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Research Article

Remark on Existence and Uniqueness of Solutions for a Coupled System of Multiterm Nonlinear Fractional Differential Equations

Huichao Zou and Yonghong Fan

School of Mathematics and Statistics Science, Ludong University, Yantai, Shandong 264025, China

Correspondence should be addressed to Huichao Zou; ldzouhuichao@163.com

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The aim of this paper is to extend the work of Sun et al. (2012) to a more general case for a wider range of function classes of f and g. Our results include the case of the previous work, which are essential improvement of the work of Sun et al. (2012), especially.

1. Introduction

Fractional calculus can give a more vivid and accurate description of problems in various fields of sciences than the traditional calculus [1–3]. Recently many complicated dynamic phenomena were modeled by fractional order calculus system and have received more and more attention; see [4–16].

In recent work [12], Sun et al. studied the existence and uniqueness of solutions for a coupled system of multiterm nonlinear fractional differential equations with an initial value condition

$$\begin{aligned} -\mathscr{D}^{\alpha} x(t) &= f\left(t, y(t), \mathscr{D}^{\beta_1} y(t), \dots, \mathscr{D}^{\beta_N} y(t)\right), \\ \mathscr{D}^{\alpha-i} x(0) &= 0, \quad i = 1, 2, \dots, n_1, \\ -\mathscr{D}^{\sigma} y(t) &= g\left(t, x(t), \mathscr{D}^{\rho_1} x(t), \dots, \mathscr{D}^{\rho_N} x(t)\right), \\ \mathscr{D}^{\sigma-j} y(0) &= 0, \quad j = 1, 2, \dots, n_2, \end{aligned} \tag{1}$$

where $t \in (0, 1]$, $\alpha > \beta_1 > \beta_2 > \cdots > \beta_N > 0$, $\sigma > \rho_1 > \rho_2 > \cdots > \rho_N > 0$, $n_1 = [\alpha] + 1$, $n_2 = [\sigma] + 1$ for $\alpha, \sigma \notin \mathbb{N}$ and $n_1 = \alpha$, $n_2 = \sigma$ for $\alpha, \sigma \in \mathbb{N}$, $\beta_q, \rho_q < 1$ for any $q \in \{1, 2, \dots, N\}$, $\mathscr{D}$ is the standard Riemann-Liouville derivative, and $f, g : [0, 1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions. In order to obtain the existence and uniqueness of solutions of (1), the following growth conditions are introduced in [12].

(H1) There exist two nonnegative functions $a(t), b(t) \in L^1[0, 1]$ such that

$$\begin{aligned} &\left|f(t, x_0, x_1, \dots, x_N)\right| \\ &\quad \le a(t) + c_0\left|x_0\right|^{\gamma_0} + c_1\left|x_1\right|^{\gamma_1} + \cdots + c_N\left|x_N\right|^{\gamma_N}, \\ &\left|G(t, x_0, x_1, \dots, x_N)\right| \\ &\quad \le b(t) + d_0\left|x_0\right|^{\theta_0} + d_1\left|x_1\right|^{\theta_1} + \cdots + d_N\left|x_N\right|^{\theta_N}, \end{aligned} \tag{2}$$

where $c_i, d_i \ge 0$, $0 < \gamma_i, \theta_i < 1$ for $i = 0, 1, 2, \dots, N$.

(H2) The functions f and g satisfy

$$\begin{aligned} \left|f(t, x_0, x_1, \dots, x_N)\right| &\le c_0\left|x_0\right|^{\gamma_0} + c_1\left|x_1\right|^{\gamma_1} + \cdots + c_N\left|x_N\right|^{\gamma_N}, \\ \left|g(t, x_0, x_1, \dots, x_N)\right| &\le d_0\left|x_0\right|^{\theta_0} + d_1\left|x_1\right|^{\theta_1} + \cdots + d_N\left|x_N\right|^{\theta_N}, \end{aligned} \tag{3}$$

where $c_i, d_i \ge 0$, $\gamma_i, \theta_i > 1$ for $i = 0, 1, 2, \dots, N$.

