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*Research Article*

# Further Refinements of Jensen's Type Inequalities for the Function Defined on the Rectangle

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We give refinement of Jensen's type inequalities given by Bakula and Pečarić (2006) for the co-ordinate convex function. Also we establish improvement of Jensen's inequality for the convex function of two variables.

## 1. Introduction

Jensen's inequality for convex functions plays a crucial role in the theory of inequalities due to the fact that other inequalities such as the arithmetic mean-geometric mean inequality, the Hölder and Minkowski inequalities, and the Ky Fan inequality, can be obtained as particular cases of it. Therefore, it is worth studying it thoroughly and refining it from different point of view. There are many refinements of Jensen's inequality; see, for example, [1–14] and the references in them.

A function $f : [a,b] \times [c,d] \to \mathbb{R}$, $[a,b] \times [c,d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$ is called convex on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$ defined as $f_y(t) = f(t,y)$ and $f_x : [c,d] \to \mathbb{R}$defined as $f_x(s) = f(x,s)$ are convex for all $x \in [a,b]$, $y \in [c,d]$. Note that every convex function $f : [a,b] \times [c,d] \to \mathbb{R}$ is co-ordinate convex, but the converse is not generally true [8].

The following theorem has been given in [4].

**Theorem 1.** *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a convex function on the co-ordinates on $[a,b] \times [c,d]$. If $\mathbf{x}$ is an n-tuple in $[a,b]$, $\mathbf{y}$ is m-tuple in $[c,d]$, $\mathbf{p}$ is a nonnegative n-tuple, and $\mathbf{w}$ is a nonnegative m-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$ and $W_m = \sum_{i=1}^{m} w_j > 0$, then*

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i, \frac{1}{W_m}\sum_{j=1}^{m}w_j y_j\right) \leq \frac{1}{2}\left\{\frac{1}{P_n}\sum_{i=1}^{n}p_i f\left(x_i, \overline{y}\right) + \frac{1}{W_m}\sum_{j=1}^{m}w_j f\left(\overline{x}, y_j\right)\right\} \leq \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_i w_j f\left(x_i, y_j\right), \tag{1}$$

*where $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$, and $\overline{y} = (1/W_m)\sum_{j=1}^{m} w_j y_j$.*

Recently Dragomir has given new refinement for Jensen inequality in [9]. The purpose of this paper is to give related refinements of Jensen's type inequalities (1) for the co-ordinate convex function. We will also discuss some particular interesting cases. We establish improvement of Jensen's inequality for the convex function defined on the rectangles. For related improvements of Jensen's inequality, see, for example, [1, 2, 9, 13, 14]. For further several related integral inequalities, see [15].

## 2. Main Results

Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be convex on the co-ordinate on $[a,b] \times [c,d]$. If $x_i \in [a,b]$, $y_j \in [c,d]$, $p_i, w_j > 0$, $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$ with $P_n = \sum_{i=1}^n p_i$, and $W_m = \sum_{j=1}^m w_j$, then for any subsets $I \subset \{1,2,\ldots,n\}$ and $J \subset \{1,2,\ldots,m\}$, we assume that $\overline{I} := \{1,2,\ldots,n\} \setminus I$ and $\overline{J} := \{1,2,\ldots,m\} \setminus J$. Define $P_I = \sum_{i\in I} p_i$, $P_{\overline{I}} = \sum_{i\in\overline{I}} p_i$, $W_J = \sum_{j\in J} w_j$, and $W_{\overline{J}} = \sum_{j\in\overline{J}} w_j$. For the function $f$ and the $n$-, $m$-tuples, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_m)$, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$, we define the following functionals:

$$\begin{aligned}
&D(f,\mathbf{p},\mathbf{x},I,y_j)\\
&\quad = \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, y_j\right) + \frac{P_{\overline{I}}}{P_n} f\left(\frac{1}{P_{\overline{I}}}\sum_{i\in \overline{I}} p_i x_i, y_j\right),\\
&D(f,\mathbf{w},\mathbf{y},J,x_i)\\
&\quad = \frac{W_J}{W_m} f\left(x_i, \frac{1}{W_J}\sum_{j\in J} w_j y_j\right) + \frac{W_{\overline{J}}}{W_m} f\left(x_i, \frac{1}{W_{\overline{J}}}\sum_{j\in \overline{J}} w_j y_j\right),\\
&D(f,\mathbf{p},\mathbf{x},I)\\
&\quad = \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, \overline{y}\right) + \frac{P_{\overline{I}}}{P_n} f\left(\frac{1}{P_{\overline{I}}}\sum_{i\in \overline{I}} p_i x_i, \overline{y}\right),\\
&D(f,\mathbf{w},\mathbf{y},J)\\
&\quad = \frac{W_J}{W_m} f\left(\overline{x}, \frac{1}{W_J}\sum_{j\in J} w_j y_j\right) + \frac{W_{\overline{J}}}{W_m} f\left(\overline{x}, \frac{1}{W_{\overline{J}}}\sum_{j\in \overline{J}} w_j y_j\right),
\end{aligned} \tag{2}$$

