mathscr{D}_a^\alpha = D^{\lceil\alpha\rceil} \mathscr{I}_a^{\lceil\alpha\rceil-\alpha}, \\ &k!\Gamma(\alpha)(\alpha+k)\binom{-\alpha}{k} = (-1)^k \Gamma(k+1+\alpha). \end{aligned} \tag{11}$$

This allows us to rewrite the statement $(I\text{-}1)$ as

$$\mathscr{I}_a^{\lceil\alpha\rceil-\alpha} f(x) = \sum_{k=0}^{\infty} \binom{\lceil\alpha\rceil-\alpha}{k} \frac{(x-a)^{k+\lceil\alpha\rceil-\alpha}}{\Gamma(k+1+\lceil\alpha\rceil-\alpha)} D_a^k f(x). \tag{12}$$

Differentiating $\lceil\alpha\rceil$ times with respect to x, we find

$$\begin{aligned} D_a^{\lceil\alpha\rceil} \mathscr{I}_a^{\lceil\alpha\rceil-\alpha} f(x) = \mathscr{D}_a^\alpha f(x) &= \sum_{k=0}^{\infty} \binom{\lceil\alpha\rceil-\alpha}{k} \\ &\times \frac{1}{\Gamma(k+1+\lceil\alpha\rceil-\alpha)} \\ &\times D_a^{\lceil\alpha\rceil} \left[(\bullet - a)^{k+\lceil\alpha\rceil-\alpha} D_a^k f\right](x). \end{aligned} \tag{13}$$

The classical version of Leibniz' formula yields

$$\begin{aligned} \mathscr{D}_a^\alpha f(x) = &\sum_{k=0}^{\infty} \binom{\lceil\alpha\rceil-\alpha}{k} \frac{1}{\Gamma(k+1+\lceil\alpha\rceil-\alpha)} \sum_{j=0}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{j} \\ &\times D_a^{\lceil\alpha\rceil-j} \left[(\bullet - a)^{k+\lceil\alpha\rceil-\alpha}\right](x)\, D_a^{k+j} f, \end{aligned} \tag{14}$$

which yields

$$\begin{aligned} \mathscr{D}_a^\alpha f(x) = &\sum_{k=0}^{\infty} \binom{\alpha-\lceil\alpha\rceil}{k} \sum_{j=0}^{\lceil\alpha\rceil} \binom{\lceil\alpha\rceil}{j} \frac{\left[(x-a)^{k+j-\alpha}\right]}{\Gamma(k+1+j-\alpha)} \\ &\times (x)\, D_a^{k+j} f. \end{aligned} \tag{15}$$

By definition, $\binom{\mu}{j} = 0$ if $\mu \in \mathbb{N}$ and $\mu < j$. Thus, we may replace the upper limit in the inner sum by ∞ without changing the expression. The substitution $j = l - k$ gives

$$\mathscr{D}_a^\alpha f(x) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha-\lceil\alpha\rceil}{k} \binom{\lceil\alpha\rceil}{l-k} \frac{\left[(x-a)^{l-\alpha}\right]}{\Gamma(l+1-\alpha)} (x)\, D_a^l f. \tag{16}$$

Using the fact that $\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} = \sum_{l=0}^{\infty} \sum_{k=0}^{l}$,

$$\mathscr{D}_a^\alpha f(x) = \sum_{l=0}^{\infty} \sum_{k=0}^{l} \binom{\alpha-\lceil\alpha\rceil}{k} \binom{\lceil\alpha\rceil}{l-k} \frac{\left[(x-a)^{l-\alpha}\right]}{\Gamma(l+1-\alpha)} (x)\, D_a^l f. \tag{17}$$

And the explicit calculation yields

$$\sum_{k=0}^{l} \binom{\alpha-\lceil\alpha\rceil}{k} \binom{\lceil\alpha\rceil}{l-k} = \binom{\lceil\alpha\rceil}{l}, \tag{18}$$

thus, $(D\text{-}1)$ follows directly. □

3. Fractional Form of Lagrange's Expansion in One Variable

We can use the standard form of Lagrange's expansion for one implicitly defined independent variable and the Definition 6 to obtain the fractional form of Lagrange's expansion as follows.

Let z be a function of (ζ, ε) and in terms of another function μ such that

$$z = z(\zeta, \varepsilon) \equiv \zeta + \varepsilon\mu(z). \tag{19}$$

Then, for any function f

$$f(z) = f(\zeta) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} \left[\mu^n(\zeta) \frac{df(\zeta)}{d\zeta}\right] \tag{20}$$

for small ε. If f is the identity

$$z = \zeta + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} \left[\mu^n(\zeta)\right], \tag{21}$$

then

$$f(z) = f(\zeta) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{d^{n-1}}{d\zeta^{n-1}} \left[\mu^n(\zeta) \frac{df(\zeta)}{d\zeta}\right]. \tag{22}$$

This classical result can be obtained using the following integral:

$$f(z) = \int \delta\left(\varepsilon\mu(\tau) - \tau + \zeta\right) f(\tau)\left(1 - \varepsilon\mu'(\tau)\right) d\tau \tag{23}$$

Now we are going to introduce a fractional form of the Lagrange inversion formula.

