

Research Article

A Note on the Asymptotic Behavior of Parabolic Monge-Ampère Equations on Riemannian Manifolds

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We study the asymptotic behavior of the parabolic Monge-Ampère equation $\partial\varphi(x, t)/\partial t = \log(\det(g(x) + \text{Hess}\varphi(x, t))/\det g(x)) - \lambda\varphi(x, t)$ in $\mathbb{M} \times (0, \infty)$, $\varphi(x, 0) = \varphi_0(x)$ in $\mathbb{M}$, where $\mathbb{M}$ is a compact complete Riemannian manifold, λ is a positive real parameter, and $\varphi_0(x) : \mathbb{M} \rightarrow \mathbb{R}$ is a smooth function. We show a meaningful asymptotic result which is more general than those in Huiskens, 1997.

1. Introduction

The main purpose of this paper is to study the asymptotic behavior of the parabolic Monge-Ampère equation:

$$\begin{aligned} \frac{\partial\varphi(x, t)}{\partial t} &= \log \left\{ \frac{\det(g(x) + \text{Hess}\varphi(x, t))}{\det g(x)} \right\} \\ &\quad - \lambda\varphi(x, t) \quad \text{in } \mathbb{M} \times (0, \infty), \\ \varphi(x, 0) &= \varphi_0(x), \quad \text{in } \mathbb{M}, \end{aligned} \quad (1)$$

where $\mathbb{M}$ is a compact complete Riemannian manifold, λ is a positive real parameter, and $\varphi_0(x) : \mathbb{M} \rightarrow \mathbb{R}$ is a smooth function. We show a meaningful precisely asymptotic result which is more general than those in [1].

Monge-Ampère equations arise naturally from some problems in differential geometry. The existence and regularity of solutions to Monge-Ampère equations have been investigated by many mathematicians [1–8]. The long time existence and convergence of solution to (1) have been investigated in [1]. To some extent, we extend asymptotic result obtained in [1] in this paper. Hence, our main result is following analogue of Theorem 1.2 of [1].

Theorem 1. Let φ be the solution of (1) with $\lambda > 0$. For $p > 1$, there exists $\delta > 0$ and $f > 0$ depending on φ_0 and $\|\nabla^\beta \varphi\|_{L^\infty}$ ($\beta = 0, 1, 2, 3$) such that

$$\begin{aligned} \int_{\mathbb{M}} (\varphi^p - \overline{\varphi^p})^2 d\mu \\ \leq f \exp \left(-2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t) \right] t \right), \end{aligned} \quad (2)$$

where $\overline{\varphi^p}$ denotes the mean value of φ^p , η_1 is the first eigenvalue of the Laplacian, and $\varepsilon(t) \triangleq \exp(-\delta t)$.

Remark 2. If $p = 1$, Theorem 1 is in accordance with Theorem 1.2 of [1].

Lemma 3 (see [1]). There exists positive constants $C_0(\varphi_0, \lambda)$ and C_1 depending on $(\mathbb{M}, g)$, φ_0 , $\|\varphi\|_{L^\infty}$, $\|\nabla^{\beta-\gamma}\varphi\|_{L^\infty}$ ($\beta = 0, 1, 2, 3; \gamma = 1, 2, 3; \gamma \leq \beta$) such that

$$\begin{aligned} |\varphi| &\leq C_0 \exp(-\lambda t), \\ |\nabla^\beta \varphi|^2 &\leq C_1 \exp(-2\lambda t). \end{aligned} \quad (3)$$

Theorem 1 is proved in Section 2.

2. Asymptotic Behavior

Proof of Theorem 1. In local coordinates, we have the following evolution equation:

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \log \left\{ \frac{\det (g + \text{Hess} \varphi)}{\det g} \right\} - \lambda \varphi \\ &= \Delta \varphi + \log \det (g_{ij} + \nabla_{ij} \varphi) \\ &\quad - \log \det g_{ij} - \Delta \varphi - \lambda \varphi \\ &= \Delta \varphi + \int_0^1 \frac{d}{ds} \left\{ \log \det (g_{ij} + s \nabla_{ij} \varphi) \right. \\ &\quad \left. - (s-1) \Delta \varphi \right\} ds - \lambda \varphi. \end{aligned} \quad (4)$$

Now, setting

$$\begin{aligned} \bar{g}_s^{ij} &\triangleq (g_{ij} + s \nabla_{ij} \varphi)^{-1}, \\ A &\triangleq \iint_0^1 s |\nabla^2 \varphi|_{\bar{g}_{ss}}^2 d\alpha ds. \end{aligned} \quad (5)$$

