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Research Article

Fixed Point Results in Quasi-Cone Metric Spaces

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The main aim of this paper is to prove fixed point theorems in quasi-cone metric spaces which extend the Banach contraction mapping and others. This is achieved by introducing different kinds of Cauchy sequences in quasi-cone metric spaces.

1. Introduction and Preliminaries

The Banach contraction principle is a fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations (see, e.g., [1–12]).

A quasi-metric is a distance function which satisfies the triangle inequality but is not symmetric: it can be regarded as an "asymmetric metric." In fact, quasi-metric space is more comprehensive than metric space. As metric space is important and has numerous applications, Huang and Zhang [13] have announced the concept of the cone metric spaces, replacing the set of real numbers by an ordered Banach space. They have proved some fixed point theorems for contractive-type mappings on cone metric spaces, whereas Rezapour and Hamlbarani [14] omitted the assumption of normality in cone metric spa ces, which is a milestone in developing fixed point theory in cone metric spaces. Since then, numerous authors have started to generalize fixed point theorems in cone metric spaces in many various directions. For some recent results (see, e.g., [15–25]) and for a current survey of the latest results in cone metric spaces, see Janković et al. [26].

Very recently, some authors generalized the contractive conditions in the literature by replacing the constants with functions. Using these generalizations, they have proved the existence and uniqueness of the fixed point in cone metric spaces; for more details see [27, 28]. Because quasi-metric space is more general than metric space and is a subject of intensive research in the context of topology and theoretical computer science, Abdeljawad and Karapinar [29] and Sonmez [30] have given a definition of quasi-cone metric space which extends the quasi-metric space.

In this paper, we also introduce the concept of a quasi-cone metric space which is somewhat different from that of Abdeljawad and Karapinar [29] and Sonmez [30]. Then we establish four kinds of Cauchy sequences in this space according to Reilly et al. [31]. Furthermore, we extend and generalize the Banach contraction principle and some results in the literature to this space. We support our results by examples. In this paper we do not impose the normality condition for the cones, and the only assumption is that the cone P is, solid; that is $\operatorname{int} P \neq \emptyset$. Now we recall some known notions, definitions, and results which will be used in this work.

Definition 1. Let E be a real Banach space and P be a subset of E. P is called a cone if and only if

(i) P is closed, $P \neq \emptyset$, $P \neq \{0\}$;

(ii) for all $x, y \in P \Rightarrow \alpha x + \beta y \in P$, where $\alpha, \beta \in \mathbb{R}^{+}$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \subset E$, we define a *partial ordering* $\preceq$ with respect to P by the following: for $x, y \in E$, we say that $x \preceq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the *interior* of P. The cone

P is called *normal* if there is a number $K > 0$ such that for all $x, y \in E$

$$0 \preceq x \preceq y \Longrightarrow \|x\| \leq K \|y\|. \quad (1)$$

The least positive number K satisfying this is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded above is convergent; that is, if x_n is a sequence such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y \quad (2)$$

for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if every decreasing sequence which is bounded below is convergent (for details, see [13]). In this paper, we always suppose that E is a real Banach space, P is a cone in E with $\operatorname{int} P \neq \emptyset$, and $\preceq$ is a partial ordering with respect to P.

Definition 2 (see [13]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

($d1$) $0 \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$;

($d2$) $d(x, y) = d(y, x)$ for all $x, y \in X$;

($d3$) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X, and (X, d) is called a cone metric space.

Now, we state our definition which is more general than cone metric space.

Definition 3. Let X be a nonempty set. Suppose the mapping $q : X \times X \to E$ satisfies

($q1$) $0 \preceq q(x, y)$ for all $x, y \in X$;

($q2$) $q(x, y) = 0 = q(y, x)$ if and only if $x = y$;

($q3$) $q(x, y) \preceq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then, q is called a quasi-cone metric on X, and (X, q) is called a quasi-cone metric space.

Remark 4. Note that in [30] Sonmez defined the quasi-cone metric space as follows.