Rewrite the integral (23) in the fractional form as

$$\begin{aligned} f(z) &= {}_z\mathcal{I}_a^{\alpha} \delta\left(\varepsilon\mu(z) - z + \zeta\right) \\ &\quad \times f(z)\left(1 - \varepsilon_z \mathcal{D}_a^{\alpha}\mu(z)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^z (z-\tau)^{\alpha-1} \delta\left(\varepsilon\mu(\tau) - \tau + \zeta\right) \\ &\quad \times f(\tau)\left(1 - \varepsilon_\tau \mathcal{D}_a^{\alpha}\mu(\tau)\right) d\tau. \end{aligned} \tag{24}$$

Writing the delta function as an integral, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi\Gamma(\alpha)} \int_a^z \int_{-\pi}^{\pi} (z-\tau)^{\alpha-1} e^{(i\xi[\varepsilon\mu(\tau)-\tau+\zeta])} \\ &\quad \times f(\tau)\left(1 - \varepsilon_\tau \mathcal{D}_a^{\alpha}\mu(\tau)\right) d\xi d\tau \\ &= \frac{1}{2\pi\Gamma(\alpha)} \sum_{n=0}^{\infty} \int_a^z \int_{-\pi}^{\pi} \frac{\left(i\xi\varepsilon\mu(\tau)\right)^n e^{ik(\zeta-\tau)}}{n!} (z-\tau)^{\alpha-1} \\ &\quad \times f(\tau)\left(1 - \varepsilon_\tau \mathcal{D}_a^{\alpha}\mu(\tau)\right) d\xi\, d\tau \\ &= \frac{1}{2\pi\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_z\mathcal{D}_a^{\alpha}\right)^n \int_a^z \int_{-\pi}^{\pi} \frac{\left(\varepsilon\mu(\tau)\right)^n e^{i\xi(\zeta-\tau)}}{n!} (z-\tau)^{\alpha-1} \\ &\quad \times f(\tau)\left(1 - \varepsilon_\tau \mathcal{D}_a^{\alpha}\mu(\tau)\right) d\xi\, d\tau. \end{aligned} \tag{25}$$

The integral over ξ then gives $\delta(\zeta - \tau) = 0$ for all $\zeta - \tau = 0$ and we have

$$\begin{aligned} f(z) &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \frac{\left(\varepsilon\mu(\zeta)\right)^n}{n!} f(\zeta)\left(1 - \varepsilon\, {}_\zeta\mathcal{D}_a^{\alpha}\mu(\zeta)\right) \\ &= \frac{1}{n!\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \left[\varepsilon^n\left(\mu(\zeta)\right)^n f(\zeta) - \varepsilon^{n+1}\right. \\ &\quad \left. \times f(\zeta)\left(\mu(\zeta)\right)^n \left({}_\zeta\mathcal{D}_a^{\alpha}\mu(\zeta)\right)\right] \\ &= \frac{1}{n!\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \left[\varepsilon^n\left(\mu(\zeta)\right)^n f(\zeta) - \varepsilon^{n+1} f(\zeta)\Gamma\right. \\ &\quad \left. \times (1+\alpha)\left(\mu(\zeta)\right)^n \left(\mu(\zeta)^{1-\alpha}\right)\right] \\ &= \frac{1}{n!\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \left[\varepsilon^n\left(\mu(\zeta)\right)^n f(\zeta) - \varepsilon^{n+1}\right. \\ &\quad \left. \times f(\zeta)\Gamma(1+\alpha)\left(\mu(\zeta)^{n+1-\alpha}\right)\right] \\ &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \left[\frac{\varepsilon^n}{n!}\left(\mu(\zeta)\right)^n f(\zeta) - \frac{\varepsilon^{n+1}\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)}\right. \\ &\quad \times \left\{{}_\zeta\mathcal{D}_a^{\alpha}\left[f(\zeta)\left(\mu(\zeta)\right)^{n+1}\right]\right. \\ &\quad \left.\left. - {}_\zeta\mathcal{D}_a^{\alpha} f(\zeta)\left(\mu(\zeta)\right)^{n+1}\right\}\right]. \end{aligned} \tag{26}$$

On extracting the first term out of summation, set $n+1 = k \Rightarrow n = k-1$, and rearranging the terms then gives the result:

$$\begin{aligned} f(z) &= \frac{1}{\Gamma(\alpha)} f(\zeta) + \sum_{k=0}^{\infty} \left({}_\zeta\mathcal{D}_a^{\alpha}\right)^{k-1} \frac{\alpha\left(\varepsilon^k\right)}{\Gamma(k+\alpha)} \\ &\quad \times \left[\left(\mu(\zeta)\right)^k {}_\zeta\mathcal{D}_a^{\alpha} f(\zeta)\right]. \end{aligned} \tag{27}$$

4. Generalized Fractional Lagrange's Expansion in One Variable

We can the Lagrange's expansion in more than one unknown function $\mu_i(z)$ as

$$z = z\left(\zeta, \varepsilon_i\right) \equiv \zeta + \sum_{i=1}^{n} \varepsilon_i \mu_i(z). \tag{28}$$

This classical result can be obtained using the following integral:

$$f(z) = \int \delta\left(\sum_{i=1}^{n} \varepsilon_i \mu_i(\tau) - \tau + \zeta\right) f(\tau)\left(1 - \sum_{i=1}^{n} \varepsilon_i \mu_i'(\tau)\right) d\tau. \tag{29}$$

Now, we are going to introduce a fractional form of the Lagrange inversion formula.

Rewrite the integral (29) in the fractional form as

$$\begin{aligned} f(z) &= {}_z\mathcal{J}_a^{\alpha}\delta\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(z) - z + \zeta\right) \\ &\quad\times f(z)\left(1 - {}_z\mathcal{D}_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(z)\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_a^z (z-\tau)^{\alpha-1}\delta\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau) - \tau + \zeta\right) \\ &\quad\times f(\tau)\left(1 - {}_\tau\mathcal{D}_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)d\tau. \end{aligned} \tag{30}$$