However, there are many functions which cannot satisfy conditions (H1) and (H2); for example,

$$g(t, x_0, x_1) = \frac{t}{6.08} + \frac{1}{25.26}\left[x_0 + e^{x_1}\right]. \tag{4}$$

Hence the results of [12] are limited only to a small subset of functions which satisfy (H1) and (H2). This paper thus aims to extend the work of Sun et al. [12] to a more general case with more general conditions on f and g. Our major contributions of this paper include three aspects.

(1) We extend the function classes to more general case; that is, the power growth assumptions (H1) and (H2) are replaced by a very general assumption where the functions $\phi(|x_0|, |x_1|, \ldots, |x_N|)$ and $\psi(|x_0|, |x_1|, \ldots, |x_N|)$ are only required to be nondecreasing function classes (see (A1)), which implies that the function classes are extended to more general case and also include the case of [12] as a special case. In mathematics and applied science, this generalization is important and interesting.

(2) In [12], the weight functions considered constants $c_0, c_1, \ldots, c_N$. But in physics, the influence of weight functions for the whole system is important, so in this work, we improve the weight functions to general Lebesgue integral functions $b(t), d(t) \in L^1[0,1]$, which is also an essential improvement.

(3) In this paper, the nonlinearities f and g are allowed to be exponential growth. However, in [12], the nonlinearities f and g are only allowed to be power growth. It is known that in most cases exponential growth is faster than power growth. From this aspect, this is also a major contribution of this paper.

The remaining part of the paper is organized as follows. In Section 2, some preliminary results including definitions, notations, and lemmas are given. Section 3 presents the main results and the proof of the results. In addition, an example is given to illustrate the application of the main results.

2. Preliminaries and Lemmas

Definition 1 (see [1–3]). The fractional integral of order $\alpha > 0$ of a function $x : (a, +\infty) \to R$ is given by

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds, \quad (5)$$

provided that the right-hand side is pointwisely defined on $(a, +\infty)$.

Definition 2 (see [1–3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function x: $(a, +\infty) \to R$ is given by

$$\mathscr{D}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds, \quad (6)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, and $t > a$, provided that the right-hand side is defined on $(a, +\infty)$.

Lemma 3 (see [1]). *Assume that $x \in L^1[0,1]$ with a fractional derivative of order $\alpha > 0$; then*

$$I^{\alpha} \mathscr{D}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad (7)$$

where $c_i \in R$, $i = 1, 2, \ldots, n$, $n = [\alpha] + 1$.

Lemma 4 (see [12]). *Suppose that $h \in L^1[0,1]$. Then the initial value problem*

$$\begin{gathered} \mathscr{D}^{\alpha} x(t) = h(t), \quad \alpha > 0, \; t \in [a, b], \\ \mathscr{D}^{\alpha} x(a) = b_k, \quad k = 1, 2, \ldots n, \end{gathered} \quad (8)$$

has a unique solution

$$x(t) = \sum_{j=1}^{n} \frac{b_j}{\Gamma(\alpha - j + 1)} (t-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \quad (9)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $\alpha = n$ for $\alpha \in \mathbb{N}$.

Let $I = [0,1]$ and let $C(I)$ be the space of all continuous functions defined on I. We define the space

$$\begin{aligned} X \times Y = \Big\{ & (x, y) \mid (x, y) \in C(I) \times C(I), \\ & \left(\mathscr{D}^{\rho_j} x(t), \mathscr{D}^{\beta_j} y(t)\right) \in C(I) \\ & \times C(I), j = 1, 2, \ldots, N \Big\} \end{aligned} \quad (10)$$

endowed with the norm $\|(x, y)\|_{X \times Y} = \max\{\|x\|_X, \|y\|_Y\}$, where

$$\begin{aligned} \|x\|_X &= \max_{t \in I} |x(t)| + \sum_{j=1}^{N} \max_{t \in I} \left|\mathscr{D}^{\rho_j} x(t)\right|, \\ \|y\|_Y &= \max_{t \in I} |y(t)| + \sum_{j=1}^{N} \max_{t \in I} \left|\mathscr{D}^{\beta_j} y(t)\right|. \end{aligned} \quad (11)$$

Then $X \times Y$ is a Banach space with norm $\|(x, y)\|_{X \times Y}$.

