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Research Article

Sensitivity of a Fractional Integrodifferential Cauchy Problem of Volterra Type

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We prove a theorem on the existence and uniqueness of a solution as well as on a sensitivity (i.e., differentiable dependence of a solution on a functional parameter) of a fractional integrodifferential Cauchy problem of Volterra type. The proof of this result is based on a theorem on diffeomorphism between Banach and Hilbert spaces. The main assumption is the Palais-Smale condition.

1. Introduction

In the paper, we consider the following fractional Integrodifferential Cauchy problem of Volterra type of order $\alpha \in (0,1)$:

$$\begin{gathered} D^{\alpha}_{a+}x(t) = \int_a^t \Phi(t,s,x(s))\,ds + g(t), \\ t \in [a,b] \ \text{ a.e.}, \\ I^{1-\alpha}_{a+}x(a) = 0, \end{gathered} \tag{1}$$

where $\Phi : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$, $P_\Delta = \{(t,s) \in [0,1]\times[0,1];\ t \geq s\}$, and $g \in L^2 = L^2([a,b],\mathbb{R}^n)$. We consider the existence and uniqueness of a solution in the space $I^{\alpha}_{a+}(L^2)$ (the range of the right-sided integral Riemann-Liouville operator $I^{\alpha}_{a+} : L^2 \to L^2$) as well as sensitivity, that is, differentiable dependence of a solution on a functional parameter $g \in L^2$.

Fractional functional systems, including Integrodifferential ones, have recently been studied by several authors. The reasons for this interest are numerous applications of fractional differential calculus in physics, chemistry, biology, economics, signal processing, image processing, aerodynamics, and so forth. Integrodifferential systems are investigated in finite and infinite dimensional spaces, with Riemann-Liouville and Caputo derivatives as well as with different types of initial and boundary conditions, local, nonlocal, involving values of solutions or their fractional integrals, delay [1–7]. Applied methods also are of different type. They are based on Banach, Brouwer, Schauder, Schaefer, Krasnoselskii fixed point theorems, nonlinear alternative Leray-Schauder type, strongly continuous operator semigroups, the reproducing kernel Hilbert space method, and so forth.

We propose a new method for the study problems of type (1), namely, a theorem on diffeomorphism between Banach and Hilbert spaces obtained by the authors in paper [8]. This theorem is based on the Palais-Smale condition. In the mentioned work, an application of this result to study problem of type (1) with $\alpha = 1$ is given. In the paper, we use the line of the proof presented therein. The main difference between cases of $\alpha \in (0,1)$ and $\alpha = 1$ is that, in the first case, the elements of the solution space $I^{\alpha}_{a+}(L^2)$ are not, in general, continuous functions on $[a,b]$ as it is when $\alpha = 1$ (cf. Remark 10).

The paper is organized as follows. In the second section, we recall some facts from the fractional calculus and formulate a theorem on diffeomorphism between Banach and Hilbert spaces. Third section is devoted to the existence and uniqueness of a solution as well as sensitivity of problem (1) (Theorem 9). Let us point that Lemma 7 in itself is a general result on the existence and uniqueness of a solution to problem (1) under a Lipschitz condition with respect to the state variable, imposed on the integrand. Strengthening the smoothness assumptions about the integrand and Palais-Smale condition allows us to prove sensitivity of (1).

To our best knowledge, sensitivity of fractional systems of type (1) has not been studied by other authors so far.

2. Preliminaries

2.1. Fractional Calculus. Let $\alpha > 0$, $h \in L^1 = L^1([a,b],\mathbb{R}^n)$. By the left-sided Riemann-Liouville fractional integral of h on the interval $[a,b]$, we mean (cf. [9]) a function $I^{\alpha}_{a+}h$ given by

$$\left(I^{\alpha}_{a+}h\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{h(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad t \in [a,b] \text{ a.e.}, \tag{2}$$

where Γ is the Euler function.

One can show that the above integral exists and is finite a.e. on $[a,b]$. Moreover, if $h \in L^p = L^p([a,b],\mathbb{R}^n)$, $1 \le p < \infty$, then $I^{\alpha}_{a+}h \in L^p$ and

$$\left\|I^{\alpha}_{a+}h\right\|_{L^p} \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\|h\|_{L^p}. \tag{3}$$

In [10], the following useful theorem is proved.

Theorem 1. *The operator $I^{\alpha}_{a+} : L^p \to L^p$, $1 \le p < \infty$, is compact, that is, it maps bounded sets onto relatively compact ones.*

Now, let $\alpha \in (0,1)$. One says that [9] $x \in L^1$ possesses the left-sided Riemann-Liouville derivative $D^{\alpha}_{a+}x$ of order $\alpha \in (0,1)$ on the interval $[a,b]$, if the integral $I^{1-\alpha}_{a+}x$ is absolutely continuous on $[a,b]$ (more precisely, if $I^{1-\alpha}_{a+}x$ has an absolutely continuous representant a.e. on $[a,b]$). By this derivative one means the classical derivative $D^1(I^{1-\alpha}_{a+}x)$, that is,

$$\left(D^{\alpha}_{a+}x\right)(t) = \frac{1}{\Gamma(1-\alpha)}D^1\left(\int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha}}d\tau\right), \quad t \in [a,b] \text{ a.e.} \tag{4}$$

One has ([9], Theorem 2.4).