A quasi-cone metric space on a nonempty X is a function $q : X \times X \to E$ such that for all $x, y, z \in X$;

(1) $q(x, y) = q(y, x) = 0 \Leftrightarrow x = y$,

(2) $q(x, y) \preceq q(x, z) + q(z, y)$.

A quasi-cone metric space is a pair (X, q) such that X is a nonempty set and q is a quasi-cone metric on X.

In fact, it has not mentioned that q takes value in P, but in this paper we require this condition.

Remark 5. Abdeljawad and Karapinar's definition of quasi-cone metric space [29] is as follows.

Let X be a nonempty set. Suppose that the mapping $q : X \times X \to E$ satisfies the following:

($q1$) $0 \preceq q(x, y)$ for all $x, y \in X$;

($q2$) $q(x, y) = 0 \Leftrightarrow x = y$;

($q3$) $q(x, y) \preceq q(x, z) + q(z, y)$ for all $x, y, z \in X$.

Then d is said to be a quasi-cone metric on X, and the pair (X, q) is called a quasi-cone metric space.

The following example indicates that our definition is more general than the one given in [29].

Example 6. Let $X = (0, \infty)$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, and $q : X \times X \to E$ defined by

$$q(x, y) = \begin{cases} \left(\dfrac{1}{y} - \dfrac{1}{x}, x\right), & \text{if } \ y < x, \\ (0, 0), & \text{if } \ y \geq x. \end{cases} \quad (3)$$

Then (X, q) satisfies our definition of a quasi-cone metric space but not the definition in [29] because if $q(x, y) = 0$ then $x = y$ or $y > x$.

Remark 7. Note that any cone metric space is a quasi-cone metric space.

2. Necessary Facts and Statements

By considering the established notions in metric spaces [31], we introduce the appropriate generalization in cone metric spaces.

Definition 8. Let (X, q) be a quasi-cone metric space. A sequence $\{x_n\}$ in X is said to be

(a) Q-Cauchy or bi-Cauchy if for each $c \in \operatorname{int} P$, there is $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) \ll c$ for all $n, m \geq n_0$;

(b) left (right) Cauchy if for any $c \in \operatorname{int} P$, there is $n_0 \in \mathbb{N}$ such that $q(x_m, x_n) \ll c$ ($q(x_n, x_m) \ll c$, resp.) for all $n \geq m \geq n_0$;

(c) weakly left (right) Cauchy if for each $c \in \operatorname{int} P$, there is $n_0 \in \mathbb{N}$ such that $q(x_{n_0}, x_n) \ll c$ ($q(x_n, x_{n_0}) \ll c$, resp.) for all $n \geq n_0$;

(d) left (right) q-Cauchy if for every $c \in \operatorname{int} P$, there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that $q(x, x_n) \ll c$ ($q(x_n, x) \ll c$, resp.) for all $n \geq n_0$.

Remark 9. These notions in quasi-cone metric space are related in the following way:

(i) Q-Cauchy $\Rightarrow$ left (right) Cauchy $\Rightarrow$ weakly left (right) Cauchy $\Rightarrow$ left (right) q-Cauchy;

(ii) a sequence is Q-Cauchy if and only if it is both left and right Cauchy.

In this paper, we use the notion of left Cauchy.

Definition 10. Let (X, q) be a quasi-cone metric space. Let $\{x_n\}_{n\geq 1}$ be a sequence in X. We say that the sequence $\{x_n\}_{n\geq 1}$ left converges to $x \in X$ if $q(x_n, x) \to 0$. One denotes this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \longrightarrow x. \quad (4)$$

We will utilize the word converges instead of left converges for simplicity.

Example 11. Let $X = [0,1]$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x, y \geq 0\}$, and $q : X \times X \to E$ defined by

$$q(x,y) = \begin{cases} (x-y, \alpha(x-y)), & \text{if } x \geq y, \\ (\alpha, 1), & \text{if } x < y, \end{cases} \quad (5)$$

where $0 \leq \alpha < 1$. q is a quasi-cone metric on X. Considering a sequence $x_n = 1/n$, then $\{x_n\}_{n\geq 1}$ is left Cauchy and is convergent to $\{0\}$ due to

$$q(x_n, x) = q\left(\frac{1}{n}, 0\right) = \left(\frac{1}{n}, \frac{\alpha}{n}\right) \longrightarrow (0,0). \quad (6)$$

On the other hand, it is not right Cauchy.