Writing the delta function as an integral, we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi\Gamma(\alpha)}\int_a^z\int_{-\pi}^{\pi}(z-\tau)^{\alpha-1}e^{(i\xi[\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)-\tau+\zeta])}f(\tau) \\ &\quad\times\left(1 - {}_\tau\mathcal{D}_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)d\xi\,d\tau \\ &= \frac{1}{2\pi\Gamma(\alpha)}\sum_{n=0}^{\infty}\int_a^z\int_{-\pi}^{\pi}\frac{1}{n!}\left(i\xi\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)^n \\ &\quad\times e^{ik(\zeta-\tau)}(z-\tau)^{\alpha-1}f(\tau) \\ &\quad\times\left(1 - {}_\tau D_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)d\xi\,d\tau \\ &= \frac{1}{2\pi\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_z\mathcal{D}_a^{\alpha}\right)^n\int_a^z\int_{-\pi}^{\pi}\frac{1}{n!}\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)^n \\ &\quad\times e^{i\xi(\zeta-\tau)}(z-\tau)^{\alpha-1}f(\tau) \\ &\quad\times\left(1 - {}_\tau\mathcal{D}_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(\tau)\right)d\xi\,d\tau. \end{aligned} \tag{31}$$

The integral over ξ then gives $\delta(\zeta-\tau)=0$ for all $\zeta-\tau=0$ and we have

$$\begin{aligned} f(z) &= \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n\frac{1}{n!}\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n \\ &\quad\times f(\zeta)\left(1 - {}_\zeta\mathcal{D}_a^{\alpha}\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right) \\ &= \frac{1}{n!\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta D_a^{\alpha}\right)\left[\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n f(\zeta) - f(\zeta)\right. \\ &\quad\left.\times\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n\times\left({}_\zeta D_a^{\alpha}\right)\sum_{i=1}^{\infty}\varepsilon_i\mu_i(\zeta)\right] \\ &= \frac{1}{n!\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n\left[\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n f(\zeta)\right. \\ &\quad\left.-f(\zeta)\Gamma(1+\alpha)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n\left(\sum_{i=1}^{n}\varepsilon_i[\mu_i(\zeta)]^{1-\alpha}\right)\right] \\ &= \frac{1}{n!\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n\left[\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n f(\zeta) - f(\zeta)\right. \\ &\quad\times\Gamma(1+\alpha) \\ &\quad\left.\times\left(\sum_{i=1}^{n}\varepsilon_i[\mu_i(\zeta)]^{n+1-\alpha}\right)\right] \\ &= \frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n\left[\frac{1}{n!}\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n f(\zeta)\right. \\ &\quad-\frac{\Gamma(1+\alpha)}{\Gamma(n+1+\alpha)} \\ &\quad\times\left\{{}_\zeta\mathcal{D}_a^{\alpha}\left[f(\zeta)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^{n+1}\right]\right. \\ &\quad\left.\left.-{}_\zeta\mathcal{D}_a^{\alpha}f(\zeta)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^{n+1}\right\}\right] \\ &= \frac{1}{\Gamma(n+1)\Gamma(\alpha)\Gamma(n+1+\alpha)}\sum_{n=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^n \\ &\quad\times\left[\Gamma(n+1+\alpha)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^n f(\zeta) - \Gamma(n+1)\right. \\ &\quad\times\Gamma(1+\alpha)\left\{{}_\zeta\mathcal{D}_a^{\alpha}\left[f(\zeta)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^{n+1}\right]\right. \\ &\quad\left.\left.-{}_\zeta\mathcal{D}_a^{\alpha}f(\zeta)\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^{n+1}\right\}\right]. \end{aligned} \tag{32}$$

On extracting the first term out of summation, set $n+1=k \Rightarrow n=k-1$, and rearranging the terms then gives the result

$$\begin{aligned} f(z) &= \frac{1}{\Gamma(\alpha)}f(\zeta) + \sum_{k=0}^{\infty}\left({}_\zeta\mathcal{D}_a^{\alpha}\right)^{k-1}\frac{\alpha}{\Gamma(k+\alpha)} \\ &\quad\times\left[\left(\sum_{i=1}^{n}\varepsilon_i\mu_i(\zeta)\right)^k {}_\zeta\mathcal{D}_a^{\alpha}f(\zeta)\right]. \end{aligned} \tag{33}$$

5. Vector and Tensor Definitions and Notation

For the treatment in higher dimensions, consider the N-dimensional space with orthogonal unit base vectors $\widehat{e}_k$, $(k = 1, 2, \ldots, n)$:

$$\widehat{e}_i \cdot \widehat{e}_j = \delta_{ij} \begin{cases} = 1 & \text{for} \quad i = j \\ = 0 & \text{for} \quad i \neq j \end{cases} \qquad (i, j = 1, 2, \ldots, n). \tag{34}$$

Let ζ, z , and the function $\mu(z)$ be N-dimensional vectors in this space such that

$$z = z(\zeta, \varepsilon) = \zeta + \varepsilon \mu(z), \tag{35}$$

where

$$\zeta = \sum_{k=1}^{n} \widehat{e}_k \zeta_k, \quad z = \sum_{k=1}^{n} \widehat{e}_k z_k, \quad \mu(z) = \sum_{k=1}^{n} \widehat{e}_k \mu_k. \tag{36}$$

For any arbitrary differentiable function $F(\zeta, \varepsilon)$, we can introduce the following fractional gradient operator as ∇_ζ^α.

Definition 10. Let Ω be a domain of $\mathbb{R}^n$. Let $F(\zeta, \varepsilon) \in AC^n(\Omega)$ is a scalar function that has absolutely continuous derivatives up to order $(n-1)$ on; then fractional gradient is defined as

$$\begin{aligned} \nabla_\zeta^\alpha F(\zeta, \varepsilon) &= {}_\zeta\mathscr{D}_a^\alpha F(\zeta, \varepsilon) = {}_{\zeta_s}\mathscr{D}_a^\alpha F(\zeta_s, \varepsilon) \widehat{e}_s \\ &= \widehat{e}_s \frac{1}{\Gamma(m-\alpha)} \left(\frac{\partial}{\partial \zeta_s}\right)^m \int_a^\zeta (\zeta - t)^{m-\alpha-1} f(\tau)\, d\tau, \\ &\qquad a \leq \zeta \leq b, \quad \alpha \in \mathbb{R}, \end{aligned} \tag{37}$$

where the partial derivatives are taken holding all other components of the argument fixed.

