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Research Article

Parameter Identification Problem for the Kirchhoff-Type Equation with Viscosity

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The constant parameter identification problem in the Kirchhoff-type equation with viscosity is studied. The problem is formulated by a minimization of quadratic cost functionals by distributive measurements. The existence of optimal parameters and necessary optimality conditions for the parameters are proved.

1. Introduction

The model of transversal vibration of a string has long history starting from D' Alembert and Euler. It is widely regarded that the model proposed by D' Alembert is simple and elementary model describing small transversal vibration of a string in which the effect of elasticity is not considered.

When we take into account the change of length of a string in its small vibration mainly due to the effect of elasticity, the classical model from D' Alembert is no more correct to cover the more realistic phenomena.

More accurate or appropriate model for the transversal vibration of an elastic string, given by

$$\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho}\left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx\right)\frac{\partial^2 y}{\partial x^2} = 0, \tag{1.1}$$

has been proposed by Kirchhoff [1]. Here L is the length of the string, h is the area of the cross section, ρ is the mass density, P_0 is the initial tension, and E is the Young's modulus of a material. For the derivations of (1.1), we can refer to the article of Ferrel and Medeiros [2].

As a general form of (1.1), we consider the following damped equation with appropriate boundary and initial conditions:

$$\frac{\partial^2 y}{\partial t^2} - m\left(\int_\Omega |\nabla y|^2 dx\right)\Delta y - \gamma \frac{\partial y}{\partial t} = f, \tag{1.2}$$

where Ω is a smooth domain in $\mathbf{R}^n$, $\gamma > 0$. Many researches have been devoted to the study of (1.2) for both damped ($\gamma > 0$) or undamped ($\gamma = 0$) cases, see Arosio [3], Spagnolo [4], Pohožaev [5], Lions [6], Nishihara, and Yamada [7] and their long roll of bibliographical references. Those researches are mainly concerned with the well-posedness of solutions in global or local sense under the various data conditions and their decays.

Especially when we take into account the viscosity effect of its vibration due to its inner friction, the damping coefficient γ in (1.2) is replaced by $\gamma\Delta$. In this case we can refer to Cavalcanti et al. [8] to show the well-posedness in the Hadamard sense under the data condition $(y(0,x),\ \partial y(0,x)/\partial t, f) \in D(\Delta)\times H_0^1(\Omega)\times L^2(0,T;L^2(\Omega))$. Making use of this result, we are going to study the constant identification problem in the equation of Kirchhoff-type equation with viscosity as follows:

$$\begin{gathered}
\frac{\partial^2 y}{\partial t^2} - \left(\alpha + \beta\int_\Omega |\nabla y|^2 dx\right)\Delta y - \gamma\Delta\frac{\partial y}{\partial t} = f \quad \text{in } Q, \\
y = 0 \quad \text{on } \Sigma, \\
y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega,
\end{gathered} \tag{1.3}$$

where $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\partial\Omega$. Here the constants α and β are physical constants explained above, and γ stands for the rate of viscosity.

Recently, Hwang and Nakagiri [9] studied optimal control problems for (1.3) under the framework of Lions [10]. And Hwang [11] studied constant parameter identification for the problem of an extensible beam equation. In this paper we will study constant parameter identification problems for (1.3) in the following way.

At first, we assume that the desired state is known, but constant parameters α, β, γ involved in the above equation are unknown. For more details, we refer to Ha and Nakagiri [12], Hwang and Nakagiri [13]. We show the existence of an optimal parameters in an admissible set and its characterizations, namely, a parameter identification problem in which we use the term optimal parameter to denote the best parameter within any admissible set for which the solution of (1.3) gives a minimum of the given functional. We take this functional by L^2-quadratic norm of observed state minus desired state that is usually regarded as a cost function in optimal control theory.

In this paper we pursue to find necessary conditions for an optimal parameters by using Gâteaux differentiability of the solution mapping and giving variational inequality via an adjoint equation. Proceeding in this way, we can obtain similar results with optimal control problems due to Lions [10]. For more detailed study, we refer to Ahmed [14] for abstract evolution equations.

We explain our identification problem precisely as follows. At first, in order to study parameter identification problem in the framework of optimal control theory due to Lions [10], we need to modify the positive constants, α, β, γ in (1.3) by $\alpha_0 + \alpha$, β, $\gamma_0 + \gamma$, respectively, where α_0 and γ_0 are fixed positive constants, and α, β, γ are nonnegative constants. Therefore, we take the set $\mathcal{P} = \{(x_1, x_2, x_3) \mid x_i \geq 0,\ i = 1,2,3\}$ as the set of parameters (α, β, γ) in (1.3). By doing this, we can guarantee the well-posedness of (1.3) in verifying the Gâteaux differentiability of the solution mapping from the set of parameters to the corresponding solution space of (1.3).

