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Research Article

The Intuitionistic Fuzzy Normed Space of Coefficients

B. T. Bilalov, S. M. Farahani, and F. A. Guliyeva

Institute of Mathematics and Mechanics of Azerbaijan, National Academy of Sciences, 1141 Baku, Azerbaijan

Correspondence should be addressed to B. T. Bilalov, bilalov.bilal@gmail.com

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Intuitionistic fuzzy normed space is defined using concepts of t-norm and t-conorm. The concepts of fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy basicity, and fuzzy space of coefficients are introduced. Strong completeness of fuzzy space of coefficients with regard to fuzzy norm and strong basicity of canonical system in this space are proved. Strong basicity criterion in fuzzy Banach space is presented in terms of coefficient operator.

1. Introduction

The fuzzy theory, dating back to Zadeh [1], has emerged as the most active area of research in many branches of mathematics and engineering. Fuzzy set theory is a powerful handset for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. The concept of fuzzy topology may have very important applications in quantum particle physics, particularly in connection with both string and $\varepsilon^{(\infty)}$ theories introduced and studied by El Naschie [2–4] and further developed in [5]. So, further development in E-Infinity may lead to a set transitional resolution of quantum entanglement [6]. A large number of research works are appearing these days which deal with the concept of fuzzy set numbers, and the fuzzification of many classical theories has also been made. The concept of Schauder basis in *intuitionistic fuzzy normed space* and some results related to this concept have recently been studied in [7–9]. These works introduced the concepts of strongly and weakly intuitionistic fuzzy (Schauder) basis in *intuitionistic fuzzy Banach spaces* (IFBS in short). Some of their properties are revealed. The concepts of *strongly* and *weakly intuitionistic fuzzy approximation properties* (*sif-AP* and *wif-AP* in short, resp.) are also introduced in these works. It is proved that if the *intuitionistic fuzzy space* has a *sif-basis*, then it has a *sif-AP*.

All the results in these works are obtained on condition that IFBS admits equivalent topology using the family of norms generated by *t-norm* and *t-conorm* (we will define them later).

In our work, we define the basic concepts of classical basis theory in *intuitionistic fuzzy normed spaces* (IFNS in short). Concepts of *weakly* and *strongly fuzzy spaces of coefficients* are introduced. *Strong completeness* of these spaces with regard to *fuzzy norm* and *strong basicity* of canonical system in them is proved. *Strong basicity* criterion in fuzzy Banach space is presented in terms of *coefficient operator*.

In Section 2, we recall some notations and concepts. In Section 3, we state our main results. We first define the *fuzzy space of coefficients* and then introduce the corresponding *fuzzy norms*. We prove that for nondegenerate system the corresponding *fuzzy space of coefficients* is *strongly fuzzy complete.* Moreover, we show that the canonical system forms a *strong basis* for this space.

2. Some Preliminary Notations and Concepts

We will use the usual notations: N will denote the set of all positive integers, R will be the set of all real numbers, C will be the set of complex numbers, and K will denote a field of scalars ($K \equiv R$, or $K \equiv \mathrm{C}$), $R_+ \equiv (0, +\infty)$. We state some concepts and facts from IFNS theory to be used later.

One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy normed space. This problem has been investigated by Park [10]. He has introduced and studied a notion of intuitionistic fuzzy metric space. We recall it.

Definition 2.1. A binary operation $* : [0,1]^2 \to [0,1]$ is a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative,

(b) $*$ is continuous,

(c) $a * 1 = a, \forall a \in [0,1]$,

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d, \forall a, b, c, d \in [0,1]$.

Example 2.2. Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min\{a; b\}$.

Definition 2.3. A binary operation $\diamond : [0,1]^2 \to [0,1]$ is a continuous t-conorm if it satisfies the following conditions:

(α) $\diamond$ is associative and commutative,

(β) $\diamond$ is continuous,

(γ) $a \diamond 0 = a, \forall a \in [0,1]$,

(η) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d, \forall a, b, c, d \in [0,1]$.

Example 2.4. Two typical examples of continuous t-conorm are $a \diamond b = \min\{a + b; 1\}$ and $a \diamond b = \max\{a; b\}$.

Definition 2.5. Let X be a linear space over a field K. Functions $\mu;\nu : X \times R \to [0,1]$ are called fuzzy norms on X if they hold the following conditions:

(1) $\mu(x;t) = 0, \forall t \leq 0, \forall x \in X$,

(2) $\mu(x;t) = 1, \forall t > 0 \Rightarrow x = 0$,

(3) $\mu(cx;t) = \mu(x;t/|c|), \forall c \neq 0$,

(4) $\mu(x;\cdot) : R \to [0,1]$ is a nondecreasing function of t for $\forall x \in X$ and $\lim_{t\to\infty}\mu(x;t) = 1, \forall x \in X$,

(5) $\mu(x;s) * \mu(y;t) \leq \mu(x+y;s+t), \forall x, y \in X, \forall s, t \in R$,

(6) $\nu(x;t) = 1, \forall t \leq 0, \forall x \in X$,

(7) $\nu(x;t) = 0, \forall t < 0 \Rightarrow x = 0$,

(8) $\nu(cx;t) = \nu(x;t/|c|), \forall c \neq 0$,

(9) $\nu(x;\cdot) : R \to [0,1]$ is a nonincreasing function of t for $\forall x \in X$ and $\lim_{t\to\infty}\nu(x;t) = 0, \forall x \in X$,

(10) $\nu(x;s) \diamond \nu(y;t) \geq \nu(x+y;s+t), \forall x, y \in X, \forall s,t \in R$,

(11) $\mu(x;t) + \nu(x;t) \leq 1, \forall x \in X, \forall t \in R$.