6. The N-Dimensional Polyadics (nth-Order Tensors)

For arbitrary n-dimensional vectors

$$A \equiv \sum_{k=1}^{n} \widehat{e}_k A_k, \qquad B \equiv \sum_{k=1}^{n} \widehat{e}_k B_k, \tag{38}$$

we use an extension of the notion of an n-dimensional dyadic (second-order tensor):

$$AB \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{e}_i \widehat{e}_j A_i B_j \tag{39}$$

to define the nth-order tensors:

$$A^{(n)} \equiv \underbrace{AAA \cdots A}_{n \text{ times}}, \qquad B^{(n)} \equiv \underbrace{BBB \cdots B}_{n \text{ times}}. \tag{40}$$

We might call $A^{(n)}$ and $B^{(n)}$ "polyadics," since the special cases for $n = 2, 3,$ and 4 are known, respectively, as dyadics, triadics, and tetradics [21]. The following defined scalar products then follow quite naturally from (34):

$$\begin{aligned} A \cdot B &\equiv \sum_{i=1}^{n} A_i B_i \\ AA : BB &\equiv A \cdot (A \cdot BB) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_i A_j B_j B_i, \\ AAA : BBB &\equiv A \cdot [A \cdot (A \cdot BBB)] = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} A_i A_j A_K B_k B_j B_i, \end{aligned} \tag{41}$$

and, in general, define the nth scalar product:

$$A^{(n)} \binom{n}{\cdot} B^{(n)} \equiv \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} A_{i_1} A_{i_2} \cdots A_{i_n} B_{i_n} B_{i_{n-1}} \cdots B_{i_1}. \tag{42}$$

Particular examples of nth order tensors to be used are

$$\begin{aligned} [\mu(\zeta)]^{(n)} &\equiv \underbrace{\mu(\zeta)\, \mu(\zeta) \cdots \mu(\zeta)}_{n \text{ times}} \\ &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \widehat{e}_{i_1} \cdots \widehat{e}_{i_n} \mu_{i_1}(\zeta) \cdots \mu_{i_n}(\zeta). \end{aligned} \tag{43}$$

Theorem 11. *Assume that $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$, and let $\Psi \in L_1[a, b]$. Then,*

$$\mathscr{I}_a^{\alpha_1} \mathscr{I}_a^{\alpha_2} \cdots \mathscr{I}_a^{\alpha_n} F(x) = \mathscr{I}^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} F(x) \tag{44}$$

holds almost everywhere on $[a, b]$. If additionally $\Psi \in C[a, b]$ or $\alpha_1 + \alpha_2 + \cdots + \alpha_n \geq 1$, then the identity holds everywhere on $[a, b]$.

Proof. We have

$$\mathscr{I}_a^\alpha F(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} F(t)\, dt, \quad a \leq x \leq b, \ \alpha \in \mathbb{R} \tag{45}$$

Thus, we can write

$$\begin{aligned} \mathscr{I}_a^{\alpha_1} \mathscr{I}_a^{\alpha_2} \cdots \mathscr{I}_a^{\alpha_n} F(x) = \frac{1}{\Gamma(\alpha_1)} \frac{1}{\Gamma(\alpha_2)} \cdots \frac{1}{\Gamma(\alpha_n)} \int_a^x (x - t_1)^{\alpha_1 - 1} \\ \times \int_a^{t_1} (t_1 - t_2)^{\alpha_2 - 1} \cdots \\ \times \int_a^{t_{n-2}} (t_{n-2} - t_{n-1})^{\alpha_{n-1} - 1} \\ \times \int_a^{t_{n-1}} (t_{n-1} - t_n)^{\alpha_n - 1} \\ \times F(t_n)\, dt_n dt_{n-1} \cdots dt_2 dt_1. \end{aligned} \tag{46}$$

Using Fubini's theorem to interchange the order of integration yields

$$\begin{aligned}
\mathcal{J}_a^{\alpha_1}\mathcal{J}_a^{\alpha_2}\cdots\mathcal{J}_a^{\alpha_n}F(x) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \\
&\quad\times\int_a^x\int_{t_1}^x\cdots\int_{t_{n-1}}^x\int_{t_n}^x(x-t_1)^{\alpha_1-1} \\
&\quad\times(t_1-t_2)^{\alpha_2-1}\cdots \\
&\quad\times(t_{n-2}-t_{n-1})^{\alpha_{n-1}-1} \\
&\quad\times(t_{n-1}-t_n)^{\alpha_n-1} \\
&\quad\times F(t_n)\,dt_1dt_2\cdots dt_{n-1}dt_n \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \\
&\quad\times\int_a^x F(t_n)\prod_{s=2}^n\int_{t_{s-1}}^x(x-t_1)^{\alpha_1-1} \\
&\quad\times(t_{s-1}-t_s)^{\alpha_s-1}dt_sdt_n.
\end{aligned}\tag{47}$$