Definition 12. A quasi-cone metric space (X,q) is called left complete if every left Cauchy sequence in X converges.

Definition 13. Let (X,q) be a quasi-cone metric space. A function $f : X \to X$ is called

(1) continuous if for any convergent sequence $\{x_n\}_{n\geq 1}$ in X with $\lim_{n\to\infty} x_n = x$, the sequence $\{f(x_n)\}_{n\geq 1}$ is convergent and $\lim_{n\to\infty} f(x_n) = f(x)$;

(2) contractive if there exists some $0 \leq \kappa < 1$ such that

$$q(f(x), f(y)) \leq \kappa q(x,y) \quad \forall x, y \in X, \quad (7)$$

and if $\kappa = 1$, then f is nonexpansive.

The following example shows that there exists a contractive function in quasi-cone metric space which is not continuous.

Example 14. Let $X = \{0,1,2,\ldots\} \cup \{\infty\}$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x, y \geq 0\}$, and $q : X \times X \to E$ defined by

$$q(x,y) = \begin{cases} (0,0), & \text{if } x \geq y, \\ \left(\left(\frac{1}{2}\right) y, y\right), & \text{if } x < y, \end{cases} \quad (8)$$

and $f : \mathrm{X} \to X$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ 1, & \text{if } x = \infty. \end{cases} \quad (9)$$

Then (X,q) is a quasi-cone metric space and f is a contractive map but not continuous due to $\lim_{x\to\infty} f(x) \neq f(\infty)$.

3. Fixed Point Theorems

In this section, we prove some fixed point results in quasi-cone metric space. Also, we generalize the contractive conditions in the literature by replacing the constants with functions. First, we state the following useful lemma.

Lemma 15. *Let (X,q) be a quasi-cone metric space and $\{x_n\}_{n\geq 1}$ a sequence in X. Suppose there exist a sequence of nonnegative real numbers $\{\lambda_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$, in which*

$$q(x_n, x_{n+1}) \leq \lambda_n M, \quad (10)$$

for some $M \in P$, and for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}_{n\geq 1}$ is left Cauchy sequence in (X,q).

Proof. For $n > m$, we get

$$\begin{aligned} q(x_m, x_n) &\leq q(x_m, x_{m+1}) + q(x_{m+1}, x_{m+2}) \\ &\quad + \cdots + q(x_{n-1}, x_n) \leq M \sum_{i=m}^{\infty} \lambda_i. \end{aligned} \quad (11)$$

Let $c \in \operatorname{int} P$ and choose $\delta > 0$ such that $c + N_\delta(0) \subset P$ where $N_\delta(0) = \{y \in E : \| y \| < \delta\}$. Since $\sum_{n=1}^{\infty} \lambda_n < \infty$, there exists a natural number n_0 such that for all $m \geq n_0$ $M \sum_{i=m}^{\infty} \lambda_i \in N_\delta(0)$, also $-M \sum_{i=m}^{\infty} \lambda_i \in N_\delta(0)$. Since $c + N_\delta(0)$ is open, therefore $c + N_\delta(0) \in \operatorname{int} P$; that is $c - M \sum_{i=m}^{\infty} \lambda_i \in \operatorname{int} P$. Thus, $M \sum_{i=m}^{\infty} \lambda_i \ll c$ for $m \geq n_0$ and so

$$q(x_m, x_n) \ll c \quad \text{for } n > m \geq n_0. \quad (12)$$

Thus, $\{x_n\}_{n\geq 1}$ is a left Cauchy sequence. □

We are now in a position to state the main fixed point theorem in the context of quasi-cone metric spaces. We will need the notion of Hausdorff in quasi-cone metric space. A quasi-cone metric space (X,d) is Hausdorff if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint.