Theorem 2. *If $h \in L^1$, then $D^{\alpha}_{a+}I^{\alpha}_{a+}h = h$ a.e. on $[a,b]$. If $x \in I^{\alpha}_{a+}(L^1)$, then $I^{\alpha}_{a+}D^{\alpha}_{a+}x = x$ a.e. on $[a,b]$.*

From the second part of the above theorem and (3), it follows that if $x \in I^{\alpha}_{a+}(L^p)$, then

$$\|x\|_{L^p} \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\left\|D^{\alpha}_{a+}x\right\|_{L^p}. \tag{5}$$

Of course, $I^{\alpha}_{a+}(L^p)$ with the norm

$$\|x\|_{I^{\alpha}_{a+}(L^p)} = \left\|D^{\alpha}_{a+}x\right\|_{L^p}, \quad x \in I^{\alpha}_{a+}\left(L^p\right) \tag{6}$$

is complete. When $p = 2$, the above norm is generated by the following scalar product:

$$\langle x, y\rangle_{I^{\alpha}_{a+}(L^2)} = \int_a^b D^{\alpha}_{a+}x(t)\,D^{\alpha}_{a+}y(t)\,dt, \quad x, y \in I^{\alpha}_{a+}\left(L^2\right). \tag{7}$$

Next, one will use the following.

Lemma 3. *If a sequence (x_n) is weakly convergent in $I^{\alpha}_{a+}(L^p)$ to some x_0, then it is convergent to x_0 with respect to the norm in L^p, and the sequence $(D^{\alpha}_{a+}x_n)$ is weakly convergent in L^p to $D^{\alpha}_{a+}x_0$.*

Proof. To prove the second part of the theorem, it is sufficient to observe that the linear operator $D^{\alpha}_{a+} : I^{\alpha}_{a+}(L^p) \to L^p$ is continuous. Consequently, it preserves weak convergence. To prove the first part, let us observe that, from Theorem 1, it follows that $I^{\alpha}_{a+} : L^p \to L^p$ maps weakly convergent sequences onto strongly convergent (with respect to the norm) ones. Thus, the sequence $(x_n) = (I^{\alpha}_{a+}D^{\alpha}_{a+}x_n)$ is convergent to $x_0 = I^{\alpha}_{a+}D^{\alpha}_{a+}x_0$ with respect to the norm in L^p. □

2.2. A Theorem on a Diffeomorphism. In [8], we proved the following theorem.

Theorem 4. *Let X be a real Banach space, let H be a real Hilbert space. If $f : X \to H$ is a C^1-mapping (i.e., differentiable in Frechet sense on X with the differential f' continuous on X) such that*

(α) *for any $y \in H$, the functional*

$$\varphi : X \ni x \longmapsto \left(\frac{1}{2}\right)\|f(x) - y\|^2 \in \mathbb{R} \tag{8}$$

satisfies Palais-Smale condition,

(β) *for any $x \in X$, $f'(x) : X \to H$ is "one-one" and "onto",*

then f is a diffeomorphism (i.e., it is "one-one", "onto" and the inverse $f^{-1} : H \to X$ is differentiable in Frechet sense).

Let us recall that a C^1-functional $\varphi : X \to \mathbb{R}$ satisfies Palais-Smale condition if any sequence (x_k) such that

$$\left|\varphi(x_k)\right| \le M \quad \forall k \in \mathbb{N}, \text{ and some } M > 0, \qquad \varphi'(x_k) \longrightarrow 0, \tag{9}$$

admits a convergent subsequence (here, $\varphi'(x_k)$ is the Frechet differential of φ at x_k).

3. Main Result

Let us consider problem (1) with $g \in L^2$. We assume that function $\Phi = (\Phi_1, \ldots, \Phi_n) : P_{\Delta} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following conditions:

(A_1) Φ is measurable in $(t,s) \in P_{\Delta}$ and continuously differentiable in Frechet sense in $x \in \mathbb{R}^n$,

(A_2) there exist functions $a, b \in L^2(P_{\Delta}) = L^2(P_{\Delta},\mathbb{R})$ such that

$$|\Phi(t,s,x)| \le a(t,s)|x| + b(t,s), \tag{10}$$

for $(t,s) \in P_{\Delta}$ a.e., $x \in \mathbb{R}^n$,

$$\|a\|_{L^2(P_{\Delta})} < \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}, \tag{11}$$

(A_3) there exists a function $c \in L^2(P_\Delta)$ such that

$$\left|\Phi_x(t,s,x)\right| \le c(t,s), \tag{12}$$

for $(t,s) \in P_\Delta$ a.e., $x \in \mathbb{R}^n$,

$$\int_a^t c^2(t,s)\,ds \le C, \tag{13}$$

for $t \in [a,b]$ a.e. and some $C > 0$.