The substitutions $t_s=t_n+y_{s-1}(x-t_n)$, $s=2,3,\ldots,n$, and $n=1,2,3,\ldots$ yields the new limits of integration as follows: when $t_s=x\Rightarrow x-t_n=y_{s-1}(x-t_n)\Rightarrow y_{s-1}=1$, and when $t_s=t_n\Rightarrow t_n-t_n=y_{s-1}(x-t_n)\Rightarrow y_{s-1}=0$:

$$\begin{aligned}
\mathcal{J}_a^{\alpha_1}\mathcal{J}_a^{\alpha_2}\cdots\mathcal{J}_a^{\alpha_n}F(x) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}\int_a^x F(t_n) \\
&\quad\times\prod_{s=2}^n\Bigg[\int_0^1((x-t_2)(1-y_1))^{\alpha_1-1} \\
&\quad\times(y_{s-1}(x-t_n))^{\alpha_s-1}(x-t_n)dy_{s-1}\Bigg]dt_n \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}\int_a^x F(t_n) \\
&\quad\times\prod_{s=2}^n\Bigg[\int_0^1((x-t_2)(1-y_1))^{\alpha_1-1} \\
&\quad\times(y_{s-1}(x-t_n))^{\alpha_s-1}(x-t_n)dy_{s-1}\Bigg]dt_n \\
&= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} \\
&\quad\times\prod_{s=2}^n\int_a^x F(t_n)(x-t_2)^{\alpha_1-1}(x-t_n)^{\alpha_s} \\
&\quad\times\left[\int_0^1(1-y_1)^{\alpha_1-1}{y_{s-1}}^{\alpha_s-1}dy_{s-1}\right]dt_n.
\end{aligned}\tag{48}$$

Iterating the Euler Beta integral $\int_0^1(1-x)^{\alpha_1-1}x^{\alpha_2-1}dx=\Gamma(\alpha_1)\Gamma(\alpha_2)/\Gamma(\alpha_1+\alpha_2)$ yields

$$\begin{aligned}
\mathcal{J}_a^{\alpha_1}\mathcal{J}_a^{\alpha_2}\cdots\mathcal{J}_a^{\alpha_n}F(x) &= \frac{1}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n)} \\
&\quad\times\int_a^x F(t_n)(x-t_n)^{\alpha_1+\alpha_2+\cdots+\alpha_n}dt_n \\
&= \mathcal{J}^{\alpha_1+\alpha_2+\cdots+\alpha_n}F(x).
\end{aligned}\tag{49}$$

hold almost everywhere on $[a,b]$.

Moreover, by the classical theorems on parameter integrals, if $\Psi\in C[a,b]$, then also $\mathcal{J}_a^{\alpha}\Psi\in C[a,b]$, and therefore $\mathcal{J}_a^{\alpha_1}\mathcal{J}_a^{\alpha_2}\cdots\mathcal{J}_a^{\alpha_n}\Psi\in C[a,b]$, and $\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\alpha_n}\Psi\in C[a,b]$ too. Thus, since these two continuous functions coincide almost everywhere, they must coincide everywhere. Finally, if $\Psi\in L_1[a,b]$ and $\alpha_1+\alpha_2+\cdots+\alpha_n\geq 1$, we have, by the result above

$$\mathcal{J}_a^{\alpha_1}\mathcal{J}_a^{\alpha_2}\cdots\mathcal{J}_a^{\alpha_n}F(x)=\mathcal{J}^{\alpha_1+\alpha_2+\cdots+\alpha_n-1}J^1F(x)\tag{50}$$

almost everywhere. Since $J^1F(x)$ is continuous, and once again we may conclude that the two functions on either side of the equality almost everywhere are continuous, thus they must be identical everywhere. □

Theorem 12. *Assume that $\alpha_1,\alpha_2,\ldots,\alpha_n\geq 0$. Moreover let $\Psi\in L_1[a,b]$. Then,*

$$\mathcal{D}_a^{\alpha_1}\mathcal{D}_a^{\alpha_2}\cdots\mathcal{D}_a^{\alpha_n}F=\mathcal{D}_a^{\alpha_1+\alpha_n+\cdots\alpha_n}F.\tag{51}$$

Proof. The key for proof is using the semigroup property of the integral operators, the assumption on F, and the definition of the Riemann-Liouville differential operator:

$$\begin{aligned}
&\mathcal{D}_a^{\alpha_1}\mathcal{D}_a^{\alpha_2}\cdots\mathcal{D}_a^{\alpha_n}F \\
&\quad=\mathcal{D}_a^{\alpha_1}\mathcal{D}_a^{\alpha_2}\cdots\mathcal{D}_a^{\alpha_n}\,\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\,\alpha_n}\Psi \\
&\quad=D^{\lceil\alpha_1\rceil}\mathcal{J}_a^{\lceil\alpha_1\rceil-\alpha_1}D^{\lceil\alpha_2\rceil}\mathcal{J}_a^{\lceil\alpha_2\rceil-\alpha_2}\cdots D^{\lceil\alpha_n\rceil}\mathcal{J}_a^{\lceil\alpha_n\rceil-\alpha_n} \\
&\qquad\times\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\,\alpha_n}\Psi \\
&\quad=D^{\lceil\alpha_1\rceil}\mathcal{J}_a^{\lceil\alpha_1\rceil-\alpha_1}D^{\lceil\alpha_2\rceil}\mathcal{J}_a^{\lceil\alpha_2\rceil-\alpha_2}\cdots D^{\lceil\alpha_n\rceil}\mathcal{J}_a^{\lceil\alpha_n\rceil} \\
&\qquad\times\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\,\alpha_{n-1}}\Psi \\
&\quad=D^{\lceil\alpha_1\rceil}\mathcal{J}_a^{\lceil\alpha_1\rceil-\alpha_1}D^{\lceil\alpha_2\rceil}\mathcal{J}_a^{\lceil\alpha_2\rceil-\alpha_2}\cdots D^{\lceil\alpha_{n-1}\rceil}\mathcal{J}_a^{\lceil\alpha_{n-1}\rceil-\alpha_{n-1}} \\
&\qquad\times\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\,\alpha_{n-1}}\Psi \\
&\quad=D^{\lceil\alpha_1\rceil}\mathcal{J}_a^{\lceil\alpha_1\rceil-\alpha_1}D^{\lceil\alpha_2\rceil}\mathcal{J}_a^{\lceil\alpha_2\rceil-\alpha_2}\cdots D^{\lceil\alpha_{n-1}\rceil}\mathcal{J}_a^{\lceil\alpha_{n-1}\rceil} \\
&\qquad\times\mathcal{J}_a^{\alpha_1+\alpha_2+\cdots+\,\alpha_{n-2}}\Psi
\end{aligned}$$