Theorem 16. *Let (X,q) be a left complete Hausdorff quasi-cone metric space and let $f : X \to X$ be a continuous function. Suppose that there exist functions $\eta, \lambda, \zeta, \mu, \xi : X \to [0,1)$ which satisfy the following for $x, y \in X$:*

(1) $\eta(f(x)) \leq \eta(x)$, $\lambda(f(x)) \leq \lambda(x)$, $\zeta(f(x)) \leq \zeta(x)$, $\mu(f(x)) \leq \mu(x)$ and $\xi(f(x)) \leq \xi(x)$;

(2) $\eta(x) + \lambda(x) + \zeta(x) + \mu(x) + 2\xi(x) < 1$;

(3) $q(f(x), f(y)) \leq \eta(x) q(x,y) + \lambda(x) q(x, f(x)) + \zeta(x) q(y, f(y)) + \mu(x) q(f(x), y) + \xi(x) q(x, f(y))$.

Then, f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and fixed, and we consider the sequence $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. If we take $x = x_{n-1}$ and $y = x_n$ in (3) we have

$$\begin{aligned} &q(x_n, x_{n+1}) \\ &\quad = q(f(x_{n-1}), f(x_n)) \\ &\quad \leq \eta(x_{n-1}) q(x_{n-1}, x_n) + \lambda(x_{n-1}) q(x_{n-1}, f(x_{n-1})) \\ &\qquad + \zeta(x_{n-1}) q(x_n, f(x_n)) + \mu(x_{n-1}) q(f(x_{n-1}), x_n) \\ &\qquad + \xi(x_{n-1}) q(x_{n-1}, f(x_n)) \end{aligned}$$

$$\begin{aligned}
&= \eta(f(x_{n-2}))\, q(x_{n-1}, x_n) + \lambda(f(x_{n-2}))\, q(x_{n-1}, x_n) \\
&\quad + \zeta(f(x_{n-2}))\, q(x_n, x_{n+1}) + \mu(f(x_{n-2}))\, q(x_n, x_n) \\
&\quad + \xi(f(x_{n-2}))\, q(x_{n-1}, x_{n+1}) \\
&\preceq \eta(x_{n-2})\, q(x_{n-1}, x_n) + \lambda(x_{n-2})\, q(x_{n-1}, x_n) \\
&\quad + \zeta(x_{n-2})\, q(x_n, x_{n+1}) + \xi(x_{n-2}) \\
&\quad \times (q(x_{n-1}, x_n) + q(x_n, x_{n+1})) \\
&\;\vdots \\
&\preceq \eta(x_0)\, q(x_{n-1}, x_n) + \lambda(x_0)\, q(x_{n-1}, x_n) \\
&\quad + \zeta(x_0)\, q(x_n, x_{n+1}) + \xi(x_0) \\
&\quad \times (q(x_{n-1}, x_n) + q(x_n, x_{n+1})).
\end{aligned} \tag{13}$$

So,

$$\begin{aligned}
&q(x_n, x_{n+1}) \\
&\quad \preceq \left(\frac{\eta(x_0) + \lambda(x_0) + \xi(x_0)}{1 - \zeta(x_0) - \xi(x_0)} \right) q(x_{n-1}, x_n) \\
&\quad = h q(x_{n-1}, x_n) \\
&\quad \preceq h^2 q(x_{n-2}, x_{n-1}) \\
&\quad\;\vdots \\
&\quad \preceq h^n q(x_0, x_1),
\end{aligned} \tag{14}$$

where $h = (\eta(x_0) + \lambda(x_0) + \xi(x_0)/1 - \zeta(x_0) - \xi(x_0))$. Thus, by Lemma 15, $\{x_n\}_{n\geq 1}$ is left Cauchy in X. Because of completeness of X and continuity of f, there exists $x^* \in X$ such that $x_n \to x^*$ and $x_{n+1} = f(x_n) \to f(x^*)$. Since X is Hausdorff, $f(x^*) = x^*$.