We shall show that the operator

$$\begin{aligned} f : I^\alpha_{a+}\left(L^2\right) \ni x \longmapsto D^\alpha_{a+}x(\cdot) \\ - \int_a^{\cdot} \Phi(\cdot, s, x(s))\,ds \in L^2 \end{aligned} \tag{14}$$

satisfies assumptions of Theorem 4 with the spaces $X = I^\alpha_{a+}(L^2)$, $Y = L^2$. Namely, we have the following.

Lemma 5. *The operator f is well-defined C^1-mapping with the differential $f'(x)$ at any $x \in I^\alpha_{a+}(L^2)$ given by*

$$\begin{aligned} f'(x) : I^\alpha_{a+}\left(L^2\right) \ni h \longmapsto D^\alpha_{a+}h(\cdot) \\ - \int_a^{\cdot} \Phi_x(\cdot, s, x(s))\,h(s)\,ds \in L^2, \end{aligned} \tag{15}$$

where Φ_x is the Jacobi matrix of Φ with respect to x.

Proof. Well-definiteness of f. Since Φ is the Caratheodory function with respect to $(t,s) \in P_\Delta$ and $x \in \mathbb{R}^n$, the function

$$P_\Delta \ni (t,s) \longmapsto \Phi(t,s,x(s)) \in \mathbb{R}^n \tag{16}$$

is measurable. From (A_2), it follows that it belongs to L^1. The Fubini theorem implies integrability of the function

$$[a,b] \ni t \longmapsto \int_a^t \Phi(t,s,x(s))\,ds \in \mathbb{R}^n. \tag{17}$$

Moreover,

$$\begin{aligned} &\left|\int_a^t \Phi(t,s,x(s))\,ds\right|^2 \\ &\le \left(\int_a^t a(t,s)\,|x(s)|\,ds + \int_a^t b(t,s)\,ds\right)^2 \\ &\le \left(\left(\int_a^t a^2(t,s)\,ds\right)^{1/2}\left(\int_a^t |x(s)|^2 ds\right)^{1/2} \right. \\ &\qquad \left. + \left(\int_a^t 1^2 ds\right)^{1/2}\left(\int_a^t b^2(t,s)\,ds\right)^{1/2}\right)^2 \\ &\le 2\left(\int_a^t a^2(t,s)\,ds\right)\left(\int_a^b |x(s)|^2 ds\right) \\ &\quad + 2(b-a)\left(\int_a^t b^2(t,s)\,ds\right). \end{aligned} \tag{18}$$

The right-hand side is integrable on $[a,b]$. So, function (17) belongs to L^2.

Differentiability of f. Continuous differentiability of the first term of f follows from the linearity and continuity of the operator $D^\alpha_{a+} : I^\alpha_{a+}(L^2) \to L^2$.

So, let one consider the second term, that is, the operator

$$g : I^\alpha_{a+}\left(L^2\right) \ni x \longmapsto \int_a^{\cdot} \Phi(\cdot, s, x(s))\,ds \in L^2. \tag{19}$$

One will check that the operator

$$g'(x) : I^\alpha_{a+}\left(L^2\right) \ni h \longmapsto \int_a^{\cdot} \Phi_x(\cdot, s, x(s))\,h(s)\,ds \in L^2 \tag{20}$$

is the Frechet differential of g at $x \in I^\alpha_{a+}(L^2)$.

First, let one observe that $g'(x)$ is well defined. Of course, the function

$$P_\Delta \ni (t,s) \longmapsto \Phi_x(t,s,x(s))\,h(s) \in \mathbb{R}^n, \tag{21}$$

where $h \in I^\alpha_{a+}(L^2)$, is measurable. By (A_3) it belongs to L^1. From the Fubini theorem it follows that the function

$$[a,b] \ni t \longmapsto \int_a^t \Phi_x(t,s,x(s))\,h(s)\,ds \tag{22}$$

is integrable. Moreover, similarly as in the case of f,

$$\begin{aligned} &\left|\int_a^t \Phi_x(t,s,x(s))\,h(s)\,ds\right|^2 \\ &\le \left(\int_a^t c(t,s)\,|h(s)|\,ds\right)^2 \\ &\le \left(\int_a^t c^2(t,s)\,ds\right)\left(\int_a^t |h(s)|^2 ds\right) \\ &\le C\left(\int_a^b |h(s)|^2 ds\right). \end{aligned} \tag{23}$$

So, function (22) belongs to L^2.

Linearity of $g'(x)$ is obvious. Its continuity follows from the following estimations (cf. (5)):

$$\begin{aligned} \left\|g'(x)\,h\right\|^2_{L^2} &= \int_a^b \left|\int_a^t \Phi_x(t,s,x(s))\,h(s)\,ds\right|^2 dt \\ &\le C(b-a)\,\|h\|^2_{L^2} \\ &\le C\frac{(b-a)^{2\alpha+1}}{\Gamma(\alpha+1)^2}\,\|h\|^2_{I^\alpha_{a+}(L^2)}. \end{aligned} \tag{24}$$

