Research Article

A New Hybrid Projection Algorithm for System of Equilibrium Problems and Variational Inequality Problems and Two Finite Families of Quasi-ϕ-Nonexpansive Mappings

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We introduce a modified Mann's iterative procedure by using the hybrid projection method for solving the common solution of the system of equilibrium problems for a finite family of bifunctions satisfying certain condition, the common solution of fixed point problems for two finite families of quasi-ϕ-nonexpansive mappings, and the common solution of variational inequality problems for a finite family of continuous monotone mappings in a uniformly smooth and strictly convex real Banach space. Then, we prove a strong convergence theorem of the iterative procedure generated by some mild conditions. Our result presented in this paper improves and generalizes some well-known results in the literature.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive numbers, respectively. We also assume that E is a real Banach space and E^* is the dual space of E. Let C be a nonempty, closed and convex subset of a real Banach space E with the dual E^*. We recall the following definitions.

A mapping $A : C \to C$ is said to be *nonexpansive* if

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1}$$

A mapping $A : C \to E^*$ is said to be *monotone* if for each $x, y \in C$, such that

$$\langle x - y, Ax - Ay \rangle \geq 0. \tag{2}$$

A mapping $A : C \to E^*$ is said to be δ-*strongly monotone* if there exists a constant $\delta > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in C. \tag{3}$$

A mapping $A : C \to E^*$ is said to be δ-*inverse strongly monotone* if there exists a constant $\delta > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \delta \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{4}$$

If A is δ-inverse strongly monotone, then it is *Lipschitz continuous* with constant $1/\delta$, that is, for all $x, y \in C$, $\|Ax - Ay\| \leq (1/\delta)\|x - y\|$.

Clearly, the class of monotone mappings include the class of δ-inverse strongly monotone mappings.

Let $A : C \to E^*$ be a monotone mapping. The variational inequality problem is to find a point $z \in C$ such that

$$\langle y - z, Az \rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The set of the solution of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$.

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E, let $T : C \to C$ be a mapping, and $F(T)$ be the set of fixed points of T.

A point $x \in C$ is said to be a *fixed point* of T if $Tx = x$. The set of the solution of the fixed point of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

A point $p \in C$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all asymptotic fixed points of T by $\widehat{F}(T)$.

A point $p \in C$ is said to be a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \to p$ and $\|x_n - Tx_n\| \to 0$. We denoted the set of all strong asymptotic fixed points of T by $\widetilde{F}(T)$.

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find a point $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C. \tag{6}$$

The set of the solution of equilibrium problem is denoted by EP(F). Numerous problems in sciences, mathematics, optimizations, and economics reduced to find a solution of equilibrium problems. The equilibrium problems include variational inequality problems and fixed point problem, and optimization problems as special cases (see, e.g., [1–3]). Recently, many authors have considered the problem for finding the common solution of fixed point problems, the common solution of equilibrium problems, and the common solution of variational inequality problems.

In 1953, Mann [4] introduced the iterative sequence $\{x_n\}_{n \in \mathbb{N}}$ which is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{7}$$

where the initial element $x_0 \in C$ is arbitrary, T is a nonexpansive mapping, and $\{\alpha_n\}$ is the sequence in $[0, 1]$ such that $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The sequence of (7) is generally referred to as the *Mann iteration*.

In 2009, Takahashi and Zembayashi [5] introduced the following iterative scheme by the shrinking projection method, and they proved that $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to $q = \Pi_{\text{EP}(F) \cap F(S)}$, under appropriate conditions.

Theorem TZ. *Let E be a uniformly smooth and uniformly convex real Banach space, and let C be a nonempty, closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let T be a relatively nonexpansive mapping from C into itself such that $F(S) \cap \text{EP}(F) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$x_0 \in C_0 = C \ chosen \ arbitrary,$$

$$y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T x_n),$$

$$u_n \in C \ such \ that \ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0,$$

$$\forall y \in C,$$

$$C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0, \tag{8}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, the sequence $\alpha_n \subset [0, 1]$ satisfies $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, and $\{r_n\}_{n \in \mathbb{N}} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. Then, the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S) \cap \text{EP}(f)} x_0$, where $\Pi_{F(S) \cap \text{EP}(F)} x_0$ is the generalized projection of E onto $F(S) \cap \text{EP}(F)$.

In 2009, Qin et al. [6] extended the iterative process (8) from a single relatively nonexpansive mapping to two relatively quasi-nonexpansive mappings. In 2011, Zegeye and Shahzad [7] introduced an iterative process for finding an element in the common fixed point set of finite family of closed relatively quasi-nonexpansive mappings, common solutions of finite family of equilibrium problems, and common solutions of the finite family of variational inequality problems for monotone mappings in Banach spaces.

Theorem ZS. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E with the dual E^*. Let $f_k : C \times C \to \mathbb{R}$, $k = 1, 2, 3, \dots, L$, be a finite family of bifunctions. Let $S_j : C \to C$, $j = 1, 2, 3, \dots, D$ be a finite family relatively quasi-nonexpansive mappings and $A_i : C \to E^*$, $i = 1, 2, 3, \dots, N$ be a finite family of continuous monotone mappings.*

For $x \in E$, they define mappings $F_{r_n}, T_{r_n} : E \to C$ by

$$F_{r_n} x := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\},$$

$$T_{r_n} x := \left\{ z \in C : f_n(z, y) \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}, \tag{9}$$

where $S_n \equiv S_n(\text{mod } D)$, $A_n \equiv A_n(\text{mod } N)$, $f_n(\cdot, \cdot) \equiv f_n(\text{mod } L)(\cdot, \cdot)$, and $\{r_n\}_{n \in \mathbb{N}} \subset [c_1, \infty)$ for some $c_1 > 0$. Assume that $\mathcal{F} := (\bigcap_{j=1}^{D} F(S_j)) \bigcap (\bigcap_{i=1}^{N} VI(C, A_i)) \bigcap (\bigcap_{k=1}^{L} \text{EP}(f_k)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C \ chosen \ arbitrary,$$

$$z_n = F_{r_n} x_n,$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1}(\alpha_0 J x_n + \alpha_1 J z_n + \alpha_2 J S_n u_n), \tag{10}$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

where the real numbers $\alpha_0, \alpha_1, \alpha_2 \in (0, 1)$ such that $\alpha_0 + \alpha_1 + \alpha_2 = 1$. Then, $\{x_n\}_{n \in \mathbb{N}}$ converges strongly to an element of $\mathcal{F}$.

In this paper, motivated and inspired by the previously mentioned above results, we introduce a new iterative procedure for solving the common solution of system of equilibrium problems for a finite family of bifunctions satisfying certain conditions and the common solution of fixed point

problems for two countable families of quasi-ϕ-nonexpansive mappings and the common solution of variational inequality problems for a finite family of monotone mappings in a uniformly smooth and strictly convex real Banach space. Then, we prove a strong convergence theorem of the iterative procedure generated by the conditions. The results obtained in this paper extend and improve several recent results in this area.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ and

$$\lim_{n\to\infty}\left\|\frac{x_n + y_n}{2}\right\| = 1. \tag{11}$$

Let $U_E = \{x \in E : \|x\| = 1\}$ be the *unit sphere* of E. Then the Banach space E is said to be *smooth* if

$$\lim_{t\to 0}\frac{\|x + ty\| - \|x\|}{t} \tag{12}$$

exists for each $x, y \in U_E$. It is said to be *uniformly smooth* if the limit (12) is attained uniformly for all $x, y \in U_E$.