Uniqueness. Let y^* be another fixed point of f, then

$$\begin{aligned}
&q(x^*, y^*) \\
&\quad = q(f(x^*), f(y^*)) \\
&\quad \preceq \eta(x^*)\, q(x^*, y^*) + \lambda(x^*)\, q(x^*, f(x^*)) \\
&\qquad + \zeta(x^*)\, q(y^*, f(y^*)) + \mu(x^*)\, q(f(x^*), y^*) \\
&\qquad + \xi(x^*)\, q(x^*, f(y^*)) \\
&\quad = \eta(x^*)\, q(x^*, y^*) + \mu(x^*)\, q(x^*, y^*) \\
&\qquad + \xi(x^*)\, q(x^*, y^*) \\
&\quad = (\eta(x^*) + \mu(x^*) + \xi(x^*))\, q(x^*, y^*).
\end{aligned} \tag{15}$$

Therefore, $q(x^*, y^*) = 0$ due to $\eta(x^*) + \mu(x^*) + \xi(x^*) < 1$. Similarly, $q(y^*, x^*) = 0$. Hence, $x^* = y^*$. □

Corollary 17. *Let (X, q) be a left complete Hausdorff quasi-cone metric space and let $f : X \to X$ be a continuous function. Suppose that there exist functions $\eta, \lambda, \mu : X \to [0, 1)$ which satisfy the following for $x, y \in X$:*

(1) $\eta(f(x)) \leq \eta(x)$, $\lambda(f(x)) \leq \lambda(x)$ *and* $\mu(f(x)) \leq \mu(x)$;

(2) $\eta(x) + 2\lambda(x) + 2\mu(x) < 1$;

(3) $q(f(x), f(y)) \preceq \eta(x)q(x, y) + \lambda(x)(q(x, f(x)) + q(y, f(y))) + \mu(x)(q(f(x), y) + q(x, f(y)))$.

Then, f has a unique fixed point.

Proof. We can prove this result by applying Theorem 16 to $\lambda(x) = \zeta(x)$ and $\mu(x) = \xi(x)$. □

Corollary 18. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and*

$$\begin{aligned}
q(f(x), f(y)) &\preceq \alpha q(x, y) + \beta q(x, f(x)) + \gamma q(y, f(y)) \\
&\quad + k q(f(x), y) + \ell q(x, f(y))
\end{aligned} \tag{16}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma, k, \ell \geq 0$ with $\alpha + \beta + \gamma + k + 2\ell < 1$. Then, f has a unique fixed point.

Proof. We can prove this result by applying Theorem 16 to $\eta(x) = \alpha$, $\lambda(x) = \beta$, $\zeta(x) = \gamma$, $\mu(x) = k$ and $\xi(x) = \ell$. □

The following corollaries generalize some results of [14] in cone metric spaces to quasi-cone metric spaces.

Corollary 19. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and*

$$q(f(x), f(y)) \preceq \alpha(q(x, f(x)) + q(y, f(y))) \tag{17}$$

for all $x, y \in X$ and $\alpha \in [0, 1/2)$. Then, f has a unique fixed point.

Corollary 20. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and*

$$q(f(x), f(y)) \preceq \gamma(q(f(x), y) + q(x, f(y))) \tag{18}$$

for all $x, y \in X$ and $\gamma \in [0, 1/2)$. Then, f has a unique fixed point.

Corollary 21. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and*

$$q(f(x), f(y)) \preceq \alpha q(x, y) + \beta q(f(x), y), \tag{19}$$

for all $x, y \in X$ and $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Then, f has a unique fixed point.

The next corollary is a generalization of Banach contraction principle.

Corollary 22. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and*

$$q(f(x), f(y)) \preceq \alpha q(x, y), \tag{20}$$

for all $x, y \in X$ and $\alpha \in [0, 1)$. Then, f has a unique fixed point.

The example in [31] shows that the Hausdorff condition is necessary for quasi-metric spaces and is so for quasi-cone metric spaces. Now, we present two examples. The first one fulfills Theorem 16in which P is normal. The second example satisfies Corollary 18 without normality of P.

Example 23. Let $X = [0, 1]$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, and $q : X \times X \to E$ such that

$$q(x, y) = \begin{cases} (x - y, \gamma(x - y)), & \text{if } x \geq y, \\ (0, 0), & \text{if } x < y, \end{cases} \tag{21}$$

where $\gamma \in [0, 1]$. Suppose $f(x) = x/4$, $\eta(x) = x/8$, $\lambda(x) = 1/3 + x/24$, $\zeta(x) = (x + x^2)/6$, $\mu(x) = 1/24$ and $\xi(x) = 0$. Then for all $x, y \in X$, we have the following.