By Lemma 4, system (1) is equivalent to the following coupled system of integral equations:

$$\begin{aligned} & x(t) \\ & = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \mathscr{D}^{\beta_1} y(s), \ldots, \mathscr{D}^{\beta_N} y(s)\right) ds, \\ & y(t) \\ & = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} g\left(s, x(s), \mathscr{D}^{\rho_1} x(s), \ldots, \mathscr{D}^{\rho_N} x(s)\right) ds. \end{aligned} \quad (12)$$

Define an operator $T : X \times Y \to X \times Y$

$$\begin{aligned} & T(x, y)(t) \\ & = (T_1 x(t), T_2 y(t)) \\ & = \Bigg(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ & \qquad \times f\left(s, y(s), \mathscr{D}^{\beta_1} y(s), \ldots, \mathscr{D}^{\beta_N} y(s)\right) ds, \end{aligned}$$

$$\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} \times g\left(s, x(s), \mathscr{D}^{\rho_1}x(s), \ldots, \mathscr{D}^{\rho_N}x(s)\right) ds\Bigg). \tag{13}$$

It is obvious that a fixed point of operator T is the solution of coupled system (1).

3. Main Result

Theorem 5. *Let $f, g : [0,1]\times\mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(A1) *there exist nonnegative functions $a, b, c, d \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that*

$$\begin{aligned} &\left|f(t, x_0, x_1, \ldots, x_N)\right| \le a(t) + b(t)\,\phi\left(|x_0|, |x_1|, \ldots, |x_N|\right), \\ &\left|g(t, x_0, x_1, \ldots, x_N)\right| \le c(t) + d(t)\,\psi\left(|x_0|, |x_1|, \ldots, |x_N|\right); \end{aligned} \tag{14}$$

(A2) *there exists a constant $R_0 > \max\{k_1, l_1\}$ such that*

$$\begin{aligned} \phi(R_0, R_0, \ldots, R_0) &\le \frac{R_0 - k_1}{k_2}, \\ \psi(R_0, R_0, \ldots, R_0) &\le \frac{R_0 - l_1}{l_2}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} k_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|a(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|a(s)|\,ds\right), \\ k_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|b(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|b(s)|\,ds\right), \\ l_1 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}|c(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}|c(s)|\,ds\right), \\ l_2 &= \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}|d(s)|\,ds + \sum_{j=1}^N \frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}|d(s)|\,ds\right). \end{aligned} \tag{16}$$

Then the coupled system (1) *has a solution.*

Proof. Define a closed ball of Banach space $X\times Y$

$$B = \left\{(x, y) \in X\times Y : \|(x, y)\|_{X\times Y} \le R_0\right\}. \tag{17}$$

We will prove that $T : B \to B$. In fact, for any $(x, y) \in B$, by (A1), we have

$$\begin{aligned} &\left|T_1 x(t)\right| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\left(s, y(s), \mathscr{D}^{\beta_1}y(s), \ldots, \mathscr{D}^{\beta_N}y(s)\right) ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|a(s)|\,ds + \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|b(s)|\,ds, \\ &\left|\mathscr{D}^{\rho_j}T_1 x(t)\right| \\ &= \left|\mathscr{D}^{\rho_j} I^{\alpha} f\left(t, y(t), \mathscr{D}^{\beta_1}y(t), \ldots, \mathscr{D}^{\beta_N}y(t)\right)\right| \\ &= \left|I^{\alpha-\rho_j} f\left(t, y(t), \mathscr{D}^{\beta_1}y(t), \ldots, \mathscr{D}^{\beta_N}y(t)\right)\right| \\ &= \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1} f\left(s, y(s), \mathscr{D}^{\beta_1}y(s), \ldots, \mathscr{D}^{\beta_N}y(s)\right) ds \\ &\le \frac{1}{\Gamma(\alpha-\rho_j)} \times \int_0^t (t-s)^{\alpha-\rho_j-1}|a(s)|\,ds + \frac{\phi(R_0, R_0, \ldots, R_0)}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}|b(s)|\,ds. \end{aligned} \tag{18}$$

Thus it follows from (18) and (A2) that

$$\begin{aligned} \|T_1 x\|_X &= \max_{t\in I}\left|T_1 x(t)\right| + \sum_{j=1}^N \max_{t\in I}\left|\mathscr{D}^{\rho_j}T_1 x(t)\right| \\ &\le k_1 + k_2\phi(R_0, R_0, \ldots, R_0) \le R_0. \end{aligned} \tag{19}$$