where $\overline{x} = (1/P_n)\sum_{i=1}^n p_i x_i$, and $\overline{y} = (1/W_m)\sum_{j=1}^m w_j y_j$.

It is worth to observe that for $I = \{k\}$, $k \in \{1,\ldots,n\}$, and $J = \{l\}$, $l \in \{1,\ldots,m\}$, we have the functionals

$$\begin{aligned}
&D_k(f,\mathbf{p},\mathbf{x},y_j)\\
&\quad := D(f,\mathbf{p},\mathbf{x},\{k\},y_j)\\
&\quad = \frac{p_k}{P_n} f(x_k, y_j) + \frac{P_n - p_k}{P_n} f\left(\frac{\sum_{i=1}^n p_i x_i - p_k x_k}{P_n - p_k}, y_j\right),\\
&D_l(f,\mathbf{w},\mathbf{y},x_i)\\
&\quad := D(f,\mathbf{w},\mathbf{y},\{l\},x_i)\\
&\quad = \frac{w_l}{W_m} f(x_i, y_l) + \frac{W_m - w_l}{W_m} f\left(x_i, \frac{\sum_{j=1}^m w_j y_j - w_l y_l}{W_m - w_l}\right),\\
&D_k(f,\mathbf{p},\mathbf{x})\\
&\quad := D(f,\mathbf{p},\mathbf{x},\{k\})\\
&\quad = \frac{p_k}{P_n} f(x_k, \overline{y}) + \frac{P_n - p_k}{P_n} f\left(\frac{\sum_{i=1}^n p_i x_i - p_k x_k}{P_n - p_k}, \overline{y}\right),\\
&D_l(f,\mathbf{w},\mathbf{y})\\
&\quad := D(f,\mathbf{w},\mathbf{y},\{l\})\\
&\quad = \frac{w_l}{W_m} f(\overline{x}, y_l) + \frac{W_m - w_l}{W_m} f\left(\overline{x}, \frac{\sum_{j=1}^m w_j y_j - w_l y_l}{W_m - w_l}\right).
\end{aligned} \tag{3}$$

The following refinement of (1) holds.

**Theorem 2.** *Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a co-ordinate convex function on $[a,b]\times[c,d]$. If $x_i \in [a,b]$, $y_j \in [c,d]$, $p_i$, $w_j > 0$, $i \in \{1,2,\ldots,n\}$, $j \in \{1,2,\ldots,m\}$ with $P_n = \sum_{i=1}^n p_i$, and $W_m = \sum_{j=1}^m w_j$, then for any subsets $I \subset \{1,2,\ldots,n\}$ and $J \subset \{1,2,\ldots,m\}$, one has*

$$\begin{aligned}
&f(\overline{x},\overline{y})\\
&\leq \frac{1}{2}\left[D(f,\mathbf{w},\mathbf{y},J) + D(f,\mathbf{p},\mathbf{x},I)\right]\\
&\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i,\overline{y}) + \frac{1}{W_m}\sum_{j=1}^m w_j f(\overline{x},y_j)\right]\\
&\leq \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^n p_i D(f,\mathbf{w},\mathbf{y},J,x_i) + \frac{1}{W_m}\sum_{j=1}^m w_j D(f,\mathbf{p},\mathbf{x},I,y_j)\right]\\
&\leq \frac{1}{P_n W_m}\sum_{i=1}^n\sum_{j=1}^m p_i w_j f(x_i,y_j),
\end{aligned} \tag{4}$$

*where $\overline{x} = (1/P_n)\sum_{i=1}^n p_i x_i$, and $\overline{y} = (1/W_m)\sum_{j=1}^m w_j y_j$.*