We rewrite (4) in more convenient notation as

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \Delta \varphi + \int_0^1 (\bar{g}_s^{ij} - g^{ij}) \nabla_{ij} \varphi d\alpha - \lambda \varphi \\ &= \Delta \varphi + \iint_0^1 \frac{d}{d\alpha} (g_{ij} + \alpha s \nabla_{ij} \varphi)^{-1} \nabla_{ij} \varphi d\alpha ds - \lambda \varphi \\ &= \Delta \varphi - \iint_0^1 (g_{ik} + \alpha s \nabla_{ik} \varphi)^{-1} (g_{jl} + \alpha s \nabla_{jl} \varphi)^{-1} \\ &\quad \times s \nabla_{kl} \varphi \nabla_{ij} \varphi d\alpha ds - \lambda \varphi \\ &= \Delta \varphi - \iint_0^1 s |\nabla^2 \varphi|_{\bar{g}_{ss}}^2 d\alpha ds - \lambda \varphi \\ &= \Delta \varphi - A - \lambda \varphi. \end{aligned} \quad (6)$$

We want to apply Gronwall inequality and hence consider the following equation:

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu &= 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \dot{\varphi} d\mu \\ &= 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} (\Delta \varphi - A - \lambda \varphi) d\mu \\ &= 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \Delta \varphi d\mu \\ &\quad - 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} A d\mu \\ &\quad - 2p\lambda \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^p d\mu. \end{aligned} \quad (7)$$

Notice that

$$\int_{\mathbb{M}} \bar{\varphi}^p (\varphi^p - \bar{\varphi}^p) d\mu = 0. \quad (8)$$

We obtain

$$-2p\lambda \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^p d\mu = -2p\lambda \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu. \quad (9)$$

Furthermore we have

$$\begin{aligned} 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \Delta \varphi d\mu &= -2p \int_{\mathbb{M}} \nabla \{ (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \} \nabla \varphi d\mu \\ &= -2p \int_{\mathbb{M}} \{ \nabla (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \\ &\quad + (p-1) \varphi^{p-2} \nabla \varphi (\varphi^p - \bar{\varphi}^p) \} \nabla \varphi d\mu \\ &= -2 \int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu - \frac{2(p-1)}{p} \\ &\quad \times \int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \\ &\quad + 2p(p-1) \bar{\varphi}^p \int_{\mathbb{M}} \varphi^{p-2} |\nabla \varphi|^2 d\mu \\ &= -\frac{2(2p-1)}{p} \int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \\ &\quad + 2p(p-1) \bar{\varphi}^p \int_{\mathbb{M}} \varphi^{p-2} |\nabla \varphi|^2 d\mu. \end{aligned} \quad (10)$$

We use the Poincare inequality

$$\|\nabla (\varphi^p - \bar{\varphi}^p)\|_{L^2} \geq \eta_1 \|(\varphi^p - \bar{\varphi}^p)\|_{L^2}. \quad (11)$$

It follows that

$$\begin{aligned} 2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} \Delta \varphi d\mu &\leq -\frac{2(2p-1)\eta_1}{p} \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\ &\quad + 2p(p-1) \bar{\varphi}^p \int_{\mathbb{M}} \varphi^{p-2} |\nabla \varphi|^2 d\mu. \end{aligned} \quad (12)$$

Moreover, we have that

$$\begin{aligned} -2p \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} A d\mu &= 2p \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \nabla_{ij} \varphi (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} d\mu \\ &\quad ds. \end{aligned}$$

