Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 971593, 10 pages
doi:10.1155/2012/971593

Research Article

Reproducing Kernel Space Method for the Solution of Linear Fredholm Integro-Differential Equations and Analysis of Stability

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Received 2 February 2012; Accepted 7 March 2012

Academic Editor: Ljubisa Kocinac

We present a numerical method to solve the linear Fredholm integro-differential equation in reproducing kernel space. A simple algorithm is given to obtain the approximate solutions of the equation. Through the comparison of approximate and true solution, we can find that the method can effectively solve the linear Fredholm integro-differential equation. At the same time the numerical solution of the equation is stable.

1. Introduction

In this paper, we consider the following first-order Fredholm type integro-differential equation:

$$u'(t) = q(t)u(t) + \int_0^1 h(t,s)u(s)ds + f(t), \quad t \in [0,1], \tag{1.1}$$

with the initial condition

$$u(0) = 0. \tag{1.2}$$

The equation is discussed by Yusufoğlu in [1], where $q(t)$, $h(t,s)$, and $f(t)$ are sufficiently regular given functions. Integro-differential system is an important tool in solving real-world problems. A wide variety of natural phenomena are modelled by Fredholm type integro-differential equations. The ordinary integro-differential system has been applied to many problems in fluid dynamics, engineering, chemical reactions, and so on.

In the recent years, there are some methods to solve the Fredholm type integro-differential equations [2–8]. At this point, a new method is presented to solve the integro-differential equations. The method is established in reproducing kernel space; the problem of solving the integro-differential problem with a perturbation can be converted into the simple problem of solving the equation. The representation of all the solutions for Fredholm type integro-differential equation is given if it has solutions. The stability is important and references are there in [9, 10]. There are many discussions about the solutions in reproducing kernel space in [11–15]. In this paper, we discuss the Fredholm type integro-differential equation. In the last section, CAS wavelet approximating methods [5], differential transformation methods [6], HPM [1], and reproducing kernel space method are compared, then we can get some effective data. The numerical experiments show that this kind of method is stable in the reproducing kernel space.

2. Two Reproducing Kernel Spaces

2.1. The Reproducing Kernel Space $W_2^2[0,1]$

The reproducing kernel space $W_2^2[0,1]$ is defined as follows:

$$W_2^2[0,1] = \{u(t) \mid u'(t) \text{ is absolutely continuous function in } [0,1],\ u''(t) \in \mathcal{L}^2[0,1]\}. \tag{2.1}$$

The inner product and norm in $W_2^2[0,1]$ are defined respectively by

$$\langle u(t), v(t)\rangle = \sum_{i=0}^{1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u''(t)v''(t)dt, \qquad u, v \in W_2^2[0,1],$$
$$\|u\|_{W_2^2[0,1]} = \langle u(t), u(t)\rangle^{1/2}, \qquad u \in W_2^2[0,1]. \tag{2.2}$$

Then $W_2^2[0,1]$ is a complete reproducing kernel space. That is, there exists a function $R_t(s)$, for each fixed $t \in [0,1]$, $R_t(s) \in W_2^2[0,1]$, and for any $u(s) \in W_2^2[0,1]$ and $s \in [0,1]$, satisfying

$$\langle u(s), R_t(s)\rangle_{W_2^2} = u(t). \tag{2.3}$$

By using Mathematica, $R_t(s)$ is given by

$$R_t(s) = \begin{cases} -\dfrac{1}{6}s\left[s^2 - 3t(2+s)\right], & t \le s, \\ -\dfrac{1}{6}t^3 + \dfrac{1}{2}t(2+t)s, & t > s. \end{cases} \tag{2.4}$$

2.2. The Reproducing Kernel Space $W_2^1[0,1]$

The construction of reproducing kernel space $W_2^1[0,1]$ can be found in [14] and its reproducing kernel function is

$$Q_s(t) = \begin{cases} 1+t, & t \le s, \\ 1+s, & t > s. \end{cases} \tag{2.5}$$

3. Analysis of the Solution of (1.1)

Let $\mathcal{L} : W_2^2[0,1] \to W_2^1[0,1]$, such that

$$\mathcal{L}u(t) = u'(t) - q(t)u(t) - \int_0^1 h(t,s)u(s)ds, \tag{3.1}$$

where $u(t) \in W_2^2[0,1]$, it is easy to know that $\mathcal{L}$ is a linear bounded operator and (1.1) can be converted into the equivalent form

$$\mathcal{L}u(t) = f(t), \quad t \in [0,1]. \tag{3.2}$$

In order to obtain the representation of all the solutions of (1.1), let $\varphi_i(t) = Q_{t_i}(t)$, $\psi_i(t) = \mathcal{L}^*\varphi_i(t)$, where $\{t_i\}_{i=1}^{\infty}$ is dense in $[0,1]$.