Let $y(q) = y(q; t, x)$ be the solution for a given $q = (\alpha, \beta, \gamma) \in \mathcal{P}$ and $\mathcal{P}_{\text{ad}} \subset \mathcal{P}$ be an admissible parameter set. We consider the following two quadratic distributive functionals:

$$J_1(q) = \int_0^T \int_\Omega \left| y(q; t, x) - Y_1(t, x) \right|^2 dx\, dt + \int_\Omega \left| y(q; T, x) - Y_1^T(x) \right|^2 dx, \tag{1.4}$$

$$J_2(q) = \int_0^T \int_\Omega \left| \frac{\partial y(q; t, x)}{\partial t} - Y_2(t, x) \right|^2 dx\, dt \quad \text{for } q \in \mathcal{P}_{\text{ad}}, \tag{1.5}$$

where $Y_i \in L^2(Q),\ i = 1,2$ and $Y_1^T \in L^2(\Omega)$ are the desired values.

The parameter identification problem for (1.3) with the cost $J = J_1$ in (1.4) or $J = J_2$ in (1.5) is to find and characterize an optimal parameters $q^* = (\alpha^*, \beta^*, \gamma^*) \in \mathcal{P}_{\text{ad}}$ satisfying that

$$J(q^*) = \inf \{ J(q) : \ q \in \mathcal{P}_{\text{ad}} \}. \tag{1.6}$$

We prove the existence of an optimal parameter q^* by using the continuity of solutions on parameters and establish the necessary optimality conditions by introducing appropriate adjoint systems for which we prove the strong Gâteaux differentiability of the nonlinear mapping $q \to y(q)$.

Another novelty of this paper is that the first-order Volterra integrodifferential equation is utilized as a proper adjoint system to establish the necessary optimality condition of the velocity's measurement case (1.5) as in [9, 13].

2. Preliminaries

We consider the following Dirichlet boundary value problem for Kirchhoff-type equation with damping term:

$$\begin{gathered} \frac{\partial^2 y}{\partial t^2} - \left(\alpha + \beta \int_\Omega |\nabla y|^2 dx \right) \Delta y - \gamma \Delta \frac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0, x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0, x) = y_1(x) \quad \text{in } \Omega, \end{gathered} \tag{2.1}$$

where f is a forcing function, y_0 and y_1 are initial data, and $\alpha, \beta, \gamma > 0$ are some physical constants. In this paper we study (2.1) in the class of *strong* solutions. For the purpose we suppose that $f \in L^2(0,T;L^2(\Omega))$, $y_0 \in D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, and $y_1 \in H_0^1(\Omega)$. The solution space $S(0,T)$ which is the space of strong solutions of (2.1) is defined by

$$S(0,T) = \left\{ g \mid g \in L^2(0,T;D(\Delta)),\ g' \in L^2(0,T;D(\Delta)),\ g'' \in L^2\left(0,T;L^2(\Omega)\right) \right\}, \tag{2.2}$$

endowed with the norm

$$\|g\|_{S(0,T)} = \left(\|g\|^2_{L^2(0,T;D(\Delta))} + \|g'\|^2_{L^2(0,T;D(\Delta))} + \|g''\|^2_{L^2(0,T;L^2(\Omega))} \right)^{1/2}. \tag{2.3}$$

Here, g' and g'' denote the first- and second-order distributional derivatives of g. The scalar products and norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by $(\phi,\psi)_2$, $|\phi|_2$ and $(\phi,\psi)_{H_0^1(\Omega)}$, $\|\phi\|$, respectively. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by $(\phi,\psi)_2$ and $|\phi|_2$. Then, the scalar product $(\phi,\psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi,\nabla\psi)_2$ and $\|\phi\| = |\nabla\phi|_2$, respectively. Finally the norm and the scalar product on $D(\Delta)$ are given by $(\Delta\phi,\Delta\psi)_2$ and $\|\phi\|_{D(\Delta)} = |\Delta\phi|_2$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle \phi,\psi \rangle$.

Definition 2.1. A function y is said to be a strong solution of (2.1) if $y \in S(0,T)$ and y satisfies

$$\begin{gathered} y''(t) - \left(\alpha + \beta|\nabla y(t)|_2^2\right)\Delta y(t) - \gamma\Delta y'(t) = f(t), \quad \text{a.e. } t \in [0,T], \\ y(0) = y_0, \qquad y'(0) = y_1. \end{gathered} \tag{2.4}$$

We remark here that $S(0,T)$ is continuously imbedded in $C([0,T];D(\Delta)) \cap C^1([0,T];H_0^1(\Omega))$ (cf. Dautray and Lions [15, page 555]).

The following variational formulation is used to define the *weak* solution of (2.1).

Definition 2.2. A function y is said to be a weak solution of (2.1) if $y \in W(0,T) \equiv \{g \mid g \in L^2(0,T;H_0^1(\Omega)),\ g' \in L^2(0,T;H_0^1(\Omega)),\ g'' \in L^2(0,T;H^{-1}(\Omega))\}$, and y satisfies

$$\begin{gathered} \langle y''(\cdot),\phi\rangle + \left(\alpha + \beta|\nabla y(\cdot)|_2^2\right)(\nabla y(\cdot),\nabla\phi) + \gamma(\nabla y'(\cdot),\nabla\phi) \\ = (f(\cdot),\phi) \quad \forall\phi \in H_0^1(\Omega) \text{ in the sense of } D'(0,T), \\ y(0) = y_0, \qquad y'(0) = y_1. \end{gathered} \tag{2.5}$$

In order to verify the well-posedness of (2.1), we refer to the results in [8, 9]. The well-posedness in the sense of Hadamard can be given as follows.