*Proof.* One-dimensional Jensen's inequality gives us

$$\begin{aligned}
f(x_i,\overline{y}) &\leq \frac{1}{W_m}\sum_{j=1}^m w_j f(x_i,y_j),\\
f(\overline{x},y_j) &\leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i,y_j).
\end{aligned} \tag{5}$$

As we have

$$\begin{aligned}
&D(f,\mathbf{p},\mathbf{x},I,y_j)\\
&\quad = \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, y_j\right) + \frac{P_{\overline{I}}}{P_n} f\left(\frac{1}{P_{\overline{I}}}\sum_{i\in \overline{I}} p_i x_i, y_j\right),
\end{aligned} \tag{6}$$

so by Jensen's inequality, we have

$$\begin{aligned} D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) &= \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, y_j\right) + \frac{P_{\bar{I}}}{P_n} f\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, y_j\right) \\ &\le \frac{P_I}{P_n}\frac{1}{P_I}\sum_{i\in I} p_i f\left(x_i, y_j\right) + \frac{P_{\bar{I}}}{P_n}\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i f\left(x_i, y_j\right) \\ &= \frac{1}{P_n}\sum_{i\in I} p_i f\left(x_i, y_j\right) + \frac{1}{P_n}\sum_{i\in \bar{I}} p_i f\left(x_i, y_j\right) \\ &= \frac{1}{P_n}\sum_{i\in I\cup\bar{I}} p_i f\left(x_i, y_j\right) \\ \Longrightarrow D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) &\le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i, y_j\right). \end{aligned} \tag{7}$$

As the function $f$ is convex on the first co-ordinate, so we have

$$\begin{aligned} D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) &= \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, y_j\right) + \frac{P_{\bar{I}}}{P_n} f\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, y_j\right) \\ &\ge f\left(\frac{P_I}{P_n}\frac{1}{P_I}\sum_{i\in I} p_i x_i + \frac{P_{\bar{I}}}{P_n}\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, y_j\right) \\ &= f\left(\frac{1}{P_n}\sum_{i\in I} p_i x_i + \frac{1}{P_n}\sum_{i\in \bar{I}} p_i x_i, y_j\right) \\ &= f\left(\frac{1}{P_n}\sum_{i\in I\cup\bar{I}} p_i x_i, y_j\right) \\ &= f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, y_j\right) \\ \Longrightarrow D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) &\ge f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, y_j\right). \end{aligned} \tag{8}$$

Now, from (7) and (8), we have

$$f\left(\bar{x}, y_j\right) \le D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i, y_j\right). \tag{9}$$

Similarly, we can write

$$f\left(x_i, \bar{y}\right) \le D\left(f,\mathbf{w},\mathbf{y},J,x_i\right) \le \frac{1}{W_m}\sum_{j=1}^{m} w_j f\left(x_i, y_j\right). \tag{10}$$

Multiplying (9) and (10), respectively, by $w_j$ and $p_i$ and summing over $i$ and $j$, we obtain

$$\begin{aligned} \frac{1}{W_m}\sum_{j=1}^{m} w_j f\left(\bar{x}, y_j\right) &\le \frac{1}{W_m}\sum_{j=1}^{m} w_j D\left(f,\mathbf{p},\mathbf{x},I,y_j\right) \\ &\le \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f\left(x_i, y_j\right), \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i, \bar{y}\right) &\le \frac{1}{P_n}\sum_{i=1}^{n} p_i D\left(f,\mathbf{w},\mathbf{y},J,x_i\right) \\ &\le \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f\left(x_i, y_j\right). \end{aligned} \tag{12}$$

Adding (11) and (12), we have

$$\begin{aligned} &\frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i, \bar{y}\right) + \frac{1}{W_m}\sum_{j=1}^{m} w_j f\left(\bar{x}, y_j\right)\right] \\ &\quad\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i D\left(f,\mathbf{w},\mathbf{y},J,x_i\right) + \frac{1}{W_m}\sum_{j=1}^{m} w_j D\left(f,\mathbf{p},\mathbf{x},I,y_j\right)\right] \\ &\quad\le \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f\left(x_i, y_j\right). \end{aligned} \tag{13}$$

Again by one-dimensional Jensen's inequality, we have

$$\begin{aligned} f\left(\bar{x}, \bar{y}\right) &\le \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i, \bar{y}\right), \\ f\left(\bar{x}, \bar{y}\right) &\le \frac{1}{W_m}\sum_{j=1}^{m} w_j f\left(\bar{x}, y_j\right). \end{aligned} \tag{14}$$