Now, one will check that $g'(x)$ is the Gateaux differential of g at x, that is,

$$\lim_{\lambda \to 0} \frac{g(x+\lambda h) - g(x)}{\lambda} = g'(x)\,h \tag{25}$$

in L^2, for any $x, h \in I^{\alpha}_{a+}(L^2)$. Indeed, let (λ_k) be a sequence of real numbers converging to 0 and consider the limit

$$\lim_{k\to\infty}\int_a^b\left|\int_a^t\left(\frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\right)ds\right|^2 dt. \tag{26}$$

From the differentiability of Φ with respect to x, it follows that, for $t \in [a,b]$ a.e., the sequence of functions

$$[a,t]\ni s\longmapsto \frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\in\mathbb{R}^n \tag{27}$$

converges pointwise a.e. on $[a,t]$ to the zero function. From the mean value theorem applied to any coordinate function

$$[0,1]\ni\tau\longmapsto\Phi_j(t,s,x(s)+\tau\lambda_k h(s))\in\mathbb{R} \tag{28}$$

$(j = 1,\dots,n)$, it follows that functions (27) indexed by $k \in \mathbb{N}$ are commonly pointwise (a.e. on $[a,t]$) bounded by an integrable function (cf. (A_3)). So,

$$\int_a^t\left(\frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\right)ds\longrightarrow 0, \tag{29}$$

for $t \in [a,b]$ a.e. Moreover, using once again the mean value theorem, one obtains

$$\begin{aligned}&\left|\int_a^t\left(\frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\right)ds\right|^2\\&\le\left(\int_a^t\left|\frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\right|ds\right)^2\\&\le(n+1)^2\left(\int_a^t c(t,s)|h(s)|\,ds\right)^2\\&\le(n+1)^2\int_a^t c^2(t,s)\,ds\int_a^b|h(s)|^2ds\\&\le(n+1)^2C\|h\|_{L^2}.\end{aligned} \tag{30}$$

Consequently,

$$\lim_{k\to\infty}\int_a^b\left|\int_a^t\left(\frac{\Phi(t,s,x(s)+\lambda_k h(s))-\Phi(t,s,x(s))}{\lambda_k}-\Phi_x(t,s,x(s))h(s)\right)ds\right|^2dt=0, \tag{31}$$

that is, (25) holds true.

Continuity of g'. Let (x_k) be a sequence converging in $I^{\alpha}_{a+}(L^2)$ to some x_0. Similarly, as mentioned above, one obtains

$$\begin{aligned}&\left\|\left(g'(x_k)-g'(x_0)\right)h\right\|^2_{L^2}\\&=\int_a^b\left|\int_a^t(\Phi_x(t,s,x_k(s))-\Phi_x(t,s,x_0(s)))h(s)\,ds\right|^2dt\\&\le\int_a^b\left(\int_a^t|\Phi_x(t,s,x_k(s))-\Phi_x(t,s,x_0(s))|\,|h(s)|\,ds\right)^2dt\\&\le\int_a^b\int_a^t|\Phi_x(t,s,x_k(s))-\Phi_x(t,s,x_0(s))|^2ds\,dt\left(\int_a^b|h(s)|^2ds\right)\\&\le\left(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right)^2\\&\quad\times\int_a^b\int_a^t|\Phi_x(t,s,x_k(s))-\Phi_x(t,s,x_0(s))|^2ds\,dt\\&\quad\times\|h\|^2_{I^{\alpha}_{a+}(L^2)}\end{aligned} \tag{32}$$

for any $h \in L^2$. Convergence

$$\int_a^b\int_a^t|\Phi_x(t,s,x_k(s))-\Phi_x(t,s,x_0(s))|^2ds\,dt\longrightarrow 0 \tag{33}$$

follows from (A_3), the Krasnoselskii theorem on the continuity of the Nemytskii operator and from (5).

So, g being continuously differentiable in Gateaux sense is continuously differentiable in Frechet sense. □

Now, one will prove.

Lemma 6. *For any fixed $y \in L^2$, the functional*

$$\varphi: I^{\alpha}_{a+}(L^2)\ni x\longmapsto\frac{1}{2}\left\|D^{\alpha}_{a+}x(\cdot)-\int_a^{\cdot}\Phi(\cdot,s,x(s))\,ds-y(\cdot)\right\|^2_{L^2}\in\mathbb{R} \tag{34}$$

satisfies Palais-Smale condition.

Proof. It is easy to see that

$$\varphi(x)\ge\frac{1}{2}\|x\|^2_{I^{\alpha}_{a+}(L^2)}+\frac{1}{2}\|y\|^2_{L^2}-\varphi_1(x)+\varphi_2(x)-\varphi_3(x), \tag{35}$$

for $x \in I_{a+}^{\alpha}(L^2)$, where

$$\varphi_1(x) = \int_a^b \left(D_{a+}^{\alpha} x(t)\right) y(t)\, dt,$$
$$\varphi_2(x) = \int_a^b \left(D_{a+}^{\alpha} x(t)\right) \int_a^t \Phi(t, s, x(s))\, ds\, dt, \tag{36}$$
$$\varphi_3(x) = \int_a^b y(t) \int_a^t \Phi(t, s, x(s))\, ds\, dt.$$