Theorem 2.3. *Assume that $f \in L^2(0,T;L^2(\Omega))$ and $y_0 \in D(\Delta)$, $y_1 \in H_0^1(\Omega)$. Then the problem* (2.1) *has a unique strong solution y in $S(0,T)$. And the solution mapping $p = (y_0,y_1,f) \to y(p)$*

of $P \equiv D(\Delta) \times H_0^1(\Omega) \times L^2(0,T;L^2(\Omega))$ *into* $S(0,T)$ *is strongly continuous. Further, for each* $p_1 = (y_0^1, y_1^1, f_1) \in P$ *and* $p_2 = (y_0^2, y_1^2, f_2) \in P$*, we have the following inequality:*

$$\begin{aligned} &\left|\nabla\left(y'(p_1;t) - y'(p_2;t)\right)\right|_2^2 + \left|\Delta\left(y(p_1;t) - y(p_2;t)\right)\right|_2^2 + \int_0^t \left|\Delta\left(y'(p_1;s) - y'(p_2;s)\right)\right|_2^2 ds \\ &\quad \le C\left(\left|\Delta\left(y_0^1 - y_0^2\right)\right|_2^2 + \left|\nabla\left(y_1^1 - y_1^2\right)\right|_2^2 + \|f_1 - f_2\|_{L^2(0,T;L^2(\Omega))}^2\right), \end{aligned} \tag{2.6}$$

where C *is a constant and* $t \in [0,T]$.

Proof (see Hwang and Nakagiri [9]). We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.6), we will write $\int_0^t |\nabla y'(p_1)|^2 ds$ instead of $\int_0^t |\nabla y'(p_1;s)|^2 ds$. □

3. Identification Problems

In this section we study the identification problem for the unknown parameters $q = (\alpha, \beta, \gamma) \in \mathcal{P}$ in the problem

$$\begin{gathered} \frac{\partial^2 y}{\partial t^2} - \left(\alpha_0 + \alpha + \beta|\nabla y|_2^2\right)\Delta y - (\gamma_0 + \gamma)\Delta\frac{\partial y}{\partial t} = f \quad \text{in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(0,x) = y_0(x), \quad \frac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \end{gathered} \tag{3.1}$$

where α_0, $\gamma_0 > 0$, $y_0 \in D(\Delta)$, $y_1 \in H_0^1(\Omega)$, and $f \in L^2(0,T;L^2(\Omega))$ are fixed. The physical constants $q = (\alpha, \beta, \gamma)$ in (3.1) are an unknown parameter that should be identified. In this setting we take $\mathcal{P} = \{(x_1, x_2, x_3) \mid x_i \ge 0, i = 1,2,3\}$ to be the space of parameters $q = (\alpha, \beta, \gamma)$ with the Euclidian norm. By Theorem 2.3 we have that for each $q \in \mathcal{P}$ there exists a unique solution $y = y(q) \in S(0,T)$ of (3.1).

At first we show the continuous dependence of solutions on parameters $q = (\alpha, \beta, \gamma)$.

Theorem 3.1. *The solution map* $q \to y(q)$ *from* $\mathcal{P} = \{(x_1, x_2, x_3) \mid x_i \ge 0, i = 1,2,3\}$ *into* $S(0,T)$ *is continuous.*

Proof. Let $q = (\alpha, \beta, \gamma)$ be arbitrarily fixed. Suppose that $q_m = (\alpha_m, \beta_m, \gamma_m) \to q = (\alpha, \beta, \gamma)$ in $\mathcal{P}$. Let $y_m = y(q_m)$ and $y = y(q)$ be the solutions of (3.1) for $q = q_m$ and for q, respectively. Since $\{q_m\}$ is bounded in $\mathcal{P}$, by Theorem 2.3, we see that

$$\left|\Delta y_m'(t)\right|_2^2 + \left|\Delta y_m(t)\right|_2^2 + \int_0^t \left|\Delta y_m'\right|_2^2 ds \le C_0 < \infty, \quad \forall t \in [0,T], \tag{3.2}$$

where $C_0 > 0$ is a constant depending only on $\alpha_0, \beta_0, \gamma_0, y_0, y_1$, and f. Applying (3.2) to (3.1), we can deduce by choosing appropriate subsequence of $\{y_m\}$ denoted again by $\{y_m\}$ that

$$y_m \longrightarrow y \quad \text{weakly in } S(0,T) \text{ as } m \longrightarrow \infty. \tag{3.3}$$

Since $D(\Delta) \hookrightarrow H_0^1(\Omega)$ is compact, we can deduce from [16, pages 273–278] that the space $S(0,T)$ is compactly imbedded in $L^2(0,T;H_0^1(\Omega))$. Therefore, we can take a subsequence $\{y_{m_k}\}$ of $\{y_m\}$, if necessary, such that

$$y_{m_k} \longrightarrow y \quad \text{strongly in } L^2\left(0,T;H_0^1(\Omega)\right) \text{ as } k \longrightarrow \infty. \tag{3.4}$$

Equation (3.4) implies that

$$\left|\nabla y_{m_k}(\cdot)\right|_2^2 \longrightarrow \left|\nabla y(\cdot)\right|_2^2 \quad \text{a.e. in } (0,T) \text{ as } k \longrightarrow \infty. \tag{3.5}$$

Taking into account (3.3) and (3.5) and coming back to (3.1), we deduce that y is the solution of (3.1) corresponding to the parameter q.