$$
\begin{aligned}
&= D^{\lceil \alpha_1 \rceil} \mathcal{J}_a^{\lceil \alpha_1 \rceil - \alpha_1} D^{\lceil \alpha_2 \rceil} \mathcal{J}_a^{\lceil \alpha_2 \rceil - \alpha_2} \cdots D^{\lceil \alpha_{n-2} \rceil} \mathcal{J}_a^{\lceil \alpha_{n-2} \rceil - \alpha_{n-2}} \\
&\quad \times \mathcal{J}_a^{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2}} \Psi \\
&\vdots \\
&= D^{\lceil \alpha_1 \rceil} \mathcal{J}_a^{\lceil \alpha_1 \rceil} \Psi = \Psi.
\end{aligned} \tag{52}
$$

The proof that $\mathcal{D}_a^{\alpha_1 + \alpha_n + \cdots + \alpha_n} F = \Psi$ is quite straightforward

$$
\begin{aligned}
\mathcal{D}_a^{\alpha_1 + \alpha_n + \cdots \alpha_n} F &= D^{\lceil \alpha_1 + \alpha_n + \cdots \alpha_n \rceil} \\
&\quad \times \mathcal{J}_a^{\lceil \alpha_1 + \alpha_n + \cdots \alpha_n \rceil - \alpha_1 - \alpha_n - \cdots - \alpha_n} \\
&\quad \times \mathcal{J}_a^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \Psi \\
\mathcal{D}_a^{\alpha_1 + \alpha_n + \ldots \alpha_n} F &= D^{\lceil \alpha_1 + \alpha_n + \ldots \alpha_n \rceil} \\
&\quad \times \mathcal{J}_a^{\lceil \alpha_1 + \alpha_n + \ldots \alpha_n \rceil} \Psi = \Psi.
\end{aligned} \tag{53}
$$

Thus,

$$
\mathcal{D}_a^{\alpha_1 + \alpha_n + \cdots + \alpha_n} F = \mathcal{D}_a^{\alpha_1} \mathcal{D}_a^{\alpha_2} \cdots \mathcal{D}_a^{\alpha_n} F. \tag{54}
$$

□

Theorem 13. *Assume that $\alpha_1, \alpha_2, \ldots \alpha_n \geq 0$. Moreover, let $\Psi \in L_1[a,b]$ and $F = \mathcal{J}_a^{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \Psi$. And let ∇_ζ^α be the fractional gradient operator. Then,*

$$
\begin{aligned}
&\nabla_\zeta^{\alpha(n)} F(\zeta) \\
&= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \widehat{e}_{i_1} \widehat{e}_{i_2} \cdots \widehat{e}_{i_n} \frac{1}{\Gamma(nm - \alpha_1 - \alpha_2 \cdots - \alpha_n)} \\
&\quad \times \left(\frac{\partial^n}{\partial \zeta_{i_1} \cdots \partial \zeta_{i_2}} \right)^m \int_a^\zeta F(\tau_n) (\zeta - \tau_n)^{nm - \alpha_1 - \alpha_2 - \cdots - \alpha_n} d\tau_n.
\end{aligned} \tag{55}
$$

Proof. By the assumption on F and the successive application of fractional gradient operator, we have $\nabla_\zeta^{\alpha(n)} F(\zeta) \equiv \nabla_\zeta^{\alpha_1} \nabla_\zeta^{\alpha_2} \cdots \nabla_\zeta^{\alpha_n} F(\zeta)$:

$$
\begin{aligned}
\nabla_\zeta^{\alpha(n)} F(\zeta) &= \sum_{i_1=1}^{n} \widehat{e}_{i_1} \frac{1}{\Gamma(m - \alpha_1)} \left(\frac{\partial}{\partial \zeta_{i_1}} \right)^m \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_1 - 1} F(\tau) \, d\tau \\
&\quad \times \sum_{i_2=1}^{n} \widehat{e}_{i_2} \frac{1}{\Gamma(m - \alpha_2)} \left(\frac{\partial}{\partial \zeta_{i_2}} \right)^m \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_2 - 1} F(\tau) \, d\tau \cdots \\
&\quad \times \sum_{i_n=1}^{n} \widehat{e}_{i_n} \frac{1}{\Gamma(m - \alpha_n)} \\
&\quad \times \left(\frac{\partial}{\partial \zeta_{i_n}} \right)^m \int_a^\zeta (\zeta - \tau)^{m - \alpha_n - 1} F(\tau) \, d\tau \\
&= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \widehat{e}_{i_1} \widehat{e}_{i_2} \cdots \widehat{e}_{i_n} \\
&\quad \times \frac{1}{\Gamma(m - \alpha_1) \Gamma(m - \alpha_2) \cdots \Gamma(m - \alpha_n)} \\
&\quad \times \left(\frac{\partial}{\partial \zeta_{i_1}} \right)^m \left(\frac{\partial}{\partial \zeta_{i_2}} \right)^m \cdots \left(\frac{\partial}{\partial \zeta_{i_n}} \right)^m \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_1 - 1} F(\tau) \, d\tau \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_2 - 1} F(\tau) \, d\tau \cdots \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_n - 1} F(\tau) \, d\tau \\
&= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \widehat{e}_{i_1} \widehat{e}_{i_2} \cdots \widehat{e}_{i_n} \\
&\quad \times \frac{1}{\Gamma(nm - \alpha_1 - \alpha_2 \cdots - \alpha_n)} \left(\frac{\partial^n}{\partial \zeta_{i_1} \cdots \partial \zeta_{i_2}} \right)^m \\
&\quad \times \int_a^\zeta (\zeta - \tau)^{m - \alpha_1 - 1} F(\tau) \, d\tau \int_a^\zeta (\zeta - \tau)^{m - \alpha_2 - 1} \\
&\quad \times F(\tau) \, d\tau \cdots \int_a^\zeta (\zeta - \tau)^{m - \alpha_n - 1} F(\tau) \, d\tau.
\end{aligned} \tag{56}
$$