From the definition of the reproducing kernel, we have

$$\begin{aligned} \langle u(t), \varphi_i(t) \rangle_{W_2^1} &= u(t_i) , \\ \psi_i(t) &= (\mathcal{L}^*\varphi_i(s))(t) \\ &= \langle (\mathcal{L}^*Q_{t_i}(s))(a), R_t(a) \rangle_{W_2^2} \\ &= \langle Q_{t_i}(s), (\mathcal{L}R_t(a))(s) \rangle_{W_2^1} \\ &= (\mathcal{L}R_t(a))(t_i), \quad i = 1,2,\ldots, \end{aligned} \tag{3.3}$$

where $\mathcal{L}^*$ is a conjugate operator of $\mathcal{L}$. Practise Gram-Schmidt orthonormalization for $\{\psi_i(t)\}_{i=1}^{\infty}$

$$\overline{\psi}_i(t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(t), \tag{3.4}$$

where β_{ik} are coefficients of Gram-Schmidt orthonormalization and $\{\overline{\psi}_i(t)\}_{i=1}^{\infty}$ is orthonormal system in $W_2^2[0,1]$.

Theorem 3.1. *If* (3.2) *has solutions, the results are proved by the following formula:*

$$u(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t). \tag{3.5}$$

Proof. The results are proved by the following formula:

$$\begin{aligned}
u(t) &= \sum_{i=1}^{\infty} \langle u(t), \overline{\varphi}_i(t) \rangle_{W_2^2} \overline{\varphi}_i(t) = \sum_{i=1}^{\infty} \left\langle u(t), \sum_{k=1}^{i} \beta_{ik} \varphi_k(t) \right\rangle_{W_2^2} \overline{\varphi}_i(t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(t), \varphi_k(t) \rangle_{W_2^2} \overline{\varphi}_i(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(t), \mathcal{L}^* \varphi_k(t) \rangle_{W_2^2} \overline{\varphi}_i(t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle \mathcal{L} u(t), \varphi_k(t) \rangle_{W_2^1} \overline{\varphi}_i(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathcal{L} u(t_k) \overline{\varphi}_i(t) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t).
\end{aligned} \tag{3.6}$$

□

Now, the approximate solution $u_n(t)$ can be obtained by the n-term truncation of (3.5), that is

$$u_n(t) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t). \tag{3.7}$$

Theorem 3.2. *Assume $u(t)$ is the solution of* (3.2) *and $r_n(t)$ is the error between the approximate solution $u_n(t)$ and the exact solution $u(t)$. Then $\|r_n(t)\|_{W_2^2} \to 0$.*

Proof. In the following we prove the sequence $r_n(t)$ is monotone decreasing in the sense of $\|\cdot\|_{W_2^2}$.

From (3.5) and (3.7), we have

$$\begin{aligned}
\|r_n\|_{W_2^2} &= \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t) \right\|_{W_2^2} = \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} f(t_k) \right)^2, \\
\|r_{n-1}\|_{W_2^2} &= \left\| \sum_{i=n}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t) \right\|_{W_2^2} = \sum_{i=n}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} f(t_k) \right)^2.
\end{aligned} \tag{3.8}$$

By (3.8), we know that $\|r_n\|_{W_2^2} \le \|r_{n-1}\|_{W_2^2}$. Then $\{r_n(t)\}_{n=1}^{\infty}$ is monotone decreasing in $\|\cdot\|_{W_2^2}$. Since the series $\sum_{i=1}^{\infty} \langle u(t), \overline{\varphi}_i(t) \rangle_{W_2^2} \overline{\varphi}_i(t)$ is convergent in $W_2^2[0,1]$, we obtain $\|r_n(t)\|_{W_2^2} \to 0$. □

4. The Stability of the Approximate Solution

In order to consider the stability of the approximate solution, we add a perturbation $\varepsilon(t)$ in the right-hand side, then (3.2) becomes

$$\mathcal{L}u(t) = f(t) + \varepsilon(t), \quad t \in [0,1]. \tag{4.1}$$

Now, we discuss the representation of the solution for (4.1).