In order to obtain strong convergency, we set $\psi_m = y_m - y$. Then, in weak sense, ψ_m satisfies

$$\begin{gathered}
\frac{\partial^2 \psi_m}{\partial t^2} - \left(\alpha_0 + \alpha_m + \beta_m\left|\nabla y_m\right|_2^2\right) \Delta\psi_m - (\gamma_0 + \gamma_m)\Delta\frac{\partial \psi_m}{\partial t} = \mathcal{F}_m \quad \text{in } Q, \\
\psi_m = 0 \quad \text{on } \Sigma, \\
\psi_m(0,x) = 0, \qquad \frac{\partial \psi_m}{\partial t}(0,x) = 0 \quad \text{in } \Omega,
\end{gathered} \tag{3.6}$$

where

$$\mathcal{F}_m = \left(\alpha_m - \alpha + \beta_m\left|\nabla y_m\right|_2^2 - \beta\left|\nabla y\right|_2^2\right)\Delta y - (\gamma - \gamma_m)\Delta\frac{\partial y}{\partial t}. \tag{3.7}$$

Using (3.5) and the Lebesgue-dominated convergence theorem, we can verify that

$$\mathcal{F}_m \longrightarrow 0 \quad \text{in } L^2\left(0,T;L^2(\Omega)\right) \text{ as } m \rightarrow \infty. \tag{3.8}$$

Multiplying (3.6) by $\Delta\psi_m'$, and integrating it over $\Omega \times [0,t]$, we have

$$\begin{aligned}
&\left|\nabla\psi_m'(t)\right|_2^2 + \left(\alpha_0 + \alpha_m + \beta_m\left|\nabla y_m(t)\right|_2^2\right)\left|\Delta\psi_m(t)\right|_2^2 + 2(\gamma_0 + \gamma_m)\int_0^t \left|\Delta\psi_m'\right|_2^2 ds \\
&\quad = -2\int_0^t (\mathcal{F}_m, \Delta\psi_m')_2 ds + \int_0^t 2\beta_m(\nabla y_m, \nabla y_m')_2\left|\Delta\psi_m\right|_2^2 ds.
\end{aligned} \tag{3.9}$$

By the Cauchy-Schwarz inequality and the fact that $y_m \in S(0,T) \hookrightarrow C([0,T];D(\Delta)) \cap C^1([0,T];H_0^1(\Omega))$, we have following inequalities:

$$\begin{aligned}\left|-2\int_0^t (\mathcal{F}_m, \nabla\psi'_m)_2 ds\right| &\le 2\int_0^t |\mathcal{F}_m|_2 |\Delta\psi'_m|_2 ds \\ &\le \frac{1}{\gamma_0}\|\mathcal{F}_m\|^2_{L^2(0,T;L^2(\Omega))} + \gamma_0\int_0^t |\Delta\psi'_m|_2^2 ds, \\ \left|\int_0^t 2\beta_m(\nabla y_m, \nabla y'_m)_2 |\Delta\psi_m|_2^2 ds\right| &\le 2\beta_m\int_0^t |\nabla y_m|_2 |\nabla y'_m|_2 |\Delta\psi_m|_2^2 ds \\ &\le 2\beta_m\|y_m\|_{C([0,T];H_0^1(\Omega))}\|y'_m\|_{C([0,T];H_0^1(\Omega))}\int_0^t |\Delta\psi_m|_2^2 ds.\end{aligned} \tag{3.10}$$

Then by (3.9) and (3.10), we can obtain

$$|\nabla\psi'_m(t)|_2^2 + |\Delta\psi_m(t)|_2^2 + \int_0^t |\Delta\psi'_m|_2^2 ds \le C\|\mathcal{F}_m\|^2_{L^2(0,T;L^2(\Omega))} + C\int_0^t |\Delta\psi_m|_2^2 ds, \tag{3.11}$$

where $C > 0$. Hence by applying Gronwall's inequality to (3.11), we have

$$|\nabla\psi'_m(t)|_2^2 + |\Delta\psi_m(t)|_2^2 + \int_0^t |\Delta\psi'_m|_2^2 ds \le C\exp(CT)\|\mathcal{F}_m\|^2_{L^2(0,T;L^2(\Omega))}. \tag{3.12}$$

Combining (3.8) and (3.12), we have

$$\begin{aligned}\psi_m &\longrightarrow 0 \quad \text{in } C([0,T];D(\Delta)), \\ \psi'_m &\longrightarrow 0 \quad \text{in } C\Big([0,T];H_0^1(\Omega)\Big) \cap L^2(0,T;D(\Delta)),\end{aligned} \tag{3.13}$$

so that

$$y_m(\cdot) \longrightarrow y(\cdot) \quad \text{strongly in } S(0,T). \tag{3.14}$$

This proves Theorem 3.1. □

As explained before, we choose the L^2 objective costs to be minimized for the identification of $q = (\alpha,\beta,\gamma)$ which are given by

$$J_1(q) = \|y(q) - Y_1\|^2_{L^2(0,T;L^2(\Omega))} + \left|y(q;T) - Y_1^T\right|_2^2, \tag{3.15}$$

$$J_2(q) = \|y'(q) - Y_2\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for } q \in \mathcal{P}_{\text{ad}}, \tag{3.16}$$

where $Y_i \in L^2(0,T;L^2(\Omega))$, $i = 1,2$, and $Y_1^T \in L^2(\Omega)$.