Of course,

$$\left|\varphi_1(x)\right| \le \left\|y\right\|_{L^2} \|x\|_{I_{a+}^{\alpha}(L^2)}. \tag{37}$$

Moreover,

$$\begin{aligned}
\left|\varphi_2(x)\right| &\le \int_a^b \left|D_{a+}^{\alpha} x(t)\right| \int_a^t \left(a(t,s)\left|x(s)\right| + b(t,s)\right) ds\, dt \\
&\le \int_a^b \left|D_{a+}^{\alpha} x(t)\right| \left(\left(\left(\int_a^t a^2(t,s)\, ds \right)^{1/2} \right.\right. \\
&\qquad \times \left(\int_a^t \left|x(s)\right|^2 ds \right)^{1/2} + (b-a)^{1/2} \\
&\qquad \left.\left. \times \left(\int_a^t b^2(t,s)\, ds \right)^{1/2} \right) \right) dt \\
&\le \|x\|_{L^2} \|x\|_{I_{a+}^{\alpha}(L^2)} \|a\|_{L^2(P_\Delta)} \\
&\quad + (b-a)^{1/2} \|x\|_{I_{a+}^{\alpha}(L^2)} \|b\|_{L^2(P_\Delta)} \\
&\le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|a\|_{L^2(P_\Delta)} \|x\|_{I_{a+}^{\alpha}(L^2)}^2 \\
&\quad + (b-a)^{1/2} \|x\|_{I_{a+}^{\alpha}(L^2)} \|b\|_{L^2(P_\Delta)},
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\left|\varphi_3(x)\right| \\
&\quad \le \int_a^b \left|y(t)\right| \int_a^t \left(a(t,s)\left|x(s)\right| + b(t,s)\right) ds\, dt \\
&\quad \le \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|x\|_{I_{a+}^{\alpha}(L^2)} \left\|y\right\|_{L^2} \|a\|_{L^2(P_\Delta)} \\
&\qquad + (b-a)^{1/2} \left\|y\right\|_{L^2} \|b\|_{L^2(P_\Delta)}.
\end{aligned} \tag{39}$$

So,

$$\begin{aligned}
\varphi(x) &\ge \frac{1}{2} \|x\|_{I_{a+}^{\alpha}(L^2)}^2 + \frac{1}{2} \left\|y\right\|_{L^2}^2 - \left\|y\right\|_{L^2} \|x\|_{I_{a+}^{\alpha}(L^2)} \\
&\quad - \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|a\|_{L^2(P_\Delta)} \|x\|_{I_{a+}^{\alpha}(L^2)}^2 \\
&\quad - (b-a)^{1/2} \|x\|_{I_{a+}^{\alpha}(L^2)} \|b\|_{L^2(P_\Delta)} \\
&\quad - \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|x\|_{I_{a+}^{\alpha}(L^2)} \left\|y\right\|_{L^2} \|a\|_{L^2(P_\Delta)} \\
&\quad - (b-a)^{1/2} \left\|y\right\|_{L^2} \|b\|_{L^2(P_\Delta)} \\
&= d_0 \|x\|_{I_{a+}^{\alpha}(L^2)}^2 + d_1 \|x\|_{I_{a+}^{\alpha}(L^2)} + d_2,
\end{aligned} \tag{40}$$

for $x \in I_{a+}^{\alpha}(L^2)$, where

$$\begin{aligned}
d_0 &= \left(\frac{1}{2} - \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \|a\|_{L^2(P_\Delta)} \right), \\
d_1 &= -\left\|y\right\|_{L^2} - (b-a)^{1/2} \|b\|_{L^2(P_\Delta)} \\
&\qquad - \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \left\|y\right\|_{L^2} \|a\|_{L^2(P_\Delta)}, \\
d_2 &= \frac{1}{2} \left\|y\right\|_{L^2}^2 - (b-a)^{1/2} \left\|y\right\|_{L^2} \|b\|_{L^2(P_\Delta)}.
\end{aligned} \tag{41}$$

Since, $d_0 > 0$ (by (A_1)), therefore φ is coercive, that is, $\varphi(x) \to \infty$ as $\|x\|_{I_{a+}^{\alpha}(L^2)} \to \infty$.

Let us fix a sequence $(x_k) \subset I_{a+}^{\alpha}(L^2)$ such that

$$\begin{gathered}
\left|\varphi(x_k)\right| \le M \quad \forall\, k \in \mathbb{N} \text{ and some } M > 0, \\
\varphi'(x_k) \longrightarrow 0.
\end{gathered} \tag{42}$$