Let E be a Banach space. Then a function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be the *modulus of smoothness* of E if

$$\rho_E(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = t\right\}. \tag{13}$$

The space E is said to be *smooth* if $\rho_E(t) > 0, \forall t > 0$ and is said to be *uniformly smooth* if and only if $\lim_{t\to 0^+}\rho_E(t)/t = 0$.

The *modulus of convexity* of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \left\|\frac{x + y}{2}\right\| : \|x\| \leq 1, \|y\| \leq 1; \|x - y\| \geq \epsilon\right\}. \tag{14}$$

A Banach space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

We recall the following definitions.

Definition 1. Let C be a nonempty set.

(1) A mapping $T : C \to C$ is said to be *closed* if for each $\{x_n\} \subset C$, $x_n \to x$ and $Tx_n \to y$ imply $Tx = y$.

(2) A mapping $T : C \to C$ is said to be *quasi-ϕ-nonexpansive* (*relatively quasi-nonexpansive*) if $F(T) \neq \emptyset$, and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T). \tag{15}$$

(3) A mapping $T : C \to C$ is said to be *relatively nonexpansive* [8, 9] if $F(T) \neq \emptyset$, $F(T) = \widehat{F}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T). \tag{16}$$

(4) A mapping $T : C \to C$ is said to be *weak relatively nonexpansive* [10] if $F(T) \neq \emptyset$, $F(T) = \widetilde{F}(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \ p \in F(T). \tag{17}$$

Remark 2. We here the following basic properties.

(1) Each relatively nonexpansive mapping is closed.

(2) The class of quasi-ϕ-nonexpansive mappings contains properly the class of weak relatively nonexpansive mappings as a subclass, but the converse may be not true.

(3) The class of weak relatively nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

(4) The class of quasi-ϕ-nonexpansive mappings contains properly the class of relatively nonexpansive mappings as a subclass, but the converse may be not true.

(5) If E is a real uniformly smooth Banach space, then J is uniformly continuous on each bounded subset of E.

(6) If E is a strictly convex reflexive Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-to-weak* continuous.

(7) If E is a smooth and strictly convex reflexive Banach space, then J is single-valued, one-to-one, and onto.

(8) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.

(9) Each uniformly convex Banach space E has the *Kadec-Klee property*, that is, for any sequence $\{x_n\} \subset E$, if $\{x_n\} \to x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.

(10) A Banach space E is strictly convex if and only if J is strictly monotone, that is,

$$\langle x - y, x^* - y^* \rangle > 0, \quad \text{whenever } x, y \in E,$$
$$x \neq y, \ x^* \in Jx, \ y^* \in Jy. \tag{18}$$

(11) Both uniformly smooth Banach spaces and uniformly convex Banach spaces are reflexive.

(12) E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a real Banach space and $\{x_n\}$ be a sequence in E. We denote by $x_n \to x$ and $x_n \rightharpoonup x$ the strong convergence and weak convergence of $\{x_n\}$, respectively. The *normalized duality mapping* J from E to 2^{E^*} is defined by

$$Jx = \left\{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\right\}, \quad \forall x \in E, \tag{19}$$

where $\langle\cdot, \cdot\rangle$ denotes the duality pairing. It is well known that if E is smooth, then J is single-valued and demicontinuous, and if E is uniformly smooth, then J is uniformly continuous on bounded subset of E. Moreover, if E is reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single-valued, one-to-one, surjective, and it is the duality

mapping from E^* to E and so $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [11, 12]). We note that in a Hilbert space H, the mapping J is the identity operator.

Now, let E be a smooth and strictly convex reflexive Banach space. As Alber (see [13]) and Kamimura and Takahashi (see [14]) did, the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \quad (20)$$

It follows from Kohsaka and Takahashi (see [15]) that $\phi(x, y) = 0$ if and only if $x = y$ and that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2. \quad (21)$$

Further suppose that C is nonempty, closed and convex subset of E. The *generalized, projection* (Alber see [13]) $\Pi_C : E \to C$ is defined by for each $x \in E$,

$$\Pi_C(x) = \arg\min_{y \in C} \phi(x, y). \quad (22)$$

Remark 3. If E is a real Hilbert space H, then $\phi(x, y) = \|x - y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

Lemma 4 (Alber [13]). *Let C be a nonempty, closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$. Then*

$$\phi(x, \Pi_C(y)) + \phi(\Pi_C(y), y) \\ \le \phi(x, y), \quad \forall x \in C, \ \forall y \in E. \quad (23)$$

Lemma 5 (Kamimura and Takahashi [14]). *Let C be a nonempty, closed and convex subset of a smooth and strictly convex reflexive Banach space E, and let $x \in E$ and $\widehat{p} \in C$. Then,*

$$\widehat{p} = \Pi_C(x) \iff \langle \widehat{p} - y, Jx - J\widehat{p} \rangle \ge 0, \quad \forall y \in C. \quad (24)$$

Lemma 6 (Qin et al. [6] and Kohsaka and Takahashi [16]). *Let E be a smooth, strictly convex and reflexive Banach space, and $A \subset E \times E^*$ is a continuous monotone mapping with $A^{-1}(0) \neq \phi$. Then, it is proved in [16] that the resolvent $J_r := (J + rA)^{-1}J$, for $r > 0$ is quasi-ϕ-nonexpansive. Moreover, if $T : E \to E$ is quasi-ϕ-nonexpansive, then using the definition of ϕ one can show that $F(T)$ is closed and convex (see [6]).*

Lemma 7 (Kamimura and Takahashi [14]). *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 8. *Let E be a uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let $\{x_n\}$ and $\{y_n\}$ be two sequences of E, and $p \in E$. If $x_n \to p$ and $\phi(x_n, y_n) \to 0$, then $y_n \to p$.*

For solving the equilibrium problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$, for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$, $\forall x, y \in C$;

(A3) $\lim_{t \to 0} F(tz + (1 - t)x, y) \le F(x, y)$, $\forall x, y, z \in C$;

(A4) for any $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 9 (Blum and Oettli [1]). *Let C be a nonempty, closed and convex subset of a smooth and strictly convex reflexive Banach space E and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions (A1)–(A4). Let $r > 0$ be any given number and $x \in E$ be any point. Then, there exists a point $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \quad \forall y \in C. \quad (25)$$

Lemma 10 (Takahashi and Zembayashi [5]). *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E, and $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions (A1)–(A4). Let $r > 0$ be any given number and $x \in E$ be any point defined a mapping $T_r : E \to C$ as follows. Then, there exists a point $z \in C$ such that*

$$T_r(x) = \left\{ z \in C : F(z, y) \\ + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \ \forall y \in C \right\}. \quad (26)$$

Then, the following conclusions hold:

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive type mapping, that is,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \\ \le \langle T_r x - T_r y, Jx - Jy \rangle, \quad \forall x, y \in E; \quad (27)$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is a closed and convex subset of C;*

(5) $\phi(p, T_r x) + \phi(T_r x, x) \le \phi(p, x)$, $\forall p \in F(T_r)$.