If $\mathcal{P}_{\rm ad}$ is compact, then for the minimizing sequence $\{q_m\}$ such as $J(q_m) \to J^* = \inf\{J(q) : q \in \mathcal{P}_{\rm ad}\}$ we can choose a subsequence $\{q_{mj}\}$ of $\{q_m\}$ such that $q_{mj} \to q^* \in \mathcal{P}_{\rm ad}$ and $y(q_{mj}) \to y(q^*)$ strongly in $S(0,T)$ by Theorem 3.1. Due to the continuous imbedding $S(0,T) \hookrightarrow C([0,T]; D(\Delta)) \cap C^1([0,T]; H_0^1(\Omega))$ we have $J^* = J(q^*)$ for the costs (3.15) and (3.16). Thus we have the following corollary.

Corollary 3.2. *If $\mathcal{P}_{ad}$ is compact, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost J_1 in* (3.15) *or J_2 in* (3.16).

Let the admissible set $\mathcal{P}_{\rm ad}$ be compact and convex in $\mathcal{P}$, and let $q^* = (\alpha^*, \beta^*, \gamma^*)$ be an optimal parameter on $\mathcal{P}_{\rm ad}$ for the cost $J(q)$. As is well known the necessary optimality condition of an optimal parameter $q^* = (\alpha^*, \beta^*, \gamma^*)$ for the cost J is given by

$$DJ(q^*)(q - q^*) \geq 0 \quad \forall q \in \mathcal{P}_{\rm ad}, \tag{3.17}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$.

The Gâteaux differentiability of the above quadratic costs $J_i(q)$, $i = 1, 2$ follows from that of the nonlinear solution mapping $q \to y(q)$ of $\mathcal{P}_{\rm ad}$ into $S(0,T)$. The following theorem proves the Gâteaux differentiability of the nonlinear solution mapping $q \to y(q)$ and gives its characterization.

Theorem 3.3. *The map $q \to y(q)$ of $\mathcal{P}_{ad}$ into $S(0,T)$ is Gâteaux differentiable at $q = q^*$ and such the Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $q - q^* \in \mathcal{P}$, say $z = Dy(q^*)(q - q^*)$, is a unique solution of the following linear problem:*

$$\begin{gathered}
\frac{\partial^2 z}{\partial t^2} - \left(\alpha_0 + \alpha^* + \beta^* \left|\nabla y^*\right|_2^2\right) \Delta z - 2\beta^* (\nabla z, \nabla y^*)_2 \Delta y^* - (\gamma_0 + \gamma^*) \Delta \frac{\partial z}{\partial t} = \mathcal{G}(q - q^*; y^*) \quad \text{in } Q, \\
z = 0 \quad \text{on } \Sigma, \\
z(0,x) = 0, \quad \frac{\partial z}{\partial t}(0,x) = 0 \quad \text{in } \Omega,
\end{gathered} \tag{3.18}$$

where $y^* = y(q^*)$ and

$$\mathcal{G}(q - q^*; y^*) = (\alpha - \alpha^*) \Delta y^* + (\beta - \beta^*) \left|\nabla y^*\right|_2^2 \Delta y^* + (\gamma - \gamma^*) \Delta \frac{\partial y^*}{\partial t}. \tag{3.19}$$

Proof. Let $\lambda \in [0,1]$, and let y_λ and y^* be the solutions of (3.1) corresponding to $q^* + \lambda(q - q^*)$ and q^*, respectively. We set $z_\lambda = \lambda^{-1}(y_\lambda - y^*)$, $\lambda \neq 0$. Then z_λ satisfies the following problem in the weak sense:

$$\begin{aligned}
&\frac{\partial^2 z_\lambda}{\partial t^2} - \left(\alpha_0 + \alpha^* + \beta^* |\nabla y^*|_2^2\right)\Delta z_\lambda - \beta^* (\nabla z_\lambda, \nabla y_\lambda + \nabla y^*)_2 \Delta y_\lambda - (\gamma_0 + \gamma^*)\Delta \frac{\partial z_\lambda}{\partial t} \\
&\qquad = \mathcal{G}(q - q^*; y_\lambda) \quad \text{in } Q, \\
&z_\lambda = 0 \quad \text{on } \Sigma, \\
&z_\lambda(0,x) = 0, \qquad \frac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \Omega,
\end{aligned} \tag{3.20}$$

where

$$\mathcal{G}(q - q^*; y_\lambda) = (\alpha - \alpha^*)\Delta y_\lambda + (\beta - \beta^*)|\nabla y_\lambda|_2^2 \Delta y_\lambda + (\gamma - \gamma^*)\Delta \frac{\partial y_\lambda}{\partial t}. \tag{3.21}$$

Since $y_\lambda \in S(0,T)$ we can easily know that $\mathcal{G}(q - q^*; y_\lambda) \in L^2(0,T; L^2(\Omega))$ and

$$\|\mathcal{G}(q - q^*; y_\lambda)\|_{L^2(0,T;L^2(\Omega))} \leq C_0 |q - q^*| \|y_\lambda\|_{S(0,T)} < C_1, \tag{3.22}$$

where C_i, $i = 0,1$ are positive constants.