The first condition and coercivity of φ imply that the sequence (x_k) is bounded. So, without loss of the generality, we may assume that it weakly converges in $I_{a+}^{\alpha}(L^2)$ to some x_0. Lemma 5 implies that φ is of class C^1 and

$$\begin{aligned}
\varphi'(x) h &= \int_a^b D_{a+}^{\alpha} x(t)\, D_{a+}^{\alpha} h(t)\, dt - \int_a^b y(t)\, D_{a+}^{\alpha} h(t)\, dt \\
&\quad - \int_a^b D_{a+}^{\alpha} h(t) \int_a^t \Phi(t, s, x(s))\, ds\, dt \\
&\quad - \int_a^b D_{a+}^{\alpha} x(t) \int_a^t \Phi_x(t, s, x(s))\, h(s)\, ds\, dt \\
&\quad + \int_a^b y(t) \int_a^t \Phi_x(t, s, x(s))\, h(s)\, ds\, dt \\
&\quad + \int_a^b \left(\int_a^t \Phi(t, s, x(s))\, ds \right) \\
&\qquad \times \left(\int_a^t \Phi_x(t, s, x(s))\, h(s)\, ds \right) dtw
\end{aligned} \tag{43}$$

for $x, h \in I_{a+}^{\alpha}(L^2)$. Consequently,

$$\begin{aligned}
&\left(\varphi'(x_k) - \varphi'(x_0)\right)(x_k - x_0) \\
&\qquad = \|x_k - x_0\|_{I_{a+}^{\alpha}(L^2)}^2 + \sum_{i=1}^{6} \psi_i(x_k),
\end{aligned} \tag{44}$$

where

$$\psi_1(x_k) = -\int_a^b \left(D^\alpha_{a+}x_k(t) - D^\alpha_{a+}x_0(t)\right) \times \left(\int_a^t \left(\Phi(t,s,x_k(s)) - \Phi(t,s,x_0(s))\right) ds\right) dt, \tag{45}$$

$$\psi_2(x_k) = -\int_a^b D^\alpha_{a+}x_k(t) \times \left(\int_a^t \Phi_x(t,s,x_k(s))\,(x_k(s) - x_0(s))\,ds\right) dt, \tag{46}$$

$$\psi_3(x_k) = \int_a^b D^\alpha_{a+}x_0(t) \times \left(\int_a^t \Phi_x(t,s,x_0(s))\,(x_k(s) - x_0(s))\,ds\right) dt, \tag{47}$$

$$\psi_4(x_k) = \int_a^b y(t) \times \left(\int_a^b \left(\Phi_x(t,s,x_k(s)) - \Phi_x(t,s,x_0(s))\right) \times (x_k(s) - x_0(s))\,ds\right) dt, \tag{48}$$

$$\psi_5(x_k) = \int_a^b \left(\int_a^t \Phi(t,s,x_k(s))\,ds\right) \times \left(\int_a^t \Phi_x(t,s,x_k(s))\,(x_k(s) - x_0(s))\,ds\right) dt, \tag{49}$$

$$\psi_6(x_k) = -\int_a^b \left(\int_a^t \Phi(t,s,x_0(s))\,ds\right) \times \left(\int_a^t \Phi_x(t,s,x_0(s))\,(x_k(s) - x_0(s))\,ds\right) dt. \tag{50}$$

The left-hand side converges to 0 because

$$\left(\varphi'(x_k) - \varphi'(x_0)\right)(x_k - x_0) = \varphi'(x_k)(x_k - x_0) - \varphi'(x_0)(x_k - x_0) \tag{51}$$

and $\varphi'(x_k) \to 0$ as well as $x_k \rightharpoonup x_0$ weakly in $I^\alpha_{a+}(L^2)$. Terms $\psi_i(x_k)$, $i = 1, \ldots, 6$, also converge to 0. This follows from the strong convergence of the sequence (x_k) to x_0 in L^2 and weak convergence of the sequence $(D^\alpha_{a+}x_k(t))$ to $D^\alpha_{a+}x_0(t)$ in L^2 (cf. Lemma 3) as well as from the Krasnoselskii theorem on the continuity of the Nemytskii operator.

Indeed, from the Krasnoselskii theorem, it follows that the sequence

$$\left(\int_a^{\cdot} \left(\Phi(\cdot,s,x_k(s)) - \Phi(\cdot,s,x_0(s))\right) ds\right) \tag{52}$$

converges pointwise a.e. on $[a,b]$ to the zero function. Moreover, in the same way as in the proof of Lemma 5, one can check that the sequence

$$\left(\left|\int_a^{\cdot} \left(\Phi(\cdot,s,x_k(s)) - \Phi(\cdot,s,x_0(s))\right) ds\right|^2\right) \tag{53}$$

is bounded on $[a,b]$ by an integrable function. This means that

$$\int_a^b \left|\int_a^t \left(\Phi(t,s,x_k(s)) - \Phi(t,s,x_0(s))\right) ds\right|^2 dt \longrightarrow 0, \tag{54}$$

that is, the sequence $(\int_a^{\cdot}(\Phi(\cdot,s,x_k(s)) - \Phi(\cdot,s,x_0(s)))ds)$ converges in L^2 to the zero function.