Lemma 11 (Zegeye and Shahzad [7]). *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E, and $A : C \times E^* \to \mathbb{R}$ be a continuous monotone mapping. Let $r > 0$ be any given number and $x \in E$ be any point defined a mapping $K_r : E \to C$ as follows:*

$$K_r(x) = \left\{ z \in C : \langle y - z, Az \rangle \\ + \frac{1}{r}\langle y - z, Jz - Jx \rangle \ge 0, \ \forall y \in C \right\}. \quad (28)$$

Then, the following conclusions hold:

(1) K_r *is single valued;*

(2) $F(K_r)$ *is a firmly nonexpansive type mapping, that is,*

$$
\begin{aligned}
&\langle K_r x - K_r y, J K_r x - J K_r y \rangle \\
&\quad \leq \langle K_r x - K_r y, Jx - Jy \rangle, \quad \forall x, y \in E;
\end{aligned}
\tag{29}
$$

(3) $F(K_r) = \mathrm{VI}(C, A)$;

(4) $\mathrm{VI}(C, A)$ *is a closed and convex subset of C;*

(5) $\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x), \forall p \in F(K_r)$.

Lemma 12 (Xu [17]). *Let E be a uniformly convex Banach space, let $B_r(0)$ be a closed ball of E, where $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$
\|\alpha x + \beta y\|^2 \leq \alpha \|x\|^2 + \beta \|y\|^2 - \alpha \beta g(\|x - y\|),
\tag{30}
$$

for all $x, y \in B_r(0)$ and all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$.

3. Main Results

In this section, we prove a strong convergence theorem which solves the problem for finding a common solution of the system of equilibrium problems and variational inequality problems and fixed point problems in Banach spaces.

Theorem 13. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) $\{F_k\}_{k=1}^M : C \times C \to \mathbb{R}$ *is a finite family of bifunctions satisfying conditions (A1)–(A4), where $k = 1, 2, 3, \dots, M$;*

(2) $\{T_i\}_{i=1}^D$ *and $\{S_i\}_{i=1}^D$ are two finite families of quasi-ϕ-nonexpansive mappings from C into E, where $i = 1, 2, 3, \dots, D$;*

(3) $\{A_n\}_{n=1}^N : C \to E^*$ *is a finite family of continuous monotone mappings, where $n = 1, 2, 3, \dots, N$;*

(4) *For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : E \to C$ by*

$$
\begin{aligned}
K_{r_n}(x) : &= \Big\{ z \in C : \langle y - z, A_n z \rangle \\
&\quad + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \Big\}, \\
T_{r_n}(x) : &= \Big\{ z \in C : F_n(z, y) \\
&\quad + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \Big\}.
\end{aligned}
\tag{31}
$$

Assume that $\Omega := (\bigcap_{i=1}^D F(T_i)) \bigcap (\bigcap_{i=1}^D F(S_i)) \bigcap (\bigcap_{k=1}^M \mathrm{SEP}(F_k)) \bigcap (\bigcap_{n=1}^N \mathrm{VI}(C, A_n))$ is a nonempty and bounded in C. Let $\{x_n\}$ be a sequence generated by

$$
\begin{aligned}
x_0 &\in C_0 = C, \ \text{chosen arbitrary}, \\
z_n &= K_{r_n} x_n, \\
u_n &= T_{r_n} x_n, \\
y_n &= J^{-1} (\alpha Jx_n + \beta JT_n z_n + \gamma JS_n u_n), \\
C_{n+1} &= \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) \}, \\
x_{n+1} &= \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,
\end{aligned}
\tag{32}
$$

where $A_n \equiv A_n \pmod{N}$, $F_n \equiv F_n \pmod{M}$, $T_n \equiv T_n \pmod{D}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$ and α, β, γ are real numbers in $(0, 1)$ such that $\alpha + \beta + \gamma = 1$. Then, the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to $\hat{p} = \Pi_\Omega(x_0)$.

Proof. We will complete this proof by seven steps below.

Step 1. We will show that C_n is closed and convex for each $n \geq 0$.
From the definition of C_n, it is obvious that C_n is closed. Moreover, since

$$
\begin{aligned}
&\phi(z, y_n) \leq \phi(z, x_n) \ \text{is equivalent to} \\
&2 \langle z, Jx_n - Jy_n \rangle - \|x_n\|^2 + \|y_n\|^2 \leq 0.
\end{aligned}
\tag{33}
$$

It follows that C_n is convex for each $n \geq 0$. Therefore, C_n is closed and convex for each $n \geq 0$.

Step 2. We will show that $\Omega \subset C_n$ for each $n \geq 0$.
From the assumption, we see that $F \subset C_0 = C$. Suppose that $F \subset C_k$ for some $k \geq 1$. Now, for $p \in \Omega$, Since S_k and T_k are quasi-ϕ-nonexpansive and by Lemmas 10 and 11, we compute

$$
\begin{aligned}
\phi(p, y_k) &= \phi(p, J^{-1}(\alpha Jx_k + \beta JT_k z_k + \gamma JS_k u_k)) \\
&= \|p\|^2 - 2 \langle p, \alpha Jx_k + \beta JT_k z_k + \gamma JS_k u_k \rangle \\
&\quad + \|\alpha Jx_k + \beta JT_k z_k + \gamma JS_k u_k\|^2 \\
&\leq \|p\|^2 - 2\alpha \langle p, Jx_k \rangle - 2\beta \langle p, JT_k z_k \rangle \\
&\quad - 2\gamma \langle p, JS_k u_k \rangle + \alpha \|Jx_k\|^2 + \beta \|JT_k z_k\|^2 \\
&\quad + \gamma \|JS_k u_k\|^2 \\
&= \alpha \phi(p, x_k) + \beta \phi(p, T_k z_k) + \gamma \phi(p, S_k u_k) \\
&\leq \alpha \phi(p, x_k) + \beta \phi(p, z_k) + \gamma \phi(p, u_k) \\
&= \alpha \phi(p, x_k) + \beta \phi(p, K_{r_k} x_k) + \gamma \phi(p, T_{r_k} x_k) \\
&\leq \alpha \phi(p, x_k) + \beta \phi(p, x_k) + \gamma \phi(p, x_k) \\
&= (\alpha + \beta + \gamma) \phi(p, x_k) \\
&= \phi(p, x_k).
\end{aligned}
\tag{34}
$$

Therefore, $p \in C_{k+1}$. By the induction, this implies that $\Omega \subset C_n$ for each $n \geq 0$.

Hence, the sequence $\{x_n\}$ is well defined.

Step 3. We will show that the sequence $\{x_n\}$ is bounded.

Let $G := \bigcap_{n=0}^{\infty} C_n$. From $\Omega \subset C_n$ for each $n \geq 0$ and $\{x_n\}$ is well defined. From the assumption of G, we see that G is closed and convex subset of C.

Let $\hat{p} = \Pi_G(x_0)$, where $\hat{p}$ is the unique element that satisfies $\inf_{x \in G} \phi(x, x_0) = \phi(\hat{p}, x_0)$. Now, we will show that $\|x_n - \hat{p}\| \to 0$ as $n \to \infty$.

From the assumption of C_n, we know that $C \supset C_1 \supset C_2 \supset C_3 \supset \cdots$ and since $x_n = \Pi_{C_n}(x_0)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$. Using Lemma 4, we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \cdots \leq \phi(\hat{p}, x_0), \qquad (35)$$

for all $\hat{p} \in \Omega \subset C_n$, where $n \geq 1$. Then, the sequence $\{\phi(x_n, x_0)\}$ is bounded. Hence, the sequence $\{x_n\}$ is also bounded.