In the same way, we also have

$$\begin{aligned} \|T_2 y\|_Y &= \max_{t\in I}\left|T_2 y(t)\right| + \sum_{j=1}^N \max_{t\in I}\left|\mathscr{D}^{\rho_j}T_2 y(t)\right| \\ &\le l_1 + l_2\psi(R_0, R_0, \ldots, R_0) \le R_0. \end{aligned} \tag{20}$$

Consequently, $\|T_1 x\|_X \le R_0$ and $\|T_2 y\|_Y \le R_0$, and then $\|T\|_{X\times Y} \le R_0$ for any $(x, y) \in B$; that is, $T : B \to B$.

By [12], we know that the operator T is completely continuous. Therefore, the Schauder fixed point theorem implies that coupled system (1) has a solution in B. The proof is completed. ◻

From Theorem 5, we easily obtain the following corollaries.

Corollary 6. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(A1) *there exist nonnegative functions $c, d \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that*

$$\begin{aligned} \left|f(t, x_0, x_1, \ldots, x_N)\right| &\le b(t)\,\phi\left(\left|x_0\right|, \left|x_1\right|, \ldots, \left|x_N\right|\right), \\ \left|g(t, x_0, x_1, \ldots, x_N)\right| &\le d(t)\,\psi\left(\left|x_0\right|, \left|x_1\right|, \ldots, \left|x_N\right|\right); \end{aligned} \tag{21}$$

(A2) *there exists a positive constant R_0 such that*

$$\begin{aligned} \phi(R_0, R_0, \ldots, R_0) &\le \frac{R_0}{k_2}, \\ \psi(R_0, R_0, \ldots, R_0) &\le \frac{R_0}{l_2}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} k_2 &= \max_{t \in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|b(s)\right| ds \right. \\ &\quad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}\left|b(s)\right| ds\right), \\ l_2 &= \max_{t \in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\left|d(s)\right| ds \right. \\ &\quad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}\left|d(s)\right| ds\right). \end{aligned} \tag{23}$$

Then the coupled system (1) *has a solution.*

Corollary 7. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(A^*1) *there exist nonnegative functions $a, c \in L^1[0,1]$ such that*

$$\begin{aligned} \left|f(t, x_0, x_1, \ldots, x_N)\right| &\le a(t), \\ \left|g(t, x_0, x_1, \ldots, x_N)\right| &\le c(t). \end{aligned} \tag{24}$$

Then the coupled system (1) *has a solution.*

Proof. In fact, let us choose $R_0 = \max\{k_1, l_1\}$, where

$$\begin{aligned} k_1 &= \max_{t \in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|a(s)\right| ds \right. \\ &\quad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}\left|a(s)\right| ds\right), \\ l_1 &= \max_{t \in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\left|c(s)\right| ds \right. \\ &\quad \left. + \sum_{j=1}^{N}\frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}\left|c(s)\right| ds\right), \end{aligned} \tag{25}$$

and construct a closed ball of Banach space $X \times Y$

$$B = \left\{(x, y) \in X \times Y : \left\|(x, y)\right\|_{X \times Y} \le R_0\right\}. \tag{26}$$

The rest of proof is similar to Theorem 5. □

Remark 8. The condition (A1) is weaker than (H1) and (H2). Clearly, $\phi(|x_0|, |x_1|, \ldots, |x_N|)$ and $\psi(|x_0|, |x_1|, \ldots, |x_N|)$ include $c_0|x_0|^{\gamma_0} + c_1|x_1|^{\gamma_1} + \cdots + c_N|x_N|^{\gamma_N}$ and $d_0|x_0|^{\theta_0} + d_1|x_1|^{\theta_1} + \cdots + d_N|x_N|^{\theta_N}$, $\theta_i, \gamma_i \neq 1$ as special cases. Moreover (A1) also includes the case $\theta_i = 1$ or/and $\gamma_i = 1$, but (H1) and (H2) do not be allowed.

Remark 9. In Corollary 7, for the special case $a, c \in C[0,1]$, clearly $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ are continuous and bounded. This leads to the Corollary 3.1 of [12]. Therefore, Corollary 3.1 of [12] is only a special case of Corollary 7.