As we have the functional

$$D\left(f,\mathbf{p},\mathbf{x},I\right) = \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, \bar{y}\right) + \frac{P_{\bar{I}}}{P_n} f\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, \bar{y}\right), \tag{15}$$

so by Jensen's inequality, we get

$$\begin{aligned} D\left(f,\mathbf{p},\mathbf{x},I\right) &= \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, \bar{y}\right) + \frac{P_{\bar{I}}}{P_n} f\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, \bar{y}\right) \\ &\le \frac{P_I}{P_n}\frac{1}{P_I}\sum_{i\in I} p_i f\left(x_i, \bar{y}\right) + \frac{P_{\bar{I}}}{P_n}\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i f\left(x_i, \bar{y}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{P_n}\sum_{i\in I} p_i f(x_i,\bar{y}) + \frac{1}{P_n}\sum_{i\in \bar{I}} p_i f(x_i,\bar{y}) \\
&= \frac{1}{P_n}\sum_{i\in I\cup\bar{I}} p_i f(x_i,\bar{y}) \\
&\Longrightarrow D(f,\mathbf{p},\mathbf{x},I) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}),
\end{aligned} \tag{16}
$$

and as the function $f$ is convex on the first co-ordinate, so we have

$$
\begin{aligned}
D(f,\mathbf{p},\mathbf{x},I) &= \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i, \bar{y}\right) \\
&\quad + \frac{P_{\bar{I}}}{P_n} f\left(\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, \bar{y}\right) \\
&\ge f\left(\frac{P_I}{P_n}\frac{1}{P_I}\sum_{i\in I} p_i x_i + \frac{P_{\bar{I}}}{P_n}\frac{1}{P_{\bar{I}}}\sum_{i\in \bar{I}} p_i x_i, \bar{y}\right) \\
&= f\left(\frac{1}{P_n}\sum_{i\in I} p_i x_i + \frac{1}{P_n}\sum_{i\in \bar{I}} p_i x_i, \bar{y}\right) \\
&= f\left(\frac{1}{P_n}\sum_{i\in I\cup\bar{I}} p_i x_i, \bar{y}\right) = f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \bar{y}\right) \\
&\Longrightarrow D(f,\mathbf{p},\mathbf{x},I) \ge f(\bar{x},\bar{y}).
\end{aligned} \tag{17}
$$

Now from (16) and (17), we have

$$
f(\bar{x},\bar{y}) \le D(f,\mathbf{p},\mathbf{x},I) \le \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}). \tag{18}
$$

Similarly, we can prove that

$$
f(\bar{x},\bar{y}) \le D(f,\mathbf{w},\mathbf{y},J) \le \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\bar{x},y_j). \tag{19}
$$

Adding (18) and (19), we get

$$
\begin{aligned}
f(\bar{x},\bar{y}) &\le \frac{1}{2}\left[D(f,\mathbf{w},\mathbf{y},J) + D(f,\mathbf{p},\mathbf{x},I)\right] \\
&\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\bar{x},y_j)\right].
\end{aligned} \tag{20}
$$

Combining (13) and (20), we have

$$
\begin{aligned}
&f(\bar{x},\bar{y}) \\
&\quad\le \frac{1}{2}\left[D(f,\mathbf{w},\mathbf{y},J) + D(f,\mathbf{p},\mathbf{x},I)\right] \\
&\quad\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\bar{x},y_j)\right] \\
&\quad\le \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i D(f,\mathbf{w},\mathbf{y},J,x_i)\right. \\
&\qquad\qquad \left. + \frac{1}{W_m}\sum_{j=1}^{m} w_j D(f,\mathbf{p},\mathbf{x},I,y_j)\right] \\
&\quad\le \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(x_i,y_j).
\end{aligned} \tag{21}
$$

□

The following cases from the above inequalities are of interest [6, 7].