Step 4. We will show that there exists $\hat{p} \in C$ such that $x_n \to \hat{p}$, as $n \to \infty$.

Since $x_n = \Pi_{C_n}(x_0)$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \qquad (36)$$

Therefore, the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing. Hence $\lim_{n \to \infty} \phi(x_n, x_0)$ exists. By the definition of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$.

It follows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0)$$
$$\leq \phi(x_m, x_0) - \phi(\Pi_{C_n} x_0, x_0) \qquad (37)$$
$$= \phi(x_m, x_0) - \phi(x_n, x_0).$$

Since $\lim_{n \to \infty} \phi(x_n, x_0)$ exists, by taking $m, n \to \infty$ in (37), we have $\phi(x_m, x_n) \to 0$.

From Lemma 7, it follows that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence.

Without a loss of generalization, we can assume that $x_n \rightharpoonup p_0 \in E$. Since $\{x_n\}$ is bounded and E is reflexive. Since $G := \bigcap_{n=0}^{\infty} C_n$ is closed and convex, it follows that $p_0 \in G$, $\forall n \geq 0$. Moreover, by using the weak lower semicontinuous of the norm on E and (35), we obtain

$$\phi(\hat{p}, x_0) \leq \phi(p_0, x_0)$$
$$\leq \liminf_{n \to \infty} \phi(x_n, x_0)$$
$$\leq \limsup_{n \to \infty} \phi(x_n, x_0) \qquad (38)$$
$$\leq \inf_{x \in G} \phi(x, x_0)$$
$$= \phi(\hat{p}, x_0).$$

This implies that

$$\lim_{n \to \infty} \phi(x_n, x_0) = \phi(\hat{p}, x_0) = \phi(p_0, x_0) = \inf_{x \in G} \phi(x, x_0). \qquad (39)$$

By using Lemma 5, we have $\langle \hat{p} - p_0, J\hat{p} - Jp_0 \rangle = 0$ and hence, $\hat{p} = p_0$. By the definition of ϕ, we get

$$\liminf_{n \to \infty} \phi(x_n, x_0) = \liminf_{n \to \infty} \left(\|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 \right)$$
$$\geq \|\hat{p}\|^2 - 2\langle \hat{p}, Jx_0 \rangle + \|x_0\|^2$$
$$= \phi(\hat{p}, x_0). \qquad (40)$$

Therefore, $\|x_n\| \to \|\hat{p}\|$. Since $x_n \rightharpoonup \hat{p}$, by the Kadec-Klee property of E, we obtain

$$\lim_{n \to \infty} x_n = \hat{p}. \qquad (41)$$

From J is uniformly continuous, we also have

$$\lim_{n \to \infty} Jx_n = J\hat{p}. \qquad (42)$$

Step 5. We will show that $y_n \to \hat{p}$, $z_n \to \hat{p}$ and $u_n \to \hat{p}$ as $n \to \infty$.

Since $x_{n+1} \in C_{n+1}$, we have $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \to 0$, as $n \to \infty$. Thus, from (21), we obtain

$$\|y_n\| \longrightarrow \|\hat{p}\|, \quad \text{as } n \longrightarrow \infty, \qquad (43)$$

and so

$$\|Jy_n\| \longrightarrow \|J\hat{p}\|, \quad \text{as } n \longrightarrow \infty. \qquad (44)$$

This implies that $\{Jy_n\}$ is bounded. Note that reflexivity of E implies reflexivity of E^*. Thus, we assume that $Jy_n \rightharpoonup y \in E^*$. Furthermore, the reflexivity of E implies there exists $x \in E$ such that $y = Jx$. Then, it follows that

$$\phi(x_{n+1}, y_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n \rangle + \|y_n\|^2$$
$$= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n \rangle + \|Jy_n\|^2. \qquad (45)$$

Taking the $\liminf_{n \to \infty}$ on both sides of (45) and using weak lower semicontinuous of norm to get that

$$0 \geq \|\hat{p}\|^2 - 2\langle \hat{p}, y \rangle + \|y\|^2$$
$$= \|\hat{p}\|^2 - 2\langle \hat{p}, Jx \rangle + \|Jx\|^2$$
$$= \|\hat{p}\|^2 - 2\langle \hat{p}, Jx \rangle + \|x\|^2 \qquad (46)$$
$$= \phi(\hat{p}, x).$$

Thus $\hat{p} = x$, and so $y = Jx = J\hat{p}$. It follows that $Jy_n \rightharpoonup J\hat{p}$. Now, from (44) and the Kadec-Klee property of E^*, we obtain

$$Jy_n \longrightarrow J\hat{p}, \quad \text{as } n \longrightarrow \infty. \qquad (47)$$

Thus the demicontinuity of J^{-1} implies that $y_n \rightarrow \widehat{p}$. Now, from (43) and the fact that E has the Kadec-Klee property, we obtain

$$\lim_{n \to \infty} y_n = \widehat{p}. \tag{48}$$

In the fact that $x_n \rightarrow \widehat{p}$ and $y_n \rightarrow \widehat{p}$ as $n \rightarrow \infty$, we get

$$\lim_{n \to \infty} \phi(p, z_n) = \phi(p, \widehat{p}). \tag{49}$$

Since $z_n = K_{r_n} x_n$, it follows from Lemma 11 that

$$\phi(z_n, x_n) = \phi\left(K_{r_n} x_n, x_n\right)$$
$$\leq \phi(p, x_n) - \phi\left(p, K_{r_n} x_n\right) \tag{50}$$
$$\leq \phi(p, x_n) - \phi(p, z_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

From (21), we obtain

$$\|z_n\| \longrightarrow \|\widehat{p}\|, \tag{51}$$

and so $\{z_n\}$ is bounded. Since E is reflexive, we assume that $z_n \rightharpoonup z \in E$. It follows that

$$\phi(z_n, x_n) = \|z_n\|^2 - 2\langle z_n, Jx_n \rangle + \|Jx_n\|^2. \tag{52}$$

Taking the $\liminf_{n \to \infty}$ on both sides of (52) and using the continuity of J, we get that

$$0 \geq \|z\|^2 - 2\langle z, J\widehat{p} \rangle + \|J\widehat{p}\|^2$$
$$= \phi(z, \widehat{p}). \tag{53}$$

This implies that $\widehat{p} = z$ and hence $z_n \rightharpoonup \widehat{p}$.