4.1. Representation of All the Solutions of (4.1)

In order to study the stability of (4.1), let $\mathcal{A}$ be a projection operator from $W_2^2[0,1]$ to Ψ,

$$\Psi = \left\{ u \mid u = \sum_{i=1}^{\infty} c_i \overline{\varphi}_i, \text{ for } \{c_i\}_{i=1}^{\infty} \in l^2 \right\}, \tag{4.2}$$

where $\{\overline{\varphi}_i\}_{i=1}^{\infty}$ is given in (3.4), and $\mathcal{A}$ satisfies $\mathcal{A}^* = \mathcal{A}$. Moreover

$$\mathcal{A}\varphi_i(s) = \sum_{k=1}^{\infty} \langle \varphi_i(s), \overline{\varphi}_k(s) \rangle \overline{\varphi}_k(s) = \varphi_i(s). \tag{4.3}$$

We have

$$\mathcal{A}u(t) = \sum_{i=1}^{\infty} \langle u(t), \overline{\varphi}_i(t) \rangle \overline{\varphi}_i(t), \tag{4.4}$$

where $u(t)$ is a solution of (3.2) in $W_2^2[0,1]$.

Define

$$u_{\mathcal{A}}(t) = \mathcal{A}u(t), \tag{4.5}$$

then we have

$$u_{\mathcal{A}}(t) = \sum_{i=1}^{\infty} \langle u(t), \overline{\varphi}_i(t) \rangle \overline{\varphi}_i(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\varphi}_i(t). \tag{4.6}$$

Theorem 4.1. *If* $\{t_i\}_{i=1}^{\infty}$ *is dense in* $[0,1]$, *then the form* (4.6) *is the solution of* (3.2) *in* Ψ.

Proof. We have

$$\begin{aligned} \mathcal{L}u_{\mathcal{A}}(t_i) &= \left\langle \mathcal{L}u_{\mathcal{A}}(t), \varphi_i(t) \right\rangle = \left\langle u_{\mathcal{A}}(t), \mathcal{L}^*\varphi_i(t) \right\rangle = \left\langle \mathcal{A}u(t), \psi_i(t) \right\rangle \\ &= \left\langle u(t), \mathcal{A}^*\psi_i(t) \right\rangle = \left\langle u(t), \mathcal{A}\psi_i(t) \right\rangle \\ &= \left\langle u(t), \psi_i(t) \right\rangle = \left\langle u(t), \mathcal{L}^*\varphi_i(t) \right\rangle \\ &= \left\langle \mathcal{L}u(t), \varphi_i(t) \right\rangle = \left\langle f(t), \varphi_i(t) \right\rangle = f(t_i), \end{aligned} \tag{4.7}$$

since $\{t_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, then $\mathcal{L}u_{\mathcal{A}}(t) = f(t)$.

It is easy to know that the solution of (3.2) is unique in Ψ (see [16]). □

The following lemma holds.

Lemma 4.2. $\Psi^{\perp} = N(\mathcal{L})$*, where* $\Psi^{\perp} = \{u(t) \mid \langle u(t), v(t) \rangle = 0,$ *for any* $v(t) \in \Psi\}$ *and* $N(\mathcal{L})$ *is a null space of* $\mathcal{L}$*, that is* $N(\mathcal{L}) = \{u \mid \mathcal{L}u = 0\}$*.*

Proof. For any $u(t) \in \Psi^{\perp}$,

$$0 = \left\langle u(t), \psi_k(t) \right\rangle = \left\langle u(t), \mathcal{L}^*\varphi_k(t) \right\rangle = \left\langle \mathcal{L}u(t), \varphi_k(t) \right\rangle = \mathcal{L}u(t_k). \tag{4.8}$$

Since $\{t_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, then $\mathcal{L}u(t) \equiv 0$. We can obtain $u(t) \in N(\mathcal{L})$. Obviously, $N(\mathcal{L}) \subset \Psi^{\perp}$. □

Then $\Psi^{\perp} = N(\mathcal{L})$.

The following theorem is obvious.

Theorem 4.3. *Let* $\{t_i\}_{i=1}^{\infty}$ *be any dense subset of* $[0,1]$*, if* (4.1) *has solutions, then its solution* $\widetilde{u}(t)$ *can be represented as follows:*

$$\widetilde{u}(t) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} f(t_k) \overline{\psi}_i(t) + \tau(t), \tag{4.9}$$

where $\tau(t) \in N(\mathcal{L})$*.*

Assume that $\{\sigma_i\}_{i=1}^{\infty}$ is a basis of $N(\mathcal{L})$. Orthonormalzing $\{\overline{\psi}_1, \overline{\psi}_2, \ldots, \sigma_1, \sigma_2, \ldots\}$, we obtain

$$\overline{\sigma}_i = \sum_{k=1}^{\infty} \beta_{ik} \overline{\psi}_k(t) + \sum_{j=1}^{i} \beta_{ij} \sigma_j, \quad i = 1, 2, \ldots. \tag{4.10}$$

Hence, $\{\overline{\psi}_1, \overline{\psi}_2, \ldots, \overline{\sigma}_1, \overline{\sigma}_2, \ldots\}$ is a normal orthogonal basis in $W_2^2[0,1]$.