Thus using the theorem, the results follow directly:

$$
\begin{aligned}
\nabla_\zeta^{\alpha(n)} F(\zeta) &= \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \widehat{e}_{i_1} \widehat{e}_{i_2} \cdots \widehat{e}_{i_n} \\
&\quad \times \frac{1}{\Gamma(nm - \alpha_1 - \alpha_2 \cdots - \alpha_n)} \left(\frac{\partial^n}{\partial \zeta_{i_1} \cdots \partial \zeta_{i_2}} \right)^m \\
&\quad \times \int_a^\zeta F(\tau_n) (\zeta - \tau_n)^{nm - \alpha_1 - \alpha_2 - \cdots - \alpha_n} d\tau_n.
\end{aligned} \tag{57}
$$

□

7. Fractional Taylor Expansion of a Function of N-Dimensional Polyadics

We have the classical Taylor expansion for the m-independent variables as

$$
\begin{aligned}
&f(x_1, x_2, \ldots, x_m) \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\sum_{i=1}^{m} (x_i - x_{i_0}) \frac{\partial}{\partial x_i} \right)^n \\
&\quad \times f(x_1, x_2, \ldots, x_m) \Big|_{x_1 = x_{1_0},\ x_2 = x_{2_0},\ \cdots x_m = x_{m_0}}
\end{aligned} \tag{58}
$$

In the light of the above definitions and theorems, we can state the following theorem.

Definition 14. Let $f(x_i) = f(x_1, \ldots, x_n) \in A^m(\Omega)$, where $\Omega = \prod_{i=1}^{n}(a_i, b_i) \subset \mathbb{R}^n$; then the fractional Riemann-Liouville multiple integrals and partial derivatives with respect to x_i are;

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a_{k_i}, x_{k_i}} f(x_i) = {} & \left(\frac{1}{\Gamma(\alpha)}\right)^{s} \int_{a_{k_1}}^{x_{k_1}} \cdots \\
& \times \int_{a_{k_s}}^{x_{k_s}} f(t_i) \prod_{i=1}^{s} \int_{t_{s-1}}^{x} \left(x_{k_i} - t_{k_i}\right)^{\alpha-1} \\
& \times dt_{k_1} \cdots dt_{k_s}
\end{aligned} \tag{59}$$

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a_{k_i}, x_{k_i}} f(x_i) = {} & \left(\frac{1}{\Gamma(n-\alpha)}\right) \left(\frac{\partial}{\partial x_i}\right)^{n} \\
& \times \int_{a_i}^{x_i} f(t_i)\ (x_i - t_i)^{n-\alpha-1}\ dt_i.
\end{aligned}$$

Theorem 15. *Let $n > 0$ and $m = \lfloor n \rfloor + 1$. Assume that $f(x_i)$ is such that $\mathcal{J}^{n-m}_{a} f(x_i) \in A^m(\Omega)$, where $\Omega = \prod_{i=1}^{n}(a_i, b_i) \subset \mathbb{R}^n$ is the domain of f. Then,*

$$\begin{aligned}
f(x_i) = {} & \frac{(x_i - a_i)^{n-m}}{\Gamma(n-m-1)} \lim_{z_i \to a_i^+} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} = f(z_i) \\
& + \sum_{k=0}^{m-1} \frac{(x_i - a_i)^{k+n-m}}{\Gamma(k+n-m-1)} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \\
& \times f(z_i) + \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} f(x_i).
\end{aligned} \tag{60}$$

Proof. Because of our assumption about f that implies the continuity of $D^{m-1}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f$, there exists some $\Psi \in L_1(\Omega)$ such that

$$D^{m-1}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(x_i) = D^{m-1}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(a_i) + J^{1}_{a_{k_i}, x_{k_i}} \Psi(x_i). \tag{61}$$

This is a classical partial differential equation of order $m-1$ for $\mathcal{J}^{m-n}_{a} f(x_i)$; its solution is easily seen to be of the form:

$$\begin{aligned}
\mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(x_i) &= \sum_{k=0}^{m-1} \frac{(x_i - a_i)^k}{k!} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \\
&= f(z_i) + J^{m}_{a_{k_i}, x_{k_i}} \Psi(x_i).
\end{aligned} \tag{62}$$

Thus, by definition of $\mathcal{D}^{n}_{a_{k_i}, x_{k_i}}$,

$$\begin{aligned}
\mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} &= f(x_i) = \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} D^{m}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(x_i) \\
&= \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} D^{m}_{x_{k_i}} \left[\sum_{k=0}^{m-1} \frac{(x_i - a_i)^k}{k!} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \right. \\
&\qquad \times \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(z_i) \\
&\qquad \left. + J^{m}_{a_{k_i}, x_{k_i}} \Psi(x_i) \right],
\end{aligned} \tag{63}$$

$$\begin{aligned}
\mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} &= f(x_i) = \sum_{k=0}^{m-1} \frac{\mathcal{J}^{n}_{a_{k_i}, x_{k_i}} D^{m}_{x_{k_i}} (x_i - a_i)^k}{k!} \\
&\qquad \times \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(z_i) \\
&\qquad + \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} D^{m}_{x_{k_i}} J^{m}_{a_{k_i}, x_{k_i}} \Psi(x_i).
\end{aligned}$$