Now, from (51) and the Kadec-Klee property of E, we obtain

$$\lim_{n \to \infty} z_n = \widehat{p}. \tag{54}$$

In the fact that $x_n \rightarrow \widehat{p}$, $y_n \rightarrow \widehat{p}$ and $z_n \rightarrow \widehat{p}$ as $n \rightarrow \infty$, we get

$$\lim_{n \to \infty} \phi(p, u_n) = \phi(p, \widehat{p}). \tag{55}$$

Since $u_n = T_{r_n} x_n$, it follows from Lemma 10 that

$$\phi(u_n, x_n) = \phi\left(T_{r_n} x_n, x_n\right)$$
$$\leq \phi(p, x_n) - \phi\left(p, T_{r_n} x_n\right) \tag{56}$$
$$\leq \phi(p, x_n) - \phi(p, u_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

From (21), we obtain

$$\|u_n\| \longrightarrow \|\widehat{p}\|, \tag{57}$$

and so $\{u_n\}$ is bounded. Since E is reflexive, we assume that $u_n \rightharpoonup u \in E$. It follows that

$$\phi(u_n, x_n) = \|u_n\|^2 - 2\langle u_n, Jx_n \rangle + \|Jx_n\|^2. \tag{58}$$

Taking the $\liminf_{n \to \infty}$ on both sides of (58) and using the continuity of J, we get that

$$0 \geq \|u\|^2 - 2\langle u, J\widehat{p} \rangle + \|J\widehat{p}\|^2$$
$$= \phi(u, \widehat{p}). \tag{59}$$

This implies that $\widehat{p} = u$ and hence $u_n \rightharpoonup \widehat{p}$.

Now, from (57) and the Kadec-Klee property of E, we obtain

$$\lim_{n \to \infty} u_n = \widehat{p}. \tag{60}$$

Step 6. We will show that $\widehat{p} \in \Omega$.

Substep 1. We will show that $\widehat{p} \in \bigcap_{n=1}^{N} \text{VI}(C, A_n)$.

From the definition of K_{r_n} of algorithm (32), we have

$$\langle y - z_n, A_n z_n \rangle + \frac{1}{r_n} \langle y - z_n, Jz_n - Jx_n \rangle \geq 0, \quad \forall y \in C. \tag{61}$$

Let $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be such that $A_{n_k} = A_1$, for all $k \in \mathbb{N}$. Then from (61), we obtain

$$\langle y - z_{n_k}, A_1 z_{n_k} \rangle + \frac{1}{r_{n_k}} \langle y - z_{n_k}, Jz_{n_k} - Jx_{n_k} \rangle \geq 0, \quad \forall y \in C, \tag{62}$$

and that is

$$\left\langle y - z_{n_k}, A_1 z_{n_k} \right\rangle + \left\langle y - z_{n_k}, \frac{Jz_{n_k} - Jx_{n_k}}{r_{n_k}} \right\rangle \geq 0, \quad \forall y \in C. \tag{63}$$

Now, we set $v_t = tv + (1 - t)\widehat{p}$, for all $t \in (0, 1]$ and $v \in C$.

Therefore, we get $v_t \in C$. From (63), it follows that

$$\left\langle v_t - z_{n_k}, A_1 v_t \right\rangle \geq \left\langle v_t - z_{n_k}, A_1 v_t \right\rangle - \left\langle v_t - z_{n_k}, A_1 z_{n_k} \right\rangle$$
$$- \left\langle v_t - z_{n_k}, \frac{Jz_{n_k} - Jx_{n_k}}{r_{n_k}} \right\rangle$$
$$= \left\langle v_t - z_{n_k}, A_1 v_t - A_1 z_{n_k} \right\rangle$$
$$- \left\langle v_t - z_{n_k}, \frac{Jz_{n_k} - Jx_{n_k}}{r_{n_k}} \right\rangle. \tag{64}$$

From the continuity of J and (41) and (54), we have $x_{n_k} \rightarrow \widehat{p}$, $z_{n_k} \rightarrow \widehat{p}$, as $k \rightarrow \infty$, we obtain

$$\frac{Jz_{n_k} - Jx_{n_k}}{r_{n_k}} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{65}$$

Since A_1 is a monotone mapping, we also have $\langle v_t - z_{n_k}, A_1 v_t - A_1 z_{n_k} \rangle \geq 0$.

Thus, it follows that

$$0 \leq \lim_{n \to \infty} \left\langle v_t - z_{n_k}, A_1 v_t \right\rangle$$
$$= \left\langle v_t - \widehat{p}, A_1 v_t \right\rangle, \tag{66}$$

and hence

$$\langle y - \hat{p}, A_1 v_t \rangle \geq 0, \quad \forall y \in C. \tag{67}$$

If $t \to 0$, we obtain

$$\langle y - \hat{p}, A_1 \hat{p} \rangle \geq 0, \quad \forall y \in C. \tag{68}$$

This implies that $\hat{p} \in \mathrm{VI}(C, A_1)$.

Similarly, let $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be such that $A_{n_k} = A_2$, for all $k \in \mathbb{N}$.

Then, we have again that $\hat{p} \in \mathrm{VI}(C, A_2)$.

Continuing in the same way, we obtain that $\hat{p} \in \mathrm{VI}(C, A_n)$, where $n = 3, 4, 5, \ldots, N$.

Hence, $\hat{p} \in \bigcap_{n=1}^{N} \mathrm{VI}(C, A_n)$.

Substep 2. We will show that $\hat{p} \in \bigcap_{k=1}^{M} \mathrm{SEP}(F_k)$.

From the definition of T_{r_n} of algorithm (32) and (A2), we have

$$\frac{1}{r_n} \langle y - u_n, J u_n - J x_n \rangle \geq -F_n(u_n, y) \geq F(y, u_n), \quad \forall y \in C. \tag{69}$$

Let $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be such that $F_{n_k} = F_1$, for all $k \in \mathbb{N}$. Then from (69), we obtain

$$\frac{1}{r_{n_k}} \langle y - u_{n_k}, J u_{n_k} - J x_{n_k} \rangle = \left\langle y - u_{n_k}, \frac{J u_{n_k} - J x_{n_k}}{r_{n_k}} \right\rangle \tag{70}$$
$$\geq F_1(y, u_{n_k}), \quad \forall y \in C.$$

From the continuity of J and (41) and (60), we have $x_{n_k} \to \hat{p}$, $u_{n_k} \to \hat{p}$, as $k \to \infty$ and we obtain

$$\frac{J u_{n_k} - J x_{n_k}}{r_{n_k}} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{71}$$

Therefore, $F_1(y, u_{n_k}) \leq 0, \ \forall y \in C$.

Now, we set $w_t = tw + (1-t)\hat{p}$, for all $t \in (0, 1]$ and $w \in C$. Consequently, we get $w_t \in C$. And so $F_1(w_t, \hat{p}) \leq 0$. Therefore, from (A1), we obtain

$$0 = F_1(w_t, w_t)$$
$$\leq tF_1(w_t, y) + (1-t)F_1(w_t, \hat{p}) \tag{72}$$
$$\leq tF_1(w_t, y).$$

Thus, $F_1(w_t, y) \geq 0, \forall y \in C$. From (A3), if $t \to 0$, then we get $F_1(\hat{p}, y) \geq 0, \forall y \in C$. This implies that $\hat{p} \in \mathrm{EP}(F_1)$.

Similarly, let $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ be such that $F_{n_k} = F_2$, for all $k \in \mathbb{N}$. Then, we have again that $\hat{p} \in \mathrm{EP}(F_2)$.

Continuing in the same way, we obtain that $\hat{p} \in \mathrm{EP}(F_k)$, where $k = 3, 4, 5, \ldots, M$. Hence, $\hat{p} \in \bigcap_{k=1}^{M} \mathrm{SEP}(F_k)$.

Substep 3. We will show that $\hat{p} \in \bigcap_{i=1}^{D} F(S_i)$.