By similar arguments in the proof of Theorem 3.1, multiplying the both sides of (3.20) by $-\Delta z'_\lambda$ and integrating it over $\Omega \times [0,t]$, we can obtain the following inequality:

$$|\Delta z_\lambda(t)|_2^2 + |\nabla z'_\lambda(t)|_2^2 + \int_0^t |\Delta z'_\lambda|_2^2 ds \leq K \|\mathcal{G}(q - q^*; y_\lambda)\|^2_{L^2(0,T;L^2(\Omega))} \tag{3.23}$$

for some $K > 0$. Therefore, combining (3.20) and (3.23), we can deduce that there exists a $z \in S(0,T)$ and a sequence $\{\lambda_k\} \subset [0,1]$ tending to 0 such that

$$z_{\lambda_k} \longrightarrow z \quad \text{weakly in } S(0,T). \tag{3.24}$$

By Theorem 3.1,

$$\nabla y_{\lambda_k} \longrightarrow \nabla y^* \quad \text{strongly in } \left[C\left([0,T]; L^2(\Omega)\right)\right]^n \text{ as } k \longrightarrow \infty \tag{3.25}$$

so that by (3.24) and by the compact imbedding theorem given in [16, pages 273–278], we can know that

$$z_{\lambda_k} \longrightarrow z \quad \text{strongly in } L^2\left(0,T; H_0^1(\Omega)\right) \text{ as } k \longrightarrow \infty. \tag{3.26}$$

Combining (3.25) and (3.26), we can have

$$\left(\nabla z_{\lambda_k}(t), \nabla y_{\lambda_k}(t)\right)_2 \longrightarrow \left(\nabla z(t), \nabla y^*(t)\right)_2 \quad \text{a.e. in } (0,T) \tag{3.27}$$

as $k \to \infty$. Therefore by Theorem 3.1, (3.27), and the Lebesgue-dominated convergence theorem we can verify that

$$(\nabla z_{\lambda_k}, \nabla y_{\lambda_k} + \nabla y^*)_2 \Delta y_{\lambda_k} \longrightarrow 2(\nabla z, \nabla y^*)_2 \Delta y^* \quad \text{strongly in } L^2\Big(0,T;L^2(\Omega)\Big) \tag{3.28}$$

as $k \to \infty$. At the same time, we can also verify that

$$\begin{aligned} &\mathcal{G}(q-q^*;y_{\lambda_k}) \longrightarrow \mathcal{G}(q-q^*;y^*) \\ &\quad \equiv (\alpha-\alpha^*)\Delta y^* + (\beta-\beta^*)|\nabla y^*|_2^2 \Delta y^* + (\gamma-\gamma^*)\Delta\frac{\partial y^*}{\partial t} \quad \text{strongly in } L^2\Big(0,T;L^2(\Omega)\Big) \end{aligned} \tag{3.29}$$

as $\lambda_k \to 0$. Hence we can see from (3.24) to (3.29) that $z_{\lambda_k} \to z = Dy(u)w$ weakly in $S(0,T)$ as $\lambda_k \to 0$ in which z is a strong solution of (3.18).

This convergency can be improved by showing the strong convergence of $\{z_\lambda\}$ in the strong topology of $S(0,T)$. Subtracting (3.18) from (3.20) and denoting $z_\lambda - z$ by ϕ_λ, we see that

$$\begin{aligned} \frac{\partial^2 \phi_\lambda}{\partial t^2} &- \Big(\alpha_0 + \alpha^* + \beta^*|\nabla y^*|_2^2\Big)\Delta\phi_\lambda - (\gamma_0+\gamma^*)\Delta\frac{\partial \phi_\lambda}{\partial t} \\ &= \mathcal{G}(q-q^*;y_\lambda) - \mathcal{G}(q-q^*;y^*) + \delta(y_\lambda,z_\lambda) \quad \text{in } Q, \\ &\phi_\lambda = 0 \quad \text{on } \Sigma, \\ &\phi_\lambda(0,x) = 0, \qquad \frac{\partial \phi_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \Omega, \end{aligned} \tag{3.30}$$

where

$$\delta(y_\lambda, z_\lambda) = \beta^*(\nabla z_\lambda, \nabla y_\lambda + \nabla y^*)_2 \Delta y_\lambda - 2\beta^*(\nabla z, \nabla y^*)_2 \Delta y^*. \tag{3.31}$$

Estimating ϕ_λ as in (3.23), we can easily deduce that

$$\begin{aligned} &|\nabla \phi_\lambda'(t)|_2^2 + |\Delta\phi_\lambda(t)|_2^2 + \int_0^t |\Delta\phi_\lambda'|_2^2 ds \\ &\quad \le C_2 \|\mathcal{G}(q-q^*;y_\lambda) - \mathcal{G}(q-q^*;y^*) + \delta(y_\lambda,z_\lambda)\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\quad \le 2C_2\Big(\|\mathcal{G}(q-q^*;y_\lambda) - \mathcal{G}(q-q^*;y^*)\|_{L^2(0,T;L^2(\Omega))}^2 + \|\delta(y_\lambda,z_\lambda)\|_{L^2(0,T;L^2(\Omega))}^2\Big), \end{aligned} \tag{3.32}$$

where C_2 is a positive constant. By virtue of (3.28), (3.29), and (3.30), we can deduce that

$$\begin{aligned} &\phi_\lambda \longrightarrow 0 \quad \text{in } C([0,T];D(\Delta)) \text{ as } \lambda \longrightarrow 0, \\ &\phi_\lambda' \longrightarrow 0 \quad \text{in } C\Big([0,T];H_0^1(\Omega)\Big) \cap L^2(0,T;D(\Delta)) \text{ as } \lambda \longrightarrow 0. \end{aligned} \tag{3.33}$$