*Remark 3.* We observe that the inequalities in (4) can be written equivalently as

$$
\begin{aligned}
&\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(x_i,y_j) \\
&\quad \ge \max_{\substack{I\subset\{1,\dots,n\}\\ J\subset\{1,\dots,m\}}} \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n} p_i D(f,\mathbf{w},\mathbf{y},J,x_i)\right. \\
&\qquad\qquad \left. + \frac{1}{W_m}\sum_{j=1}^{m} w_j D(f,\mathbf{p},\mathbf{x},I,y_j)\right], \\
&\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\bar{x},y_j) \\
&\quad \le \min_{\substack{I\subset\{1,\dots,n\}\\ J\subset\{1,\dots,m\}}} \left[\frac{1}{P_n}\sum_{i=1}^{n} p_i D(f,\mathbf{w},\mathbf{y},J,x_i)\right. \\
&\qquad\qquad \left. + \frac{1}{W_m}\sum_{j=1}^{m} w_j D(f,\mathbf{p},\mathbf{x},I,y_j)\right], \\
&\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i,\bar{y}) + \frac{1}{W_m}\sum_{j=1}^{m} w_j f(\bar{x},y_j) \\
&\quad \ge \max_{\substack{I\subset\{1,\dots,n\}\\ J\subset\{1,\dots,m\}}} \left[D(f,\mathbf{w},\mathbf{y},J) + D(f,\mathbf{p},\mathbf{x},I)\right],
\end{aligned}
$$

$$f(\overline{x},\overline{y}) \leq \min_{\substack{I\subset\{1,\ldots,n\}\\ J\subset\{1,\ldots,m\}}} \frac{1}{2}\left[D(f,\mathbf{w},\mathbf{y},J)+D(f,\mathbf{p},\mathbf{x},I)\right]. \tag{22}$$

These inequalities imply the following results:

$$\begin{aligned}
&\frac{1}{P_nW_m}\sum_{i=1}^{n}\sum_{j=1}^{m}p_iw_jf(x_i,y_j) \\
&\geq \max_{\substack{k\in\{1,\ldots,n\}\\ l\in\{1,\ldots,m\}}} \frac{1}{2}\left[\frac{1}{P_n}\sum_{i=1}^{n}p_iD_l(f,\mathbf{w},\mathbf{y},x_i) + \frac{1}{W_m}\sum_{j=1}^{m}w_jD_k(f,\mathbf{p},\mathbf{x},y_j)\right], \\
&\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i,\overline{y})+\frac{1}{W_m}\sum_{j=1}^{m}w_jf(\overline{x},y_j) \\
&\leq \min_{\substack{k\in\{1,\ldots,n\}\\ l\in\{1,\ldots,m\}}} \left[\frac{1}{P_n}\sum_{i=1}^{n}p_iD_l(f,\mathbf{w},\mathbf{y},x_i) + \frac{1}{W_m}\sum_{j=1}^{m}w_jD_k(f,\mathbf{p},\mathbf{x},y_j)\right], \\
&\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i,\overline{y})+\frac{1}{W_m}\sum_{j=1}^{m}w_jf(\overline{x},y_j) \\
&\geq \max_{\substack{k\in\{1,\ldots,n\}\\ l\in\{1,\ldots,m\}}} \left[D_l(f,\mathbf{w},\mathbf{y})+D_k(f,\mathbf{p},\mathbf{x})\right], \\
&f(\overline{x},\overline{y}) \leq \min_{\substack{k\in\{1,\ldots,n\}\\ l\in\{1,\ldots,m\}}} \frac{1}{2}\left[D_l(f,\mathbf{w},\mathbf{y})+D_k(f,\mathbf{p},\mathbf{x})\right].
\end{aligned} \tag{23}$$