From algorithm (32) and Lemma 12, we compute

$$\phi(p, y_n) = \phi\left(p, J^{-1}\left(\alpha J x_n + \beta J T_n z_n + \gamma J S_n u_n\right)\right)$$
$$= \|p\|^2 - 2\langle p, \alpha J x_n + \beta J T_n z_n + \gamma J S_n u_n \rangle$$
$$\quad + \|\alpha J x_n + \beta J T_n z_n + \gamma J S_n u_n\|^2$$
$$\leq \|p\|^2 - 2\alpha \langle p, J x_n \rangle - 2\beta \langle p, J T_n z_n \rangle$$
$$\quad - 2\gamma \langle p, J S_n u_n \rangle + \alpha \|x_n\|^2 + \beta \|T_n z_n\|^2$$
$$\quad + \gamma \|S_n u_n\|^2 - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right)$$
$$\quad - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$= \alpha\phi(p, x_n) + \beta\phi(p, T_n z_n) + \gamma\phi(p, S_n u_n)$$
$$\quad - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right) - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$\leq \alpha\phi(p, x_n) + \beta\phi(p, z_n) + \gamma\phi(p, u_n)$$
$$\quad - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right) - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$= \alpha\phi(p, x_n) + \beta\phi\left(p, K_{r_n} x_n\right) + \gamma\phi\left(p, T_{r_n} x_n\right)$$
$$\quad - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right) - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$\leq \alpha\phi(p, x_n) + \beta\phi(p, x_n) + \gamma\phi(p, x_n)$$
$$\quad - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right) - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$= (\alpha + \beta + \gamma)\phi(p, x_n)$$
$$\quad - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right)$$
$$\quad - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$= \phi(p, x_n) - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right)$$
$$\quad - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$\leq \phi(p, x_n). \tag{73}$$

From (73), we have

$$\phi(p, y_n) \leq \phi(p, x_n) - \alpha\beta g\left(\|J x_n - J T_n z_n\|\right)$$
$$\quad - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right) \tag{74}$$
$$\leq \phi(p, x_n) - \alpha\gamma g\left(\|J x_n - J S_n u_n\|\right).$$

From $x_n \to \hat{p}$, $y_n \to \hat{p}$, and $\alpha, \gamma > 0$, we obtain

$$\alpha\gamma g\left(\|J x_n - J S_n u_n\|\right)$$
$$\leq \phi(p, x_n) - \phi(p, y_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{75}$$

Therefore,

$$g\left(\|J x_n - J S_n u_n\|\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{76}$$

It follows from the property of g that

$$\|Jx_n - JS_n u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{77}$$

From (42), we have $Jx_n \to J\widehat{p}$, as $n \to \infty$. Then,

$$JS_n u_n \longrightarrow J\widehat{p}, \quad \text{as } n \longrightarrow \infty, \tag{78}$$

and so

$$\|S_n u_n\| \longrightarrow \|\widehat{p}\|, \quad \text{as } n \longrightarrow \infty. \tag{79}$$

Moreover, the demicontinuity of J^{-1} implies that $S_n u_n \to \widehat{p}$ as $n \to \infty$. Thus, the Kadec-Klee property of E, we obtain

$$S_n u_n \longrightarrow \widehat{p}, \quad \text{as } n \longrightarrow \infty. \tag{80}$$

Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that $S_{n_k} = S_1$, for all $k \in \mathbb{N}$.

Then, from (60), we have $u_{n_k} \to \widehat{p}$, as $n \to \infty$. It follows from (80) and the closedness of S_1 that

$$\widehat{p} = \lim_{k\to\infty} S_{n_k} u_{n_k} = \lim_{k\to\infty} S_1 u_{n_k} = S_1 \widehat{p}. \tag{81}$$

This implies that $\widehat{p} \in F(S_1)$.

Similarly, let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that $S_{n_k} = S_2$, for all $k \in \mathbb{N}$. Then, we have again that $\widehat{p} \in F(S_2)$.

Continuing in the same way, we obtain that $\widehat{p} \in F(S_i)$, where $i = 3, 4, 5, \ldots, D$.

Hence, $\widehat{p} \in \bigcap_{i=1}^{D} F(S_i)$.

Substep 4. We will show that $\widehat{p} \in \bigcap_{i=1}^{D} F(T_i)$.

From (73), we have

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, x_n) - \alpha\beta g\left(\|Jx_n - JT_n z_n\|\right) \\
&\quad - \alpha\gamma g\left(\|Jx_n - JS_n u_n\|\right) \\
&\leq \phi(p, x_n) - \alpha\beta g\left(\|Jx_n - JT_n z_n\|\right).
\end{aligned} \tag{82}$$

From $x_n \to \widehat{p}$, $y_n \to \widehat{p}$, and $\alpha, \beta > 0$, we obtain

$$\begin{aligned}
&\alpha\beta g\left(\|Jx_n - JT_n z_n\|\right) \\
&\leq \phi(p, x_n) - \phi(p, y_n) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{83}$$

Therefore,

$$g\left(\|Jx_n - JT_n z_n\|\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{84}$$

It follows from the property of g that

$$\|Jx_n - JT_n z_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{85}$$

From (42), we have $Jx_n \to J\widehat{p}$, as $n \to \infty$. Then,

$$JT_n z_n \longrightarrow J\widehat{p}, \quad \text{as } n \longrightarrow \infty, \tag{86}$$

and so

$$\|T_n z_n\| \longrightarrow \|\widehat{p}\|, \quad \text{as } n \longrightarrow \infty. \tag{87}$$

Moreover, the demicontinuity of J^{-1} implies that $T_n z_n \to \widehat{p}$ as $n \to \infty$. Thus, the Kadec-Klee property of E, we obtain

$$T_n z_n \longrightarrow \widehat{p}, \quad \text{as } n \longrightarrow \infty. \tag{88}$$

Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that $T_{n_k} = T_1$, for all $k \in \mathbb{N}$.

Then, from (54), we have $z_{n_k} \to \widehat{p}$, as $n \to \infty$. It follows from (88) and the closedness of T_1 that

$$\widehat{p} = \lim_{k\to\infty} T_{n_k} z_{n_k} = \lim_{k\to\infty} T_1 z_{n_k} = T_1 \widehat{p}. \tag{89}$$

This implies that $\widehat{p} \in F(T_1)$.

Similarly, let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be such that $T_{n_k} = T_2$, for all $k \in \mathbb{N}$. Then, we have again that $\widehat{p} \in F(T_2)$.

Continuing in the same way, we obtain that $\widehat{p} \in F(T_i)$, where $i = 3, 4, 5, \ldots, D$.

Hence, $\widehat{p} \in \bigcap_{i=1}^{D} F(T_i)$.

From Substeps (6.1)–(6.4), we can conclude that

$$\begin{aligned}
\widehat{p} \in \Omega := &\left(\bigcap_{i=1}^{D} F(T_i)\right) \bigcap \left(\bigcap_{i=1}^{D} F(S_i)\right) \bigcap \left(\bigcap_{k=1}^{M} \text{SEP}(F_k)\right) \\
&\bigcap \left(\bigcap_{n=1}^{N} \text{VI}(C, A_n)\right).
\end{aligned} \tag{90}$$

Step 7. Finally, we will show that $\widehat{p} = \Pi_\Omega(x_0)$.