(1) $\eta(f(x)) = x/32 \leq x/8 = \eta(x)$, $\lambda(f(x)) = 1/3 + x/96 \leq 1/3 + x/24 = \lambda(x)$, $\zeta(f(x)) = (x/4 + x^2/16)/6 \leq (x+x^2)/6 = \zeta(x)$, $\mu(f(x)) = \mu(x) = 1/24$, and $\xi(f(x)) = \xi(x) = 0$.

(2) $x/8 + (1/3 + x/24) + (x + x^2)/6 + 1/24 < 1$.

(3) Condition (3) of Theorem 16 is satisfied. For $x \geq y$, we have

$$\begin{aligned} &q(f(x), f(y)) \\ &\quad \preceq \eta(x) q(x, y) \\ &\qquad + \lambda(x) q(x, f(x)) + \zeta(x) q(y, f(y)) \\ &\qquad + \mu(x) q(f(x), y) + \xi(x) q(x, f(y)) \end{aligned} \tag{22}$$

due to

$$\begin{aligned} &\left(\frac{x}{4} - \frac{y}{4}, r\left(\frac{x}{4} - \frac{y}{4}\right)\right) \\ &\quad \preceq \left(\frac{25x}{96} - \frac{y}{24} + \frac{x^2 y}{8}\right. \\ &\qquad \left. + \frac{5x^2}{32}, r\left(\frac{25x}{96} - \frac{y}{24} + \frac{x^2 y}{8} + \frac{5x^2}{32}\right)\right) \end{aligned} \tag{23}$$

and for $x < y$ it is trivial.

Therefore, $\{0\}$ is a fixed point.

Example 24. Let $X = [0, \infty)$, $E = C^1_{\mathbb{R}}[0, 1] \times C^1_{\mathbb{R}}[0, 1]$, $P = \{(\phi, \varphi) \in E : \phi(t) \text{ and } \varphi(t) \geq 0,\ t \in [0, 1]\}$, and $q : X \times X \to E$ defined by

$$q(x, y) = \begin{cases} \left(\left(x^2 - y^2\right)\phi, \left(x^2 - y^2\right)\phi\right), & \text{if } x \geq y, \\ (0, 0), & \text{if } x \leq y, \end{cases} \tag{24}$$

where $\phi(t) = e^t$. Suppose $f(x) = (1/8)x$. If we take $\alpha = 1/16$, $\beta = 1/3$, $\gamma = 1/4$, $k = 1/6$, and $\ell = 1/16$, then all the assumptions of Corollary 18 are satisfied. Thus, $\{0\}$ is a fixed point.

Theorem 25. *Let (X, q) be a left complete Hausdorff quasi-cone metric space, and let $f : X \to X$ be a continuous function and for all $x, y \in X$*

$$\begin{aligned} &\alpha q(f(x), f(y)) + \beta q(x, f(x)) \\ &\quad + \gamma q(y, f(y)) + k q(x, f(y)) + \ell q(y, f(x)) \\ &\quad \preceq s q(x, y) + t q\left(x, f^2(x)\right), \end{aligned} \tag{$*$}$$

where $s \geq \beta \geq \ell$, $\gamma \geq k \geq t$, $\alpha+k > 0$, and $0 \leq (s-\ell)/(\alpha+k) < 1$. Then, f has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary and fixed and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. If we take $x = x_{n-1}$ and $y = x_n$ in $(*)$, we have

$$\begin{aligned} &\alpha q(f(x_{n-1}), f(x_n)) \\ &\quad + \beta q(x_{n-1}, f(x_{n-1})) + \gamma q(x_n, f(x_n)) \\ &\quad + k q(x_{n-1}, f(x_n)) + \ell q(x_n, f(x_{n-1})) \\ &\quad \preceq s q(x_{n-1}, x_n) + t q\left(x_{n-1}, f^2(x_{n-1})\right). \end{aligned} \tag{25}$$