Moreover, from the above, we also have

$$\begin{aligned}
&\max_{I\subset\{1,\ldots,n\}} D(f,\mathbf{p},\mathbf{x};I) \geq \max_{k\in\{1,\ldots,n\}} D_k(f,\mathbf{p},\mathbf{x}), \\
&\max_{J\subset\{1,\ldots,m\}} D(f,\mathbf{w},\mathbf{y};J) \geq \max_{l\in\{1,\ldots,m\}} D_l(f,\mathbf{w},\mathbf{y}), \\
&\max_{I\subset\{1,\ldots,n\}} D(f,\mathbf{p},\mathbf{x};I,y_j) \geq \max_{k\in\{1,\ldots,n\}} D_k(f,\mathbf{p},\mathbf{x},\mathbf{y}_j), \\
&\max_{J\subset\{1,\ldots,m\}} D(f,\mathbf{w},\mathbf{y};J,x_i) \geq \max_{l\in\{1,\ldots,m\}} D_l(f,\mathbf{w},\mathbf{y},x_i), \\
&\min_{k\in\{1,\ldots,n\}} D_k(f,\mathbf{p},\mathbf{x}) \geq \min_{I\subset\{1,\ldots,n\}} D(f,\mathbf{p},\mathbf{x};I), \\
&\min_{l\in\{1,\ldots,m\}} D_l(f,\mathbf{w},\mathbf{y}) \geq \min_{J\subset\{1,\ldots,m\}} D(f,\mathbf{w},\mathbf{y};J), \\
&\min_{k\in\{1,\ldots,n\}} D_k(f,\mathbf{p},\mathbf{x},y_j) \geq \min_{I\subset\{1,\ldots,n\}} D(f,\mathbf{p},\mathbf{x};I,y_j), \\
&\min_{l\in\{1,\ldots,m\}} D_l(f,\mathbf{w},\mathbf{y},x_i) \geq \min_{J\subset\{1,\ldots,m\}} D(f,\mathbf{w},\mathbf{y};J,x_i).
\end{aligned} \tag{24}$$

We discuss the following particular cases of the above inequalities which is of interest [6].

In the case when $p_i = 1$ and $w_j = 1$ for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$, consider the natural numbers $k$, $l$ with $1 \leq k \leq n-1$ and $1 \leq l \leq m-1$ and define

$$\begin{aligned}
&D_k(f,\mathbf{x},y_j) = \frac{k}{n}f\left(\frac{1}{k}\sum_{i=1}^{k}x_i,y_j\right)+\frac{n-k}{n}f\left(\frac{1}{n-k}\sum_{i=k+1}^{n}x_i,y_j\right), \\
&D_l(f,\mathbf{y},x_i) = \frac{l}{m}f\left(x_i,\frac{1}{l}\sum_{j=1}^{l}y_j\right)+\frac{m-l}{m}f\left(x_i,\frac{1}{m-l}\sum_{j=l+1}^{m}y_j\right), \\
&D_k(f,\mathbf{x}) = \frac{k}{n}f\left(\frac{1}{k}\sum_{i=1}^{k}x_i,\overline{y}\right)+\frac{n-k}{n}f\left(\frac{1}{n-k}\sum_{i=k+1}^{n}x_i,\overline{y}\right), \\
&D_l(f,\mathbf{y}) = \frac{l}{m}f\left(\overline{x},\frac{1}{l}\sum_{j=1}^{l}y_j\right)+\frac{m-l}{m}f\left(\overline{x},\frac{1}{m-l}\sum_{j=l+1}^{m}y_j\right).
\end{aligned} \tag{25}$$

We can give the following result.

**Corollary 4.** *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a co-ordinate convex function on $[a,b]\times[c,d]$. If $x_i \in [a,b]$ and $y_j \in [c,d]$, then for any $k \in \{1,\ldots,n-1\}$ and $l \in \{1,\ldots,m-1\}$, one has*

$$\begin{aligned}
&f\left(\frac{1}{n}\sum_{i=1}^{n}x_i,\frac{1}{m}\sum_{j=1}^{m}y_j\right) \\
&\leq \frac{1}{2}\left[D_l(f,\mathbf{y})+D_k(f,\mathbf{x})\right] \\
&\leq \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}f(x_i,\overline{y})+\frac{1}{m}\sum_{j=1}^{m}f(\overline{x},y_j)\right] \\
&\leq \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}D_l(f,\mathbf{y},x_i)+\frac{1}{m}\sum_{j=1}^{m}D_k(f,\mathbf{x},y_j)\right] \\
&\leq \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}f(x_i,y_j).
\end{aligned} \tag{26}$$

In particular, we have the bounds

$$\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} f\left(x_i, y_j\right) \geq \max_{\substack{k\in\{1,\dots,n\}\\ l\in\{1,\dots,m\}}} \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n} D_l\left(f, \mathbf{y}, x_i\right) + \frac{1}{m}\sum_{j=1}^{m} D_k\left(f, \mathbf{x}, y_j\right)\right],$$

$$\frac{1}{n}\sum_{i=1}^{n} f\left(x_i, \overline{y}\right) + \frac{1}{m}\sum_{j=1}^{m} f\left(\overline{x}, y_j\right) \leq \min_{\substack{k\in\{1,\dots,n\}\\ l\in\{1,\dots,m\}}} \left[\frac{1}{n}\sum_{i=1}^{n} D_l\left(f, \mathbf{y}, x_i\right) + \frac{1}{m}\sum_{j=1}^{m} D_k\left(f, \mathbf{x}, y_j\right)\right],$$