Similarly, if $\chi_k(t,s)$, $k \in \mathbb{N}$, are functions belonging to $L^2(P_\Delta)$, commonly bounded on P_Δ by a function $\chi \in L^2(P_\Delta)$, then the sequence

$$\left(\int_a^{\cdot} \chi_k(\cdot,s)\,(x_k(s) - x_0(s))\,ds\right) \tag{55}$$

converges pointwise a.e. on $[a,b]$ to the zero function. Moreover,

$$\begin{aligned} &\left|\int_a^t \chi(t,s)\,(x_k(s) - x_0(s))\,ds\right|^2 \\ &\quad \le \left(\int_a^t |\chi_k(t,s)|\,|(x_k(s) - x_0(s))|\,ds\right)^2 \\ &\quad \le \left(\int_a^t |\chi(t,s)|^2 ds\right)\left(\int_a^b |x_k(s) - x_0(s)|^2 ds\right) \\ &\quad \le \text{const}\left(\int_a^t |\chi(t,s)|^2 ds\right), \end{aligned} \tag{56}$$

where const is a constant which bounds the sequence (x_k) in L^2. So, sequence (55) converges in L^2 to the zero function. Applying these facts to the functions

$$\begin{aligned} \chi_k(t,s) &= \Phi_x(t,s,x_k(s)), \\ \chi_k(t,s) &= \Phi_x(t,s,x_0(s)), \\ \chi_k(t,s) &= \Phi_x(t,s,x_k(s)) - \Phi_x(t,s,x_0(s)), \end{aligned} \tag{57}$$

and using (54) one asserts that $\psi_i(x_k) \to 0$, for $i = 1, \ldots, 6$.

Consequently, $\|x_k - x_0\|^2_{I^\alpha_{a+}(L^2)} \to 0$, that is, φ satisfies Palais-Smale condition. □

Now, one will show that f satisfies assumption (β) from Theorem 4. More precisely, one will prove a more general result, namely.

Lemma 7. *If $\Psi = \Psi(t,s,h) : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$ is*

(B_1) *measurable in $(t,s) \in P_\Delta$,*

(B_2) *there exists a function $d \in L^2(P_\Delta)$ such that*

$$\left|\Psi(t,s,h_1) - \Psi(t,s,h_2)\right| \le d(t,s)\left|h_1 - h_2\right| \tag{58}$$

for $(t,s) \in P_\Delta$ a.e., $h_1, h_2 \in \mathbb{R}^n$,

$$\int_a^t d^2(t,s)\,ds \le D \tag{59}$$

for $t \in [a,b]$ a.e. and some $D > 0$,

(B_3) $\Psi(\cdot,\cdot,0) \in L^2(P_\Delta)$,

then the operator

$$\Lambda : I_{a+}^{\alpha}\left(L^2\right) \ni h \longmapsto D_{a+}^{\alpha} h(\cdot) - \int_a^{\cdot} \Psi(\cdot,s,h(s))\,ds \in L^2 \tag{60}$$

is well defined, "one-one" and "onto".

Proof. First, let us observe that Λ is well defined. Indeed, for any $h \in L^2$ (in particular, for $h \in I_{a+}^{\alpha}(L^2)$), one has

$$\begin{aligned}\left|\int_a^t \Psi(t,s,h(s))\,ds\right| &\le \int_a^t |\Psi(t,s,h(s)) - \Psi(t,s,0)|\,ds \\ &\quad + \int_a^t |\Psi(t,s,0)|\,ds \\ &\le \int_a^t d(t,s)\,|h(s)|\,ds \\ &\quad + \int_a^t |\Psi(t,s,0)|\,ds \\ &\le \left(\int_a^t d^2(t,s)\,ds\right)^{1/2}\left(\int_a^b |h(s)|^2 ds\right)^{1/2} \\ &\quad + \int_a^t |\Psi(t,s,0)|\,ds,\end{aligned} \tag{61}$$

for $t \in [a,b]$ a.e., and the right-hand side belongs to L^2.

Now, let one consider some auxiliary problem

$$\begin{gathered} D_{a+}^{\alpha} h(t) = v(t), \quad t \in [a,b] \text{ a.e.}, \\ I_{a+}^{1-\alpha} h(a) = 0. \end{gathered} \tag{62}$$

with a fixed $v \in L^2$. Of course, problem (62) has a unique solution $h_v(t) = I_{a+}^{\alpha} v(t)$ in the space $I_{a+}^{\alpha}(L^2)$ (cf. [11]).

To end the proof, it is sufficient to show that the operator

$$K : L^2 \ni v \longmapsto \int_a^{\cdot} \Psi(\cdot,s,h_v(s))\,ds + g(\cdot) \in L^2 \tag{63}$$

with any fixed $g \in L^2$ possesses a unique fixed point.

One will show that there exist constants $\kappa \in (0,1)$, $l \in \mathbb{N}$ such that

$$\left\|Kv_1 - Kv_2\right\|_l \le \kappa \left\|v_1 - v_2\right\|, \tag{64}$$

for any $v_1, v_2 \in L^2$, where $\|\cdot\|_l$ is the Bielecki norm in L^2 given by