$D^{m}_{x_{k_i}}$ annihilates every summand in the sum. And due to Theorem 8, we obtain

$$\mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} f(x_i) = \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} = \Psi(x_i). \tag{64}$$

Next, we apply the operator $\mathcal{D}^{m-n}_{a_{k_i}, x_{k_i}}$ to (63), finding

$$\begin{aligned}
f(x_i) &= \sum_{k=0}^{m-1} \frac{\mathcal{D}^{m-n}_{a_{k_i}, x_{k_i}} (x_i - a_i)^k}{k!} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \\
&= f(z_i) + \mathcal{D}^{m-n}_{a_{k_i}, x_{k_i}} J^{m}_{a_{k_i}, x_{k_i}} \Psi(x_i) \\
&= \sum_{k=0}^{m-1} \frac{\mathcal{D}^{m-n}_{a_{k_i}, x_{k_i}} (x_i - a_i)^k}{k!} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \\
&= f(z_i) + D^{1}_{x_{k_i}} \mathcal{J}^{1-m+n}_{a_{k_i}, x_{k_i}} J^{m}_{a_{k_i}, x_{k_i}} \Psi(x_i) \\
&= \sum_{k=0}^{m-1} \frac{(x_i - a_i)^{k+n-m}}{\Gamma(k+n-m-1)} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \\
&= f(z_i) + \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \Psi(x_i).
\end{aligned} \tag{65}$$

Using (64), we obtain

$$\begin{aligned}
\mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \Psi(x_i) &= \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} f(x_i) \\
&= f(x_i) - \sum_{k=0}^{m-1} \frac{(x_i - a_i)^{k+n-m}}{\Gamma(k+n-m-1)} \lim_{z_i \to a_i^+} D^{k}_{x_{k_i}} \\
&\qquad \times \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(z_i).
\end{aligned} \tag{66}$$

Upon extracting the first term out of summation, we can write

$$f(x_i) = \frac{(x_i - a_i)^{n-m}}{\Gamma(n-m-1)} \lim_{z_i \to a_i^+} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} f(z_i) + \sum_{k=1}^{m-1} \frac{(x_i - a_i)^{k+n-m}}{\Gamma(k+n-m-1)} \lim_{z_i \to a_i^+} D^k_{x_{k_i}} \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} \times f(z_i) + \mathcal{J}^{n}_{a_{k_i}, x_{k_i}} \mathcal{D}^{n}_{a_{k_i}, x_{k_i}} \times f(x_i). \quad (67)$$

□

This result can be written conveniently in the Polyadics notation $A^{(n)}$, extending the upper limit of summation to ∞ to absorb the remainder, as

$$F\left(r^{(n)}\right) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+n-m-1)} \times \lim_{z^{(n)} \to a^{(n)+}} \left(\left(r^{(n)} - a^{(n)} \right)^{n-m} \binom{n}{\cdot} \nabla_\zeta^{\alpha(n)} \right)^k \times \mathcal{J}^{m-n}_{a_{k_i}, x_{k_i}} = F\left(z^{(n)}\right). \quad (68)$$

8. Fractional N-Dimensional Lagrange Expansion

Theorem 16. *Let $F(z) = F(w) + \mu(z, \varepsilon), F, \mu \in A^m(\Omega)$ such that*

$$F(z) = F(w) + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} [\mu(z)]^{(n)} \binom{n}{\cdot} \nabla_w^{(n)} F(w), \quad (69)$$

then it is still possible to invert or solve the equation for $F(z)$ such that $w = F(z), F \in A^m(\Omega)$ on a neighborhood of $f(a)$ using the fractional calculus as follows:

$$F(z) = \frac{1}{\Gamma(\alpha - m - 1)} \mathcal{J}^{m-\alpha}_{a_{k_i}, x_{k_i}} = F(a) + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon^n}{\Gamma(k+\alpha-m-1)} \times \lim_{w \to a^+} \left([\mu(w-a)^{n-m}]^{(n)} \binom{n}{\cdot} \nabla_\zeta^{\alpha(n)} \right)^k \times \mathcal{J}^{m-\alpha}_{a_{k_i}, x_{k_i}} = F(w), \quad (70)$$

where $\alpha > 0$ and $m = \lfloor \alpha \rfloor + 1$.

Proof. It is possible to invert or solve the equation for $F(z)$ in a neighborhood of $F(a)$. Using the fractional Taylor expansion.

$$F(z) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon^n}{\Gamma(k+\alpha-m-1)} \times \lim_{w \to a^+} \left([\mu(w-a)^{n-m}]^{(n)} \binom{n}{\cdot} \nabla_\zeta^{\alpha(n)} \right)^k \mathcal{J}^{m-\alpha}_{a_{k_i}, x_{k_i}} \times F(w). \quad (71)$$

Extracting the first term out of the summations, we obtain the result directly

$$F(z) = \frac{1}{\Gamma(\alpha - m - 1)} \mathcal{J}^{m-\alpha}_{a_{k_i}, x_{k_i}} = F(a) + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{\varepsilon^n}{\Gamma(k+\alpha-m-1)} \times \lim_{w \to a^+} \left([\mu(w-a)^{n-m}]^{(n)} \binom{n}{\cdot} \nabla_\zeta^{\alpha(n)} \right)^k \mathcal{J}^{m-\alpha}_{a_{k_i}, x_{k_i}} \times F(w). \quad (72)$$

□

Acknowledgments

The author is deeply indebted and thankful to the deanship of the scientific research and its helpful and distinct team of employees at Taibah University, Al-Madinah Al-Munawwarah, Saudi Arabia This research work was supported by a Grant no. 3017/1434. Also he wishes to express his deep gratitude and indebtedness to the referees of his manuscript for all very fruitful discussions, constructive comments, remarks, and suggestions.

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