Finally, by means of (3.30) and (3.33) it is followed that

$$z_\lambda(\cdot) \longrightarrow z(\cdot) \quad \text{strongly in } (0,T) \text{ as } \lambda \longrightarrow 0. \tag{3.34}$$

This completes the proof. □

3.1. Case of Distributive and Terminal Value Observations

The cost functional J_1 in (3.15) is represented by

$$J_1(q) = \int_0^T \left|y(q;t) - Y_1(t)\right|_2^2 dt + \left|y(q;T) - Y_1^T\right|_2^2, \quad q \in \mathcal{P}. \tag{3.35}$$

Then it is easily verified that the optimality condition (3.17) is written as

$$\int_0^T \left(y(q^*;t) - Y_1(t), Dy(q^*)(q-q^*)(t)\right)_2 dt + \left(y(q;T) - Y_1^T, Dy(q^*)(q-q^*)(T)\right)_2 \geq 0, \quad \forall q \in \mathcal{P}_{\text{ad}}, \tag{3.36}$$

where $q^* = (\alpha^*, \beta^*, \gamma^*)$ is the optimal parameter for (3.35), and $z = Dy(q^*)(q-q^*)$ is a solution of (3.18). The necessary condition for the optimal parameter $q^* = (\alpha^*, \beta^*, \gamma^*)$ is given in the following theorem.

Theorem 3.4. *The optimal parameter $q^* = (\alpha^*, \beta^*, \gamma^*)$ for (3.35) is characterized by the following system of equations and inequality:*

$$\begin{gathered}
\frac{\partial^2 y^*}{\partial t^2} - \left(\alpha_0 + \alpha^* + \beta^* \left|\nabla y^*\right|_2^2\right)\Delta y^* - (\gamma_0 + \gamma^*)\Delta \frac{\partial y^*}{\partial t} = f \quad \text{in } Q, \\
y^* = 0 \quad \text{on } \Sigma, \\
y^*(0,x) = y_0(x), \qquad \frac{\partial y^*}{\partial t}(0,x) = y_1(x) \quad \text{in } \Omega, \\
\frac{\partial^2 p}{\partial t^2} - \left(\alpha_0 + \alpha^* + \beta^* \left|\nabla y^*\right|_2^2\right)\Delta p - 2\beta^*(\nabla p, \nabla y^*)_2 \Delta y^* + (\gamma_0 + \gamma^*)\Delta \frac{\partial p}{\partial t} = y^* - Y_1 \quad \text{in } Q, \\
p = 0 \quad \text{on } \Sigma, \\
p(T,x) = 0, \qquad \frac{\partial p}{\partial t}(T,x) = -y^*(T) + Y_1^T \quad \text{in } \Omega,
\end{gathered} \tag{3.37}$$

$$\int_Q p\mathcal{G}(q-q^*; y^*)\,dx\,dt \geq 0, \quad \forall q = (\alpha, \beta, \gamma) \in \mathcal{P}_{ad}. \tag{3.38}$$

Proof. Since $y^* - Y_1 \in L^2(0,T;L^2(\Omega))$ and $y^*(T) - Y_1^T \in L^2(\Omega)$, it is verified by the time reversion $t \to T-t$, and there is a unique weak solution $p \in W(0,T)$ of (3.37) (cf. [15, pages 558–574]).

Multiplying both sides of the weak form of (3.37) by $z = Dy(q^*)(q-q^*)$ and integrating it by parts on $[0,T]$, we have that

$$\begin{aligned}
&\int_0^T (y^* - Y_1, z)_2 dt \\
&\quad = \int_0^T \left(p'' - \left(\alpha_0 + \alpha^* + \beta^* |\nabla y^*|_2^2\right)\Delta p - 2\beta^* (\nabla p, \nabla y^*)_2 \Delta y^* + (\gamma_0 + \gamma^*)\Delta p', z\right)_2 dt \\
&\quad = \int_0^T \left(p, z'' - \left(\alpha_0 + \alpha^* + \beta^* |\nabla y^*|_2^2\right)\Delta z - 2\beta^* (\nabla z, \nabla y^*)_2 \Delta y^* - (\gamma_0 + \gamma^*)\Delta z'\right)_2 dt \\
&\qquad - \left(y(q;T) - Y_1^T, Dy(q^*)(q-q^*)(T)\right)_2 \\
&\quad = \int_0^T (p, \mathcal{G}(q-q^*; y^*))_2 dt - \left(y(q;T) - Y_1^T, Dy(q^*)(q-q^*)(T)\right)_2.
\end{aligned} \tag{3.39}$$

Therefore, (3.39) and (3.36) imply that the required optimality condition (3.36) is equivalent to the condition (3.38). This proves Theorem 3.4. □

3.2. *Case of Velocity Observations*

The cost functional J_2 in (3.16) is represented by

$$J_2(q) = \int_0^T |y'(q;t) - Y_2(t)|_2^2 dt, \quad q \in \mathcal{P}. \tag{3.40}$$

The optimality condition (3.17) for (3.40) is given by

$$\int_0^T \left(y'(q^*;t) - Y_2(t), Dy(q^*)(q-q^*)'(t)\right)_2 dt \geq 0, \quad \forall q \in \mathcal{P}_{\text{ad}}, \tag{3.41}$$

where $z = Dy(q^*)(q-q^*)$ is a solution of (3.18).