$$\begin{aligned}
&= -2p \int_0^1 \int_{\mathbb{M}} \nabla_i (\bar{g}_s^{ij} - g^{ij}) \nabla_j \varphi (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} d\mu \\
&\quad - 2p \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \nabla_j \varphi \nabla_i (\varphi^p - \bar{\varphi}^p) \varphi^{p-1} d\mu \\
&\quad - 2p \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \nabla_j \varphi (\varphi^p - \bar{\varphi}^p) \nabla_i (\varphi^{p-1}) d\mu \\
&= -2 \int_0^1 \int_{\mathbb{M}} \nabla_i (\bar{g}_s^{ij} - g^{ij}) \nabla_j (\varphi^p - \bar{\varphi}^p) (\varphi^p - \bar{\varphi}^p) d\mu \\
&\quad - 2 \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \nabla_j (\varphi^p - \bar{\varphi}^p) \nabla_i (\varphi^p - \bar{\varphi}^p) d\mu \\
&\quad - 2p(p-1) \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \varphi^{2(p-1)} \nabla_i \varphi \nabla_j \varphi d\mu \\
&\quad + 2p(p-1) \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \varphi^{p-2} \nabla_i \varphi \nabla_j \varphi \bar{\varphi}^p d\mu \\
&= -2 \int_0^1 \int_{\mathbb{M}} \nabla_i (\bar{g}_s^{ij} - g^{ij}) \nabla_j (\varphi^p - \bar{\varphi}^p) (\varphi^p - \bar{\varphi}^p) d\mu \\
&\quad - \frac{2(2p-1)}{p} \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \nabla_j (\varphi^p - \bar{\varphi}^p) \nabla_i \\
&\quad \times (\varphi^p - \bar{\varphi}^p) d\mu \\
&\quad + 2p(p-1) \bar{\varphi}^p \int_0^1 \int_{\mathbb{M}} (\bar{g}_s^{ij} - g^{ij}) \varphi^{p-2} \nabla_i \varphi \nabla_j \varphi d\mu \\
&\leq 2 \int_0^1 \int_{\mathbb{M}} |\nabla (\bar{g}_s^{ij} - g^{ij})| \cdot |\nabla (\varphi^p - \bar{\varphi}^p)| \cdot |\varphi^p - \bar{\varphi}^p| d\mu \\
&\quad + \frac{2(2p-1)}{p} \int_0^1 \int_{\mathbb{M}} |\bar{g}_s^{ij} - g^{ij}| \cdot |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \\
&\quad + 2p(p-1) |\bar{\varphi}^p| \int_0^1 \int_{\mathbb{M}} |(\bar{g}_s^{ij} - g^{ij})| \cdot |\varphi^{p-2}| \cdot |\nabla \varphi|^2 d\mu \\
&\leq 2C \sup |\nabla^3 \varphi| \\
&\quad \times \left(\int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu + \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \right) \\
&\quad + \frac{2C(2p-1)}{p} \sup |\nabla^2 \varphi| \int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \\
&\quad + 2p(p-1) C |\bar{\varphi}^p| \sup |\nabla^2 \varphi| \int_{\mathbb{M}} |\varphi^{p-2}| \cdot |\nabla \varphi|^2 d\mu.
\end{aligned} \tag{13}$$

where C is always a constant that may change from line to line.

Substituting (9), (12), and (13) in the right-hand side of (7)

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\
&\leq -2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda \right] \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\
&\quad + 2p(p-1) \bar{\varphi}^p \int_{\mathbb{M}} \varphi^{p-2} |\nabla \varphi|^2 d\mu
\end{aligned}$$

$$\begin{aligned}
&+ 2C \sup |\nabla^3 \varphi| \left(\int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \right. \\
&\quad \left. + \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \right) \\
&+ \frac{2C(2p-1)}{p} \sup |\nabla^2 \varphi| \int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \\
&+ 2p(p-1) C |\bar{\varphi}^p| \sup |\nabla^2 \varphi| \int_{\mathbb{M}} |\varphi^{p-2}| \cdot |\nabla \varphi|^2 d\mu.
\end{aligned} \tag{14}$$

By Lemma 3, that is, the exponential decay of $|\nabla^\beta \varphi|_{L^\infty}$ ($\beta = 0, 1, 2, 3$), it is easy to obtain the following.

For any $\varepsilon > 0$, there exists a $T(\varepsilon)$ such that

$$\begin{aligned}
&\frac{\partial}{\partial t} \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\
&\leq -2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t) \right] \\
&\quad \times \int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\
&\quad + 2\varepsilon \left(\int_{\mathbb{M}} |\nabla (\varphi^p - \bar{\varphi}^p)|^2 d\mu \right. \\
&\quad \left. + |\bar{\varphi}^p| \int_{\mathbb{M}} |\varphi^{p-2}| \cdot |\nabla \varphi|^2 d\mu \right).
\end{aligned} \tag{15}$$

The Gronwall inequality yields

$$\begin{aligned}
&\int_{\mathbb{M}} (\varphi^p - \bar{\varphi}^p)^2 d\mu \\
&\leq f \exp \left(-2 \left[\frac{(2p-1)\eta_1}{p} + p\lambda - \varepsilon(t) \right] t \right),
\end{aligned} \tag{16}$$

where the constant $f > 0$ depending on φ_0 and $\|\nabla^\beta \varphi\|_{L^\infty}$ ($\beta = 0, 1, 2, 3$).

Thus, the proof of Theorem 1 is completed. $\square$

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