In the following, we focus on the uniqueness of the solution of the system (1).

Theorem 10. *Let $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ be continuous. Assume that*

(B1) *there exist nonnegative functions $a, c \in L^1[0,1]$ and nonnegative nondecreasing functions ϕ, ψ with respect to each variable x_i, $i = 0, 1, 2, \ldots, N$, such that*

$$\begin{aligned} &\left|f(t, u_0, u_1, \ldots, u_N) - f(t, v_0, v_1, \ldots, v_N)\right| \\ &\quad \le a(t)\,\phi\left(\left|u_0 - v_0\right|, \left|u_1 - v_1\right|, \ldots, \left|u_N - v_N\right|\right), \\ &\left|g(t, u_0, u_1, \ldots, u_N) - g(t, v_0, v_1, \ldots, v_N)\right| \\ &\quad \le b(t)\,\psi\left(\left|u_0 - v_0\right|, \left|u_1 - v_1\right|, \ldots, \left|u_N - v_N\right|\right); \end{aligned} \tag{27}$$

(B2) *for any $s > 0$,*

$$\phi(s, s, \ldots, s) \le s, \qquad \psi(s, s, \ldots, s) \le s, \tag{28}$$

and $\max\{k_1^2, l_1^2\} < 1$, where

$$k_1 = \max_{t\in I}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\right),$$
$$l_1 = \max_{t\in I}\left(\frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}c(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\sigma-\beta_j)}\int_0^t (t-s)^{\sigma-\beta_j-1}c(s)\,ds\right). \tag{29}$$

Then coupled system (1) *has a unique solution.*

Proof. We prove that the operator $T : X\times Y \to X\times Y$ is contraction. To do this, let (x_1, y_1), $(x_2, y_2) \in X\times Y$; we have

$$\begin{aligned}
&\left|T_1x_2(t) - T_1x_1(t)\right| \\
&= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times f\left(s, y_2(s), \mathscr{D}^{\beta_1}y_2(s), \dots, \mathscr{D}^{\beta_N}y_2(s)\right)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \times f\left(s, y_1(s), \mathscr{D}^{\beta_1}y_1(s), \dots, \mathscr{D}^{\beta_N}y_1(s)\right)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times\phi\left(\left|y_2(s)-y_1(s)\right|, \left|\mathscr{D}^{\beta_1}y_2(s) - \mathscr{D}^{\beta_1}y_1(s)\right|, \dots, \left|\mathscr{D}^{\beta_N}y_2(s) - \mathscr{D}^{\beta_N}y_1(s)\right|\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times\phi\left(\left\|y_2-y_1\right\|, \left\|\mathscr{D}^{\beta_1}y_2 - \mathscr{D}^{\beta_1}y_1\right\|, \dots, \left\|\mathscr{D}^{\beta_N}y_2 - \mathscr{D}^{\beta_N}y_1\right\|\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s) \times\phi\left(\left\|y_2-y_1\right\|_Y, \left\|y_2-y_1\right\|_Y, \dots, \left\|y_2-y_1\right\|_Y\right)ds \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds\left\|y_2-y_1\right\|_Y,
\end{aligned}$$

$$\begin{aligned}
&\left|\mathscr{D}^{\rho_j}T_1x_2(t) - \mathscr{D}^{\rho_j}T_1x_1(t)\right| \\
&= \left|I^{\alpha-\rho_j}\left(f\left(t, y_2(t), \mathscr{D}^{\beta_1}y_2(t), \dots, \mathscr{D}^{\beta_N}y_2(t)\right) - f\left(t, y_1(t), \mathscr{D}^{\beta_1}y_1(t), \dots, \mathscr{D}^{\beta_N}y_1(t)\right)\right)\right| \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times\int_0^t (t-s)^{\alpha-\rho_j-1} \times\left|f\left(t, y_2(t), \mathscr{D}^{\beta_1}y_2(t), \dots, \mathscr{D}^{\beta_N}y_2(t)\right) - f\left(t, y_1(t), \mathscr{D}^{\beta_1}y_1(t), \dots, \mathscr{D}^{\beta_N}y_1(t)\right)\right|ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times\int_0^t (t-s)^{\alpha-\rho_j-1}a(s) \times\phi\left(\left\|y_2-y_1\right\|, \left\|\mathscr{D}^{\beta_1}y_2 - \mathscr{D}^{\beta_1}y_1\right\|, \dots, \left\|\mathscr{D}^{\beta_N}y_2 - \mathscr{D}^{\beta_N}y_1\right\|\right)ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)} \times\int_0^t (t-s)^{\alpha-\rho_j-1}a(s) \times\phi\left(\left\|y_2-y_1\right\|_Y, \left\|y_2-y_1\right\|_Y, \dots, \left\|y_2-y_1\right\|_Y\right)ds \\
&\le \frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\left\|y_2-y_1\right\|_Y.
\end{aligned} \tag{30}$$