From $x_n = \Pi_{C_n}(x_0)$, we have

$$\langle Jx_0 - Jx_n, x_n - p \rangle \geq 0, \quad \forall p \in \Omega. \tag{91}$$

Taking $n \to \infty$ in (91), one has

$$\langle Jx_0 - J\widehat{p}, \widehat{p} - p \rangle \geq 0, \quad \forall p \in \Omega. \tag{92}$$

Now, we have $\widehat{p} \in \Omega$ and by Lemma 5, we get

$$\widehat{p} = \Pi_\Omega(x_0). \tag{93}$$

This completes the proof of Theorem 13. □

If we set $M = N = D = 1$ in Theorem 13, then we obtain the following result.

Corollary 14. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) *$F : C \times C \to \mathbb{R}$ is a bifunction satisfying conditions (A1)–(A4);*

(2) *T and S are two quasi-ϕ-nonexpansive mappings from C into E;*

(3) *$A : C \to E^*$ is a continuous monotone mapping.*

(4) For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : E \rightarrow C$ as follows:

$$K_{r_n}(x) : = \left\{ z \in C : \langle y - z, Az \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\};$$

$$T_{r_n}(x) : = \left\{ z \in C : F(z, y) \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}.$$

(94)

Assume that $\Omega := F(T) \cap F(S) \cap \mathrm{EP}(F) \cap \mathrm{VI}(C, A)$ is a nonempty and bounded in C and let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C, \ \text{chosen arbitrary},$$

$$z_n = K_{r_n} x_n,$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1} \left(\alpha J x_n + \beta J T z_n + \gamma J S u_n \right),$$

$$C_{n+1} = \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) \},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

(95)

where $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$, and α, β, γ are real numbers in $(0, 1)$ such that $\alpha + \beta + \gamma = 1$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\widehat{p} = \Pi_\Omega(x_0)$.

If we set $T_i = I$, for any $i = 1, 2, 3, \ldots, D$ in Theorem 13, then we obtain the following result which extends and improves the result's Zegeye and Shahzad [7].

Corollary 15 (Zegeye and Shahzad [7]). *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has Kadec-Klee property. Suppose that*

(1) *$\{F_k\}_{k=1}^{M} : C \times C \rightarrow \mathbb{R}$ is a finite family of bifunctions satisfying conditions (A1)–(A4), where $k = 1, 2, 3, \ldots, M$;*

(2) *$\{S_i\}_{i=1}^{D} : C \rightarrow E$ is a finite family of quasi-ϕ-nonexpansive mappings, where $i = 1, 2, 3, \ldots, D$;*

(3) *$\{A_n\}_{n=1}^{N} : C \rightarrow E^*$ is a finite family of continuous monotone mappings, where $n = 1, 2, 3, \ldots, N$;*

(4) For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : E \rightarrow C$ by

$$K_{r_n}(x) := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\};$$

$$T_{r_n}(x) := \left\{ z \in C : F_n(z, y) \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}.$$

(96)

Assume that $\Omega := (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{k=1}^{M} \mathrm{SEP}(F_k)) \bigcap (\bigcap_{n=1}^{N} \mathrm{VI}(C, A_n))$ is a nonempty and bounded in C and let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C, \ \text{chosen arbitrary},$$

$$z_n = K_{r_n} x_n,$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1} \left(\alpha J x_n + \beta J z_n + \gamma J S_n u_n \right),$$

$$C_{n+1} = \{ z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) \},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

(97)

where $A_n \equiv A_n \pmod{N}$, $F_n \equiv F_n \pmod{M}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$, and α, β, γ are real numbers in $(0,1)$ such that $\alpha + \beta + \gamma = 1$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\widehat{p} = \Pi_\Omega(x_0)$.

4. Deduced Theorems

If we set $F_k \equiv 0$, for any $k = 1, 2, 3, \ldots, M$ in Theorem 13, then we obtain the following result.

Corollary 16. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) *$\{T_i\}_{i=1}^{D}$ and $\{S_i\}_{i=1}^{D}$ are finite families of quasi-ϕ-nonexpansive mappings from C into E, where $i = 1, 2, 3, \ldots, D$;*

(2) *$\{A_n\}_{n=1}^{N} : C \rightarrow E^*$ is a finite family of continuous monotone mappings, where $n = 1, 2, 3, \ldots, N$;*

(3) *For $x \in E$, we define the mappings $K_{r_n} : E \rightarrow C$ by*

$$K_{r_n}(x) := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}.$$

(98)

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{n=1}^{N} VI(C, A_n))$ *is a nonempty and bounded in C and let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C_0 = C, \text{ chosen arbitrary,}$$

$$z_n = K_{r_n} x_n,$$

$$y_n = J^{-1} (\alpha J x_n + \beta J T_n z_n + \gamma J S_n x_n), \qquad (99)$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,$$

where $A_n \equiv A_n \pmod{N}$, $T_n \equiv T_n \pmod{D}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* $n \in \mathbb{N}$, *and* α, β, γ *are real numbers in (0,1) such that* $\alpha + \beta + \gamma = 1$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\widehat{p} = \Pi_{\Omega}(x_0)$.

If we set $A_n \equiv 0$, for any $n = 1, 2, 3, \ldots, N$ in Theorem 13, then we obtain the following result.

Corollary 17. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) $\{F_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ *is a finite family of bifunctions satisfying conditions (A1)–(A4), where* $k = 1, 2, 3, \ldots, M$;

(2) $\{T_i\}_{i=1}^{D}$ *and* $\{S_i\}_{i=1}^{D}$ *are finite families of quasi-ϕ-nonexpansive mappings from C into E, where* $i = 1, 2, 3, \ldots, D$;

(3) *For* $x \in E$, *one defines the mappings* $T_{r_n} : E \to C$ *by*

$$T_{r_n}(x) := \left\{ z \in C : F_n(z, y) + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \ge 0, \ \forall y \in C \right\}. \qquad (100)$$

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{k=1}^{M} SEP(F_k))$ *is a nonempty and bounded in C and let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C_0 = C, \text{ chosen arbitrary,}$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1} (\alpha J x_n + \beta J T_n x_n + \gamma J S_n u_n), \qquad (101)$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,$$

where $F_n \equiv F_n \pmod{M}$, $T_n \equiv T_n \pmod{D}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* $n \in \mathbb{N}$, *and* α, β, γ *are real numbers in (0, 1) such that* $\alpha + \beta + \gamma = 1$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\widehat{p} = \Pi_{\Omega}(x_0)$.

If we set $F_k \equiv 0$, for any $k = 1, 2, 3, \ldots, M$ and $A_n \equiv 0$, for any $n = 1, 2, 3, \ldots, N$ in Theorem 13, then we obtain the following result.

Corollary 18. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that* $\{T_i\}_{i=1}^{D}$ *and* $\{S_i\}_{i=1}^{D}$ *are finite families of quasi-ϕ-nonexpansive mappings from C into E, where* $i = 1, 2, 3, \ldots, D$.

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i))$ *is a nonempty and bounded in C and let* $\{x_n\}$ *be a sequence generated by*

$$x_0 \in C_0 = C, \text{ chosen arbitrary,}$$

$$y_n = J^{-1} (\alpha J x_n + \beta J T_n x_n + \gamma J S_n x_n),$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n)\}, \qquad (102)$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \ge 0,$$

where $T_n \equiv T_n \pmod{D}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$ *and* $n \in \mathbb{N}$, *and* α, β, γ *are real numbers in (0,1) such that* $\alpha + \beta + \gamma = 1$.