Remark 3.5. As indicated in [13], if we derive a formal second-order adjoint system of this quasilinear system related to the velocity observation with the cost (3.40), then it is hard to explain whether it is well-posed or not. In order to overcome this difficulty, we follow the idea given in Hwang and Nakagiri [17] in which it is adopted that the first-order integrodifferential system as an appropriate adjoint-system of a quasilinear system instead of the formal second-order adjoint system.

For this reason, we introduce an adjoint-system represented by the following first-order integrodifferential equation:

$$\frac{\partial p}{\partial t}+\int_t^T\left\{\left(\alpha_0+\alpha^*+\beta^*|\nabla y^*|_2^2\right)\Delta p+2\beta^*(\nabla p,\nabla y^*)_2\Delta y^*\right\}ds+(\gamma_0+\gamma^*)\Delta p=\frac{\partial y^*}{\partial t}-Y_2 \quad \text{in } Q,$$
$$p=0 \quad \text{on } \Sigma,$$
$$p(T,x)=0 \quad \text{in } \Omega. \tag{3.42}$$

Since $(\partial y^*/\partial t)-Y_2\in L^2(Q)=L^2(0,T;L^2(\Omega))$, by reversing the direction of time $t\to T-t$ and applying the result of [15, pages 656–662] to the problem (3.42), we can assert that (3.42) admits a unique weak solution p satisfying

$$p \in W\left(H_0^1(\Omega),L^2(\Omega)\right)\cap C\left([0,T];H_0^1(\Omega)\right), \tag{3.43}$$

where the solution space $W(H_0^1(\Omega),L^2(\Omega))$ is defined by

$$W\left(H_0^1(\Omega),L^2(\Omega)\right)=\left\{g\mid g\in L^2\left(0,T;H_0^1(\Omega)\right),\ g'\in L^2\left(0,T;L^2(\Omega)\right)\right\}. \tag{3.44}$$

Theorem 3.6. *The optimal parameter* $q^*=(\alpha^*,\beta^*,\gamma^*)$ *for* (3.40) *is characterized by the following system of equations and inequality:*

$$\frac{\partial^2 y^*}{\partial t^2}-\left(\alpha_0+\alpha^*+\beta^*|\nabla y^*|_2^2\right)\Delta y^*-(\gamma_0+\gamma^*)\Delta\frac{\partial y^*}{\partial t}=f \quad \textit{in } Q,$$
$$y^*=0 \quad \textit{on } \Sigma,$$
$$y^*(0,x)=y_0(x),\quad \frac{\partial y^*}{\partial t}(0,x)=y_1(x) \quad \textit{in } \Omega,$$
$$\frac{\partial p}{\partial t}+\int_t^T\left\{\left(\alpha_0+\alpha^*+\beta^*|\nabla y^*|_2^2\right)\Delta p+2\beta^*(\nabla p,\nabla y^*)_2\Delta y^*\right\}ds+(\gamma_0+\gamma^*)\Delta p=\frac{\partial y^*}{\partial t}-Y_2 \quad \textit{in } Q,$$
$$p=0 \quad \textit{on } \Sigma,$$
$$p(T,x)=0 \quad \textit{in } \Omega, \tag{3.45}$$

$$\int_Q p\,\mathcal{G}(q-q^*;y^*)\,dx\,dt\le 0,\quad \forall q=(\alpha,\beta,\gamma)\in\mathcal{P}_{ad}. \tag{3.46}$$

Proof. Multiplying both sides of the weak form of (3.45) by $z' = Dy(q^*)(q - q^*)'$, taking dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and integrating it by parts on $[0,T]$, we have that

$$\begin{aligned}
&\int_0^T (y^{*\prime} - Y_2, z')_2 dt \\
&\quad = \int_0^T \left\langle p' + \int_t^T \left\{ \left(\alpha_0 + \alpha^* + \beta^* |\nabla y^*|_2^2\right) \Delta p + 2\beta^* (\nabla p, \nabla y^*)_2 \Delta y^* \right\} ds + (\gamma_0 + \gamma^*) \Delta p, z' \right\rangle dt \\
&\quad = -\int_0^T \left(p, z'' - \left(\alpha_0 + \alpha^* + \beta^* |\nabla y^*|_2^2\right) \Delta z - 2\beta^* (\nabla z, \nabla y^*)_2 \Delta y^* - (\gamma_0 + \gamma^*) \Delta z'\right)_2 dt \\
&\quad = -\int_0^T (p, \mathcal{G}(q - q^*; y^*))_2 dt.
\end{aligned} \tag{3.47}$$

Thus, (3.47) and (3.41) imply that the required optimality condition is given by (3.46). □

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