$$\frac{1}{n}\sum_{i=1}^{n} f\left(x_i, \overline{y}\right) + \frac{1}{m}\sum_{j=1}^{m} f\left(\overline{x}, y_j\right) \geq \max_{\substack{k\in\{1,\dots,n\}\\ l\in\{1,\dots,m\}}} \left[D_l\left(f, \mathbf{y}\right) + D_k\left(f, \mathbf{x}\right)\right],$$

$$f\left(\overline{x}, \overline{y}\right) \leq \min_{\substack{k\in\{1,\dots,n\}\\ l\in\{1,\dots,m\}}} \frac{1}{2}\left[D_l\left(f, \mathbf{y}\right) + D_k\left(f, \mathbf{x}\right)\right]. \quad (27)$$

*Remark 5.* Note that if we substitute $m = 1$, $f(x, y_1) \to f(x)$, $D(f, \mathbf{w}, \mathbf{y}, J, x_i) = f(\overline{x})$, $D(f, \mathbf{p}, \mathbf{x}, I, y_j) = D(f, \mathbf{p}, \mathbf{x}, I)$, and $D(f, \mathbf{w}, \mathbf{y}, J) = f(\overline{x})$ in Theorem 2, we get the following result of Dragomir [9] for convex function defined on the interval and $\sum_{i=1}^{n} p_i = P_n > 0$,

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{P_I}{P_n} f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right) + \frac{P_{\overline{I}}}{P_n} f\left(\frac{1}{P_{\overline{I}}}\sum_{i\in \overline{I}} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f\left(x_i\right). \quad (28)$$

The following refinement of Hölder inequality holds.

**Corollary 6.** *Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be two positive $n$-tuples. Then for $(1/p) + (1/q) = 1$, $p, q > 1$, one has*

$$\sum_{i=1}^{n} x_i y_i \leq \left[\left(\sum_{i=1}^{n} y_i^q\right)^{p/q} \left\{\left(\sum_{i\in I} y_i^q\right)^{-p/q} \left(\sum_{i\in I} x_i y_i\right)^p + \left(\sum_{i\in \overline{I}} y_i^q\right)^{-p/q} \left(\sum_{i\in \overline{I}} x_i y_i\right)^p\right\}\right]^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q}. \quad (29)$$

*Proof.* Using the functions $f(x) = x^p$, $p > 1$, $x_i \to x_i y_i^{-q/p}$, and $p_i \to y_i^q$ in (28), we get (29). □

*Remark 7.* As mentioned above from the inequalities in (29), we can write

$$\left(\sum_{i=1}^{n} x_i^p\right)^{1/p} \left(\sum_{i=1}^{n} y_i^q\right)^{1/q} \geq \max_{I\subset\{1,\dots,n\}} \left[\left(\sum_{i=1}^{n} y_i^q\right)^{p/q} \left\{\left(\sum_{i\in I} y_i^q\right)^{-p/q} \left(\sum_{i\in I} x_i y_i\right)^p + \left(\sum_{i\in \overline{I}} y_i^q\right)^{-p/q} \left(\sum_{i\in \overline{I}} x_i y_i\right)^p\right\}\right]^{1/p},$$

$$\sum_{i=1}^{n} x_i y_i \leq \min_{I\subset\{1,\dots,n\}} \left[\left(\sum_{i=1}^{n} y_i^q\right)^{p/q} \left\{\left(\sum_{i\in I} y_i^q\right)^{-p/q} \left(\sum_{i\in I} x_i y_i\right)^p + \left(\sum_{i\in \overline{I}} y_i^q\right)^{-p/q} \left(\sum_{i\in \overline{I}} x_i y_i\right)^p\right\}\right]^{1/p}. \quad (30)$$

The following improvement of Jensen's inequality is valid.