Then, the sequence $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\widehat{p} = \Pi_{\Omega}(x_0)$.

5. Some Applications

5.1. Application to Weak Relatively Nonexpansive Mappings. If we change the condition (2) in Theorem 13 as follows: $\{T_i\}_{i=1}^{D}$ and $\{S_i\}_{i=1}^{D}$ are finite families of weak relatively nonexpansive mappings. From Remark 2(2), every weak relatively nonexpansive mappings is quasi-ϕ-nonexpansive mappings. Then, we obtain the following result.

Corollary 19. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) $\{F_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ *is a finite family of bifunctions satisfying conditions (A1)–(A4), where* $k = 1, 2, 3, \ldots, M$;

(2) $\{T_i\}_{i=1}^{D}$ *and* $\{S_i\}_{i=1}^{D}$ *are finite families of weak relatively nonexpansive mappings from C into E, where* $i = 1, 2, 3, \ldots, D$;

(3) $\{A_n\}_{n=1}^{N} : C \to E^*$ *is a finite family of continuous monotone mappings, where* $n = 1, 2, 3, \ldots, N$;

(4) *For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : E \to C$ by*

$$K_{r_n}(x) := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\};$$

$$T_{r_n}(x) := \left\{ z \in C : F_n(z, y) \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}.$$

$$(103)$$

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{k=1}^{M} \text{SEP}(F_k)) \bigcap (\bigcap_{n=1}^{N} \text{VI}(C, A_n))$ is a nonempty and bounded in C and let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C, \ \text{chosen arbitrary},$$

$$z_n = K_{r_n} x_n,$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1}\left(\alpha J x_n + \beta J T_n z_n + \gamma J S_n u_n\right),$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

$$(104)$$

where $A_n \equiv A_n \pmod N$, $F_n \equiv F_n \pmod M$, $T_n \equiv T_n \pmod D$, $S_n \equiv S_n \pmod D$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$, and α, β, γ are real numbers in (0,1) such that $\alpha + \beta + \gamma = 1$.
Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\hat{p} = \Pi_{\Omega}(x_0)$.

5.2. Application to Relatively Nonexpansive Mappings. If we change the condition (2) in Theorem 13 as follows: $\{T_i\}_{i=1}^{D}$ and $\{S_i\}_{i=1}^{D}$ are finite families of relatively nonexpansive mappings. From Remark 2(3) and (2) every relatively nonexpansive mappings is weak relatively nonexpansive mappings and every weak relatively nonexpansive mappings is quasi-ϕ-nonexpansive mappings. Then, we obtain the following result.

Corollary 20. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which has the Kadec-Klee property. Suppose that*

(1) *$\{F_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ is a finite family of bifunctions satisfying conditions (A1)–(A4), where $k = 1, 2, 3, \ldots, M$;*

(2) *$\{T_i\}_{i=1}^{D}$ and $\{S_i\}_{i=1}^{D}$ are finite families of relatively nonexpansive mappings from C into E, where $i = 1, 2, 3, \ldots, D$;*

(3) *$\{A_n\}_{n=1}^{N} : C \to E^*$ is a finite family of continuous monotone mappings, where $n = 1, 2, 3, \ldots, N$;*

(4) *For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : E \to C$ by*

$$K_{r_n}(x) := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\};$$

$$T_{r_n}(x) := \left\{ z \in C : F_n(z, y) \right.$$

$$\left. + \frac{1}{r_n} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C \right\}.$$

$$(105)$$

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{k=1}^{M} \text{SEP}(F_k)) \bigcap (\bigcap_{n=1}^{N} \text{VI}(C, A_n))$ is a nonempty and bounded in C and let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C, \ \text{chosen arbitrary},$$

$$z_n = K_{r_n} x_n,$$

$$u_n = T_{r_n} x_n,$$

$$y_n = J^{-1}\left(\alpha J x_n + \beta J T_n z_n + \gamma J S_n u_n\right),$$

$$C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\},$$

$$x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

$$(106)$$

where $A_n \equiv A_n \pmod N$, $F_n \equiv F_n \pmod M$, $T_n \equiv T_n \pmod D$, $S_n \equiv S_n \pmod D$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$, and α, β, γ are real numbers in (0,1) such that $\alpha + \beta + \gamma = 1$.
Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\hat{p} = \Pi_{\Omega}(x_0)$.

5.3. Application to Hilbert Spaces. If $E = H$, a real Hilbert space, then E is uniformly smooth and strictly convex real Banach space. In this case, $J = I$ and $\Pi_C = P_C$. Then, we obtain the following result.

Corollary 21. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that*

(1) *$\{F_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ is a finite family of bifunctions satisfying conditions (A1)–(A4), where $k = 1, 2, 3, \ldots, M$;*

(2) *$\{T_i\}_{i=1}^{D}$ and $\{S_i\}_{i=1}^{D}$ are finite families of nonexpansive mappings from C into C, where $i = 1, 2, 3, \ldots, D$;*

(3) *$\{A_n\}_{n=1}^{N} : C \to H$ is a finite family of continuous monotone mappings, where $n = 1, 2, 3, \ldots, N$;*

(4) *For $x \in E$, one defines the mappings $K_{r_n}, T_{r_n} : H \to C$ by*

$$K'_{r_n}(x) := \left\{ z \in C : \langle y - z, A_n z \rangle \right.$$

$$+ \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \Big\};$$

$$T'_{r_n}(x) := \left\{ z \in C : F_n(z, y) \right.$$

$$+ \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \Big\}.$$

(107)

Assume that $\Omega := (\bigcap_{i=1}^{D} F(T_i)) \bigcap (\bigcap_{i=1}^{D} F(S_i)) \bigcap (\bigcap_{k=1}^{M} \text{SEP}(F_k)) \bigcap (\bigcap_{n=1}^{N} \text{VI}(C, A_n))$ is a nonempty and bounded in C and let $\{x_n\}$ be a sequence generated by

$$x_0 \in C_0 = C, \ \text{chosen arbitrary},$$

$$z_n = K'_{r_n} x_n,$$

$$u_n = T'_{r_n} x_n,$$

$$y_n = \alpha x_n + \beta T_n z_n + \gamma S_n u_n,$$

$$C_{n+1} = \{ z \in C_n : \|z - y_n\| \leq \|z - x_n\| \},$$

$$x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \geq 0,$$

(108)

where $A_n \equiv A_n \pmod{N}$, $F_n \equiv F_n \pmod{M}$, $T_n \equiv T_n \pmod{D}$, $S_n \equiv S_n \pmod{D}$, $\{r_n\} \subset [a, \infty)$ for some $a > 0$ and $n \in \mathbb{N}$, and α, β, γ are real numbers in $(0,1)$ such that $\alpha + \beta + \gamma = 1$.
Then, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to an element of Ω.

Remark 22. Our theorem extends and improves the corresponding results in [5–7] in the following aspect.

(a) For the mapping, we extend the mappings from nonexpansive mappings, relatively nonexpansive mappings, and weak relatively nonexpansive mappings to more general than quasi-ϕ-nonexpansive mappings.

(b) For the common solution, we extend the common solution of a single finite family of quasi-ϕ-nonexpansive mappings to two finite families of quasi-ϕ-nonexpansive mappings.

Acknowledgments

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