Thus it follows from (30) and (B2) that

$$\begin{aligned}
&\left\|T_1x_2 - T_1x_1\right\|_X \\
&= \max_{t\in I}\left|T_1x_2(t) - T_1x_1(t)\right| + \sum_{j=1}^{N}\max_{t\in I}\left|\mathscr{D}^{\rho_j}\left(T_1x_2(t) - T_1x_1(t)\right)\right| \\
&\le \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}a(s)\,ds + \sum_{j=1}^{N}\frac{1}{\Gamma(\alpha-\rho_j)}\int_0^t (t-s)^{\alpha-\rho_j-1}a(s)\,ds\right)\left\|y_2-y_1\right\|_Y \\
&\le k_1\left\|y_2-y_1\right\|_Y.
\end{aligned} \tag{31}$$

Similarly, we can get

$$\begin{aligned}
&\left|T_1 y_2(t) - T_1 y_1(t)\right| \\
&\quad \le \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s)\, ds \left\|x_2 - x_1\right\|_X, \\
&\left|\mathscr{D}^{\beta_j} T_1 y_2(t) - \mathscr{D}^{\beta_j} T_1 y_1(t)\right| \\
&\quad \le \frac{1}{\Gamma\left(\sigma-\beta_j\right)} \int_0^t (t-s)^{\sigma-\beta_j-1} c(s)\, ds \left\|x_2 - x_1\right\|_X, \\
&\left\|T_1 y_2 - T_1 y_1\right\|_Y \\
&\quad = \max_{t\in I}\left|T_1 y_2(t) - T_1 y_1(t)\right| \\
&\qquad + \sum_{j=1}^{N} \max_{t\in I}\left|\mathscr{D}^{\rho_j}\left(T_1 y_2(t) - T_1 y_1(t)\right)\right| \\
&\quad \le \left(\frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} c(s)\, ds \right.\\
&\qquad \left. + \sum_{j=1}^{N} \frac{1}{\Gamma\left(\sigma-\beta_j\right)} \int_0^t (t-s)^{\sigma-\beta_j-1} c(s)\, ds \right) \left\|x_2 - x_1\right\|_X \\
&\quad \le l_1 \left\|x_2 - x_1\right\|_X.
\end{aligned} \tag{32}$$

Hence, for the Euclidean distance d on $\mathbb{R}^2$, we get

$$\begin{aligned}
&d\left(T\left(x_2, y_2\right), T\left(x_1, y_1\right)\right) \\
&\quad = \sqrt{\left\|T_1 x_2 - T_1 x_1\right\|_X^2 + \left\|T_1 y_2 - T_1 y_1\right\|_Y^2} \\
&\quad \le \sqrt{k_1^2 \left\|x_2 - x_1\right\|_X^2 + l_1^2 \left\|y_2 - T_1\right\|_Y^2} \\
&\quad \le \sqrt{\max\left\{k_1^2, l_1^2\right\}} \sqrt{\left\|x_2 - x_1\right\|_X^2 + \left\|y_2 - T_1\right\|_Y^2} \\
&\quad = \sqrt{\max\left\{k_1^2, l_1^2\right\}}\, d\left(\left(x_2, y_2\right), \left(x_1, y_1\right)\right).
\end{aligned} \tag{33}$$

Thus T is a contraction since $\sqrt{\max\{k_1^2, l_1^2\}} < 1$.