According to Theorem 4.3, we can obtain the solution of (4.1). Hence, through the Gram-Schmidt process, we have $\{\overline{\psi}_i\}_{i=1}^{\infty} \cup \{\overline{\sigma}_i\}_{i=1}^{\infty}$ are the complete orthonormal system in $W_2^2[0,1]$, and $\{\overline{\sigma}_i\}_{i=1}^{\infty}$ is a complete orthonormal system in $N(\mathcal{L})$.

4.2. The Stability of the Solution for (3.2) *in Reproducing Kernel Space*

Let the space Ψ be complete. $\mathcal{L}_\Psi$ be a restricted operator of $\mathcal{L}$ in Ψ, we have the converse operator $\mathcal{L}_\Psi^{-1} : W_2^2[0,1] \to \Psi$ exists and is bounded.

Lemma 4.4. *If $u_{\mathcal{A}}(t)$ is given by* (4.6)*, then $u_{\mathcal{A}}(t)$ is the minimal norm solution.*

Proof. Let $u(t)$ be a solution of (3.2). We have

$$u(t) = u_{\mathcal{A}}(t) + v(t), \tag{4.11}$$

where $u_{\mathcal{A}}(t) \in \Psi$ and $v(t) \in \Psi^\perp$. The following

$$\|u\|^2 = \langle u_{\mathcal{A}} + v, u_{\mathcal{A}} + v\rangle = \|u_{\mathcal{A}}\|^2 + \langle u_{\mathcal{A}}, v\rangle + \langle v, u_{\mathcal{A}}\rangle + \|v\|^2 = \|u_{\mathcal{A}}\|^2 + \|v\|^2 \geq \|u_{\mathcal{A}}\|^2 \tag{4.12}$$

holds.

It is pointed that $u_{\mathcal{A}}$ is the minimal norm solution of (3.2). □

Theorem 4.5. *If the* (4.1) *has solutions and let $u_{\mathcal{A}}(t)$ be the minimal norm solution, then*

$$\|u_{\mathcal{A}}(t) - u_{\mathcal{A},n}(t)\| \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{4.13}$$

where $u_{\mathcal{A},n}(t)$ is a truncation of $u_{\mathcal{A}}(t)$. Hence, $u_{\mathcal{A}}(t)$ is stable in $W_2^2[0,1]$.

Proof. Let $\mathcal{L}_\Psi u_{\mathcal{A},n}(t) = f_n(t)$ and $f(t) = f_n(t) + \varepsilon_n(t)$, where $\varepsilon_n(t)$ is a perturbation and $\varepsilon_n(t) \to 0$ $(n \to \infty)$ in $\|\cdot\|_{W_2^2}$.

On the one hand, since $\mathcal{L}_\Psi^{-1}\varepsilon_n(t) \in \Psi$, $\mathcal{L}_\Psi^*\varphi_k = \psi_k$, it follows that

$$\begin{aligned}
\mathcal{L}_\Psi^{-1}\varepsilon_n(t) &= \sum_{i=1}^{\infty}\left\langle \mathcal{L}_\Psi^{-1}\varepsilon_n(t), \overline{\psi}_i(t)\right\rangle\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\left\langle \mathcal{L}_\Psi^{-1}\varepsilon_n(t), \sum_{k=1}^{i}\beta_{ik}\psi_k(t)\right\rangle\overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle \mathcal{L}_\Psi^{-1}\varepsilon_n(t), \psi_k(t)\right\rangle\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle \varepsilon_n(t), \mathcal{L}_\Psi^{-1*}\psi_k(t)\right\rangle\overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle \varepsilon_n(t), \mathcal{L}_\Psi^{-1*}\mathcal{L}_\Psi^*\varphi_k(t)\right\rangle\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle \varepsilon_n(t), \varphi_k(t)\rangle\overline{\psi}_i(t) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\varepsilon_n(t_k)\overline{\psi}_i(t).
\end{aligned} \tag{4.14}$$

On the other hand, $f, f_n \in W_2^1[0,1]$, from the form (4.6), we have

$$u_{\mathcal{A}}(t) - u_{\mathcal{A},n}(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left[f(t) - f_n(t)\right]\overline{\psi}_i(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\varepsilon_n(t_k)\overline{\psi}_i(t), \tag{4.15}$$

Table 1: Comparison of absolute errors of approximate solutions, for example, obtained by reproducing kernel space methods, CAS wavelet approximating [5], differential transformation [6] methods and HPM [1].