$$\|v\|_l^2 = \int_a^b e^{-lt} |v(t)|^2 dt, \qquad v \in L^2. \tag{65}$$

Indeed, one has

$$\begin{aligned}&\left\|Kv_1 - Kv_2\right\|_l^2 \\ &\quad = \int_a^b e^{-lt}\left|Kv_1(t) - Kv_2(t)\right|^2 dt \\ &\quad \le \int_a^b e^{-lt}\left(\int_a^t \left|\Psi\left(t,s,h_{v_1}(s)\right) - \Psi\left(t,s,h_{v_2}(s)\right)\right| ds\right)^2 dt \\ &\quad \le \int_a^b e^{-lt}\left(\int_a^t d(t,s)\left|h_{v_1}(s) - h_{v_2}(s)\right| ds\right)^2 dt \\ &\quad = \int_a^b e^{-lt}\left(\int_a^t d(t,s)\left|I_{a+}^{\alpha}(v_1 - v_2)(s)\right| ds\right)^2 dt \\ &\quad \le \int_a^b e^{-lt} \int_a^t d^2(t,s)\,ds \int_a^t \left|I_{a+}^{\alpha}(v_1 - v_2)(s)\right|^2 ds\,dt \\ &\quad \le D\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)^2 \int_a^b e^{-lt} \int_a^t \left|(v_1 - v_2)(s)\right|^2 ds\,dt \\ &\quad \le D\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)^2 \frac{1}{l} \int_a^b e^{-lt}\left|(v_1 - v_2)(t)\right|^2 dt \\ &\quad = \frac{D}{l}\left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right)^2 \left\|v_1 - v_2\right\|_l^2,\end{aligned} \tag{66}$$

for v_1, $v_2 \in L^2$. It is sufficient to choose $l \in \mathbb{N}$, such that $\sqrt{D/k}((b-a)^\alpha/\Gamma(\alpha+1)) < 1$. □

Applying Lemma 7 to the function

$$\Psi(t,s,h) = \Phi_x(t,s,x(s))\,h \tag{67}$$

with a fixed $x \in I_{a+}^{\alpha}(L^2)$, one obtains the following.

Lemma 8. *Operator f satisfies (β).*

Theorem 4 and Lemmas 5, 6, and 8 imply the following.

Theorem 9. *Problem* (1) *possesses a unique solution $x_g \in I_{a+}^{\alpha}(L^2)$, for any $g \in L^2$, and the operator*

$$L^2 \ni g \longmapsto x_g \in I_{a+}^{\alpha}\left(L^2\right) \tag{68}$$

is differentiable in Frechet sense.

Remark 10. When $\alpha \in (1/2, 1)$, all elements of $I_{a+}^{\alpha}(L^2)$ are continuous (cf. [9], Theorem 3.6). Consequently, growth condition (A_3) can be slightly weakened just like in [8].

4. Conclusions

In the paper, sensitivity of a fractional Integrodifferential Cauchy problem of Volterra type has been investigated. Namely, it has been proved that problem (1) possesses (under the appropriate assumptions) a unique solution $x_g \in I^{\alpha}_{a+}(L^2)$ for any fixed functional parameter $g \in L^2$ and the dependence (68) is differentiable in Frechet sense. In the next paper, sensitivity of such a problem with an integral term of Fredholm type as well as of a problem containing the both terms will be considered.

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References

[1] B. Ahmad and J. Nieto, "Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions," *Boundary Value Problems*, vol. 2011, p. 36, 2011.

[2] S. Bushnaq, S. Momani, and Y. Zhou, "A reproducing kernel Hilbert space method for solving integro-differential equations of fractional order," *Journal of Optimization Theory and Applications*, vol. 156, no. 1, pp. 96–105, 2013.

[3] B. Gayathri, R. Murugesu, and J. Rajasingh, "Existence of solutions of some impulsive fractional integrodifferential equations," *International Journal of Mathematical Analysis*, vol. 6, no. 17, pp. 825–836, 2012.

[4] M. M. Matar, "Boundary value problem for some fractional integrodifferential equations with nonlocal conditions," *International Journal of Nonlinear Science*, vol. 11, no. 1, pp. 3–9, 2011.

[5] W. Sudsutad and J. Tariboon, "Existence results of fractional integro-differential equations with m-point multi-term fractional order integral boundary conditions," *Boundary Value Problems*, vol. 2012, p. 94, 2012.

[6] J. Wang and W. Wei, "Nonlinear delay integrodifferential systems with Caputo fractional derivative in infinite-dimensional spaces," *Annales Polonici Mathematici*, vol. 105, pp. 209–223, 2012.

[7] Z. Yan, "Existence results for fractional functional integrodifferential equations with nonlocal conditions in Banach spaces," *Annales Polonici Mathematici*, vol. 97, no. 3, pp. 285–299, 2010.

[8] D. Idczak, A. Skowron, and S. Walczak, "On the diffeomorphisms between Banach and Hilbert spaces," *Advanced Nonlinear Studies*, vol. 12, pp. 89–100, 2012.

[9] S. G. Samko, A. A. Kilbas, and O. I. Marichev :, *Fractional Integrals and Derivatives-Theory and Applications*, Gordon and Breach, Amsterdam, The Netherlands, 1993.

[10] M. W. Michalski, *Derivatives of noninteger order and their applications [Ph.D. thesis]*, Polish Academy of Sciences, Warsaw, Poland, 1993.

[11] D. Idczak and R. Kamocki, "On the existence and unqueness and formula for the solution of R-L fractional Cauchy problem in R^n," *Fractional Calculus and Applied Analysis*, vol. 14, no. 4, pp. 538–553, 2011.