**Theorem 8.** *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be convex on the coordinates of $[a,b]\times[c,d] \subseteq \mathbb{R}^2$. If $\mathbf{x}$ is an $n$-tuple in $[a,b]$, $\mathbf{y}$ is an $m$-tuple in $[c,d]$, $\mathbf{p}$ is a nonnegative $n$-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$, and $\mathbf{w}$ is a nonnegative $m$-tuple such that $W_m = \sum_{j=1}^{m} w_j > 0$, then*

$$\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f\left(x_i, y_j\right) - f\left(\overline{x}, \overline{y}\right) \geq \left|\frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left|f\left(x_i, y_j\right) - f\left(\overline{x}, \overline{y}\right)\right| - \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left|\frac{\partial f_+\left(\overline{x}, \overline{y}\right)}{\partial w}\left(x_i - \overline{x}\right) + \frac{\partial f_+\left(\overline{x}, \overline{y}\right)}{\partial z}\left(y_j - \overline{y}\right)\right|\right|, \quad (31)$$

*where $\overline{x} = (1/P_n)\sum_{i=1}^{n} p_i x_i$, and $\overline{y} = (1/W_m)\sum_{j=1}^{m} w_j y_j$.*

*Proof.* Since $f$ is convex on $[a,b]\times[c,d]$, therefore we have

$$f\left(x, y\right) - f\left(w, z\right) \geq \frac{\partial f_+\left(w, z\right)}{\partial w}\left(x - w\right) + \frac{\partial f_+\left(w, z\right)}{\partial z}\left(y - z\right). \quad (32)$$

From the above inequality, we have

$$\begin{aligned} f(x,y) - f(w,z) &- \frac{\partial f_+(w,z)}{\partial w}(x-w) - \frac{\partial f_+(w,z)}{\partial z}(y-z) \\ &= \left| f(x,y) - f(w,z) - \frac{\partial f_+(w,z)}{\partial w}(x-w) \right. \\ &\quad \left. - \frac{\partial f_+(w,z)}{\partial z}(y-z) \right| \\ &\geq \left| \left| f(x,y) - f(w,z) \right| \right. \\ &\quad \left. - \left| \frac{\partial f_+(w,z)}{\partial w}(x-w) + \frac{\partial f_+(w,z)}{\partial z}(y-z) \right| \right|. \end{aligned} \tag{33}$$

Let $x \to x_i$, $y \to y_j$, $w \to \sum_{i=1}^{n} p_i x_i / P_n := \overline{x}$, and $z \to \sum_{j=1}^{n} w_j y_j / W_m := \overline{y}$, then (33) becomes

$$\begin{aligned} f(x_i,y_j) &- f(\overline{x},\overline{y}) \\ &- \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) - \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \\ &\geq \left| \left| f(x_i,y_j) - f(\overline{x},\overline{y}) \right| \right. \\ &\quad \left. - \left| \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) + \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \right| \right|. \end{aligned} \tag{34}$$

Multiplying (34) by $p_i$ and $w_j$ and summing over $i$ and $j$, we have

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(x_i,y_j) &- \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(\overline{x},\overline{y}) \\ &- \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) \\ &- \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \\ &\geq \left| \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| f(x_i,y_j) - f(\overline{x},\overline{y}) \right| \right. \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) \right. \\ &\quad \left. \left. + \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \right| \right|. \end{aligned} \tag{35}$$

One has

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) &= \sum_{j=1}^{m} w_j \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}\left( \sum_{i=1}^{n} p_i x_i - P_n \cdot \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \right) = 0, \\ \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) &= 0. \end{aligned} \tag{36}$$

Therefore (35) becomes

$$\begin{aligned} \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(x_i,y_j) &- \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(\overline{x},\overline{y}) \\ &\geq \left| \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| f(x_i,y_j) - f(\overline{x},\overline{y}) \right| \right. \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) \right. \\ &\quad \left. \left. + \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \right| \right|. \end{aligned} \tag{37}$$

Multiplying both hand sides by $1/P_n W_m$, we have

$$\begin{aligned} \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j f(x_i,y_j) &- f(\overline{x},\overline{y}) \\ &\geq \left| \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| f(x_i,y_j) - f(\overline{x},\overline{y}) \right| \right. \\ &\quad - \frac{1}{P_n W_m}\sum_{i=1}^{n}\sum_{j=1}^{m} p_i w_j \left| \frac{\partial f_+(\overline{x},\overline{y})}{\partial w}(x_i-\overline{x}) \right. \\ &\quad \left. \left. + \frac{\partial f_+(\overline{x},\overline{y})}{\partial z}(y_j-\overline{y}) \right| \right|. \end{aligned} \tag{38}$$

This completes the proof. □

## Conflict of Interests

The authors declare that they have no conflict of interests regarding publication of this paper.

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