Rewriting this inequality as

$$\begin{aligned} &\alpha q(x_n, x_{n+1}) + \beta q(x_{n-1}, x_n) \\ &\quad + \gamma q(x_n, x_{n+1}) + k q(x_{n-1}, x_{n+1}) + \ell q(x_n, x_n) \\ &\quad \preceq s q(x_{n-1}, x_n) + t q(x_{n-1}, x_{n+1}) \end{aligned} \tag{26}$$

implies that

$$\begin{aligned} &(\alpha + \gamma) q(x_n, x_{+1}) + (k - t) q(x_{n-1}, x_{n+1}) \\ &\quad \preceq (s - \beta) q(x_{n-1}, x_n). \end{aligned} \tag{27}$$

Since $k \geq t$, we have

$$(\alpha + \gamma) q(x_n, x_{n+1}) \preceq (s - \beta) q(x_{n-1}, x_n). \tag{28}$$

Therefore, we obtain

$$q(x_n, x_{n+1}) \preceq \frac{s - \beta}{\alpha + \gamma} q(x_{n-1}, x_n), \tag{29}$$

due to $\beta \geq \ell$, $\gamma \geq k$, and $\alpha+\gamma > 0$, and we get $(s-\beta)/(\alpha+\gamma) \leq (s - \ell)/(\alpha + k)$. Therefore,

$$\begin{aligned} q(x_n, x_{n+1}) &\preceq \frac{s - \ell}{\alpha + k} q(x_{n-1}, x_n) \\ &\preceq \left(\frac{s - \ell}{\alpha + k}\right)^2 q(x_{n-2}, x_{n-1}) \\ &\ \ \vdots \\ &\preceq \left(\frac{s - \ell}{\alpha + k}\right)^n q(x_0, x_1). \end{aligned} \tag{30}$$

Thus, by Lemma 15, $\{x_n\}_{n\geq 1}$ is left Cauchy in X. Because of completeness of X and continuity of f, there exists $x^* \in X$ such that $x_n \to x^*$ and $x_{n+1} = f(x_n) \to f(x^*)$. Since X is Hausdorff, $f(x^*) = x^*$.

Uniqueness. Let y^* be another fixed point. Putting $x = x^*$ and $y = y^*$ in $(*)$, we obtain

$$\begin{aligned} &\alpha q(f(x^*), f(y^*)) \\ &\quad + \beta q(x^*, f(x^*)) + \gamma q(y^*, f(y^*)) \\ &\quad + kq(x^*, f(y^*)) + \ell q(y^*, f(x^*)) \\ &\leq sq(x^*, y^*) + tq(x^*, f^2(x^*)). \end{aligned} \tag{31}$$

Hence,

$$(\alpha + k) q(x^*, y^*) + \ell q(y^*, x^*) \leq sq(x^*, y^*). \tag{32}$$

Similarly, applying $(*)$ with $x = y^*$ and $y = x^*$, we have

$$(\alpha + k) q(y^*, x^*) + \ell q(x^*, y^*) \leq sq(y^*, x^*). \tag{33}$$

Adding up the above two inequalities, we get

$$\begin{aligned} &(\alpha + k)(q(x^*, y^*) + q(y^*, x^*)) \\ &\quad + \ell(q(y^*, x^*) + q(x^*, y^*)) \\ &\leq s(q(x^*, y^*) + q(y^*, x^*)). \end{aligned} \tag{34}$$

Subsequently, we obtain

$$\begin{aligned} &(\alpha + k)(q(x^*, y^*) + q(y^*, x^*)) \\ &\leq (s - \ell)(q(x^*, y^*) + q(y^*, x^*)). \end{aligned} \tag{35}$$

Thus,

$$q(x^*, y^*) + q(y^*, x^*) \leq \frac{s - \ell}{\alpha + k}(q(x^*, y^*) + q(y^*, x^*)). \tag{36}$$

Hence, $q(x^*, y^*) + q(y^*, x^*) = 0$ due to $0 \leq (s - \ell)/(\alpha + k) < 1$. Therefore, $q(x^*, y^*) = q(y^*, x^*) = 0$ and $x^* = y^*$. □

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