By Banach contraction principle, T has a unique fixed point, which is a solution of the coupled system (1). The proof is completed. □

An Example. Consider the existence of solutions for the following coupled system of multiterm nonlinear fractional differential equations:

$$\begin{aligned}
&-\mathscr{D}^{3.5} x(t) = \frac{t}{6.08} + \frac{1}{25.26}\left[y(t) + e^{(\mathscr{D}^{0.8} y(t))}\right], \\
&\qquad \mathscr{D}^{3.5} x(0) = 0, \quad i = 1, 2, \ldots, 4, \\
&-\mathscr{D}^{4.2} y(t) = \frac{10000}{5501}\left[t^{-1/2} x^{0.2}(t) + t^2 \left(\mathscr{D}^{0.5} x(t)\right)^{0.5}\right], \\
&\qquad \mathscr{D}^{4.2-j} y(0) = 0, \quad j = 1, 2, \ldots, 5,
\end{aligned} \tag{34}$$

where $t \in (0, 1]$.

Let

$$\begin{aligned}
f\left(t, x_0, x_1\right) &= \frac{t}{6.08} + \frac{1}{25.26}\left[x_0 + e^{x_1}\right], \\
g\left(t, x_0, x_1\right) &= t^{-1/2} x_0^{0.2} + t^2 x_1^{0.5},
\end{aligned} \tag{35}$$

and choose

$$\begin{aligned}
a(t) &= \frac{t}{6.08}, \qquad b(t) = \frac{1}{25.26}, \\
\phi\left(x_0, x_1\right) &= x_0 + e^{x_1}, \qquad c(t) = 0, \\
d(t) &= \frac{10000}{5501}\left[t^{-1/2} + t^2\right], \\
\psi\left(x_0, x_1\right) &= x_0^{0.2} + x_1^{0.5}.
\end{aligned} \tag{36}$$

Then

$$\begin{aligned}
f\left(t, x_0, x_1\right) &\le a(t) + b(t)\, \phi\left(x_0, x_1\right), \\
g\left(t, x_0, x_1\right) &\le c(t) + d(t)\, \psi\left(x_0, x_1\right);
\end{aligned} \tag{37}$$

consequently, (A1) holds.

In the following, we check the condition (A1). Since

$$\begin{aligned}
k_1 &= \max\left(\frac{1}{\Gamma(3.5)} \int_0^t \frac{(t-s)^{2.5} s}{6.08} ds + \frac{1}{\Gamma(3)} \int_0^t \frac{(t-s)^2 s}{6.08} ds\right) \\
&= 0.01, \\
k_2 &= \max\left(\frac{1}{\Gamma(3.5)} \int_0^t \frac{(t-s)^{2.5}}{25.26} ds + \frac{1}{\Gamma(3)} \int_0^t \frac{(t-s)^2}{25.26} ds\right) \\
&= 0.01, \\
l_1 &= 0, \\
l_2 &= \frac{10000}{5501} \\
&\quad \times \max\left(\frac{1}{\Gamma(4.2)} \int_0^t (t-s)^{3.2}\left(s^{-1/2} + s^2\right) ds \right.\\
&\qquad \left. + \frac{1}{\Gamma(3.4)} \int_0^t (t-s)^{2.4}\left(s^{-1/2} + s^2\right) ds\right) = 1,
\end{aligned} \tag{38}$$

take $R_0 = 5$; we have

$$\begin{aligned}
\phi\left(R_0, R_0\right) &= R_0 + e^{R_0} = 5 + e^5 \\
&= 153.44 < \frac{R_0 - k_1}{k_2} = \frac{5 - 0.01}{0.01} = 499, \\
\psi\left(R_0, R_0\right) &= R_0^{0.2} + R_0^{0.5} = 5^{0.2} + 5^{0.5} \\
&= 3.6158 < \frac{R_0 - l_1}{l_2} = 5,
\end{aligned} \tag{39}$$

which implies that (A2) is satisfied. Hence, by Theorem 5, the coupled system of fractional differential equation (34) has a solution.

Remark 11. In the coupled system of fractional differential equation (34), the nonlinear function f involves exponential growth, but the results of [12] are only allowed to be power growth; that is, (34) cannot be solved by using the results of [12]. So the results obtained in this paper give a significant improvement of the previous work in [12].

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