Node	Method in [5]	Method in [6]	Method in [1]	Reproducing kernel methods
0.1	$1.34917637e-03$	$1.00118319e-02$	$2.304814815e-04$	$4.77378e-05$
0.2	$1.15960044e-03$	$2.78651355e-02$	$9.259259259e-04$	$4.80048e-05$
0.3	$5.67152531e-03$	$5.08730892e-02$	$2.083333333e-03$	$4.97945e-05$
0.4	$5.93405645e-02$	$7.55356316e-02$	$3.703703704e-03$	$5.31631e-05$
0.5	$1.32330751e-02$	$9.71888592e-02$	$5.787037037e-03$	$5.81740e-05$
0.6	$4.39287720e-02$	$1.09551714e-01$	$8.333333333e-03$	$6.48985e-05$
0.7	$1.41201624e-02$	$1.04133232e-01$	$1.134259259e-02$	$7.34170e-05$
0.8	$1.34514117e-02$	$6.94512700e-02$	$1.481481481e-02$	$8.38198e-05$
0.9	$1.32045209e-02$	$1.00034260e-02$	$1.875000000e-02$	$9.62085e-05$

where

$$u_{\mathcal{A},n}(t) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik} f_n(t_k)\overline{\psi}_i(t), \tag{4.16}$$

and then

$$u_{\mathcal{A}}(t) - u_{\mathcal{A},n}(t) = \mathcal{L}_{\Psi}^{-1}\varepsilon_n(t). \tag{4.17}$$

From the continuity of $\mathcal{L}_{\Psi}^{-1}$ and $\varepsilon_n(t) \to 0$ $(n \to \infty)$ in $\|\cdot\|_{W_2^2}$, we have

$$\lim_{n\to\infty}\|u_{\mathcal{A}}(t) - u_{\mathcal{A}n}(t)\|_{W_2^2} \le \left\|\mathcal{L}_{\Psi}^{-1}\right\|_{W_2^2}\lim_{n\to\infty}\|\varepsilon_n(t)\|_{W_2^2} = 0. \tag{4.18}$$

□

5. Numerical Experiments

To illustrate the effectiveness of the above method, we give an example as follows:

Example 5.1. Consider the following first-order Fredholm type integro-differential equation (see [1]):

$$\begin{gathered} u'(t) = (t+1)e^t - t + \int_0^1 tu(s)ds, \quad t \in [0,1], \\ u(0) = 0. \end{gathered} \tag{5.1}$$

The analytic solution of this equation is $u(t) = te^t$.

From Tables 1, 2, and 3, we can see that the absolute error, relative error and the root-mean-square are small.

Table 2: Relative error of approximate solutions with $\varepsilon = 0.0001$.

Node	Approximate solution $u_{100}(t)$	True solution $u(t)$	Relative error
0.1	0.110575	0.110517	$5.25129e-4$
0.2	0.244350	0.244281	$2.83292e-4$
0.3	0.405040	0.404958	$2.03706e-4$
0.4	0.596828	0.596730	$1.64163e-4$
0.5	0.824476	0.824361	$1.40317e-4$
0.6	1.093410	1.093270	$1.24119e-4$
0.7	1.409790	1.409630	$1.12167e-4$
0.8	1.780620	1.780430	$1.02793e-4$
0.9	2.213850	2.213640	$9.50945e-5$
1	2.718520	2.718280	$8.85463e-5$

Table 3: The root-mean-square, for example, with $\varepsilon = 0.0001$.

$\sqrt{\sum_{i=1}^{100} \frac{[\tilde{u}(0.01i) - \tilde{u}_n(0.01i)]^2}{100}}$	$1.38171e-04$
$\sqrt{\sum_{i=1}^{100} \frac{[\tilde{u}'(0.01i) - \tilde{u}'_n(0.01i)]^2}{100}}$	$1.67175e-04$

6. Conclusions

From the previous numerical results, we can see that the error is quite small and the numerical solution is stable when the right-hand side with a small perturbation. It illustrates that the method is given in the paper is valid.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (61071181), the Excellent Middle-Young Scientific Research Award Foundation of Heilongjiang Province (QC2010036), academic foundation for youth of Harbin Normal University (10KXQ-05), and science and technology development project of Harbin Normal University (08XYG-13).

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