Then the 5-tuple $(X;\mu;\nu;*;\diamond)$ is said to be an intuitionistic fuzzy normed space (shortly IFNS).

Example 2.6. Let $(X;\|\cdot\|)$ be a normed space. Denote $a * b = ab$ and $a \diamond b = \min\{a+b;1\}$, for $\forall a,b \in [0,1]$, and define μ and ν as follows:

$$\mu(x;t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$
$$\nu(x;t) = \begin{cases} \dfrac{t}{t+\|x\|}, & t > 0, \\ 1, & t \leq 0. \end{cases} \tag{2.1}$$

Then $(X;\mu;\nu;*;\diamond)$ is an IFNS.

The above concepts allow to introduce the following kinds of convergence (or topology) in IFNS.

Definition 2.7. Let $(X;\mu;\nu)$ be a fuzzy normed space, and let $\{x_n\}_{n\in N} \subset X$ be some sequence, then it is said to be strongly intuitionistic fuzzy convergent to $x \in X$ (denoted by $x_n \xrightarrow{s} x$, $n \to \infty$ or $s\text{-}\lim_{n\to\infty}x_n = x$ in short) if and only if for $\forall\varepsilon > 0, \exists n_0 = n_0(\varepsilon) : \mu(x_n - x;t) \geq 1-\varepsilon$, $\nu(x_n - x;t) \leq \varepsilon, \forall n \geq n_0, \forall t \in R$.

Definition 2.8. Let $(X;\mu;\nu)$ be a fuzzy normed space, and let$\{x_n\}_{n\in N} \subset X$ be some sequence, then it is said to be weakly intuitionistic fuzzy convergent to $x \in X$ (denoted by $x_n \xrightarrow{w} x$, $n \to \infty$, or $w\text{-}\lim_{n\to\infty}x_n = x$ in short) if and only if for $\forall t \in R_+, \forall\varepsilon > 0, \exists n_0 = n_0(\varepsilon;t) : \mu(x_n - x;t) \geq 1-\varepsilon$, $\nu(x_n - x;t) \leq \varepsilon, \forall n \geq n_0$. More details on these concepts can be found in [10–19].

Let $(X;\mu;\nu)$ be an IFNS, and let $M \subset X$ be some set. By $L[M]$, we denote the linear span of M in X. The weakly (strongly) intuitionistic fuzzy convergent closure of $L[M]$ will be denoted by $\overline{L_s[M]}$ ($\overline{L_w[M]}$). If X is complete with respect to the weakly (strongly) intuitionistic fuzzy convergence, then we will call it intuitionistic fuzzy weakly (strongly) Banach space (IFB_wS or X_w (IFB_sS or X_s) in short). Let X be an IFB_sS (IFB_wS). We denote by X_s^* (X_w^*) the linear space of linear and continuous in IFB_sS (IFB_wS) functionals over the same field K.

Now, we define the corresponding concepts of basis theory for IFNS. Let $\{x_n\}_{n\in N} \subset X$ be some system.

Definition 2.9. System $\{x_n\}_{n\in N}$ is called s-complete (w-complete) in X_s (in X_w) if $\overline{L_s[\{x_n\}_{n\in N}]} \equiv X_s$ ($\overline{L_w[\{x_n\}_{n\in N}]} \equiv X_w$).

Definition 2.10. System $\{x_n^*\}_{n\in N} \subset X_s^*$ ($\{x_n^*\}_{n\in N} \subset X_w^*$) is called s-biorthogonal (w-biorthogonal) to the system $\{x_n\}_{n\in N}$ if $x_n^*(x_k) = \delta_{nk}$, $\forall n, k \in N$, where δ_{nk} is the Kronecker symbol.

Definition 2.11. System $\{x_n\}_{n\in N} \subset X_s$ ($\{x_n\}_{n\in N} \subset X_w$) is called s-linearly (w-linearly) independent in X if $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ in X_s (in X_w) implies $\lambda_n = 0$, $\forall n \in N$.

Definition 2.12. System $\{x_n\}_{n\in N} \subset X_s$ ($\{x_n\}_{n\in N} \subset X_w$) is called *s-basis* (*w-basis*) for X_s (for X_w) if $\forall x \in X$, $\exists!\{\lambda_n\}_{n\in N} \subset K : \sum_{n=1}^{\infty} \lambda_n x_n = x$ in X_s (in X_w).

We will also need the following concept.

Definition 2.13. System $\{x_n\}_{n\in N} \subset X$ is called nondegenerate if $x_n \neq 0$, $\forall n \in N$.

3. Main Results

3.1. Space of Coefficients

Let X be an IFNS, and let $\{x_n\}_{n\in N} \subset X$ be some system.

Assume that

$$\mathcal{K}_{\overline{x}}^{w} \equiv \left\{ \{\lambda_n\}_{n\in N} \subset C : \sum_{n=1}^{\infty} \lambda_n x_n \text{ converges in } X_w \right\},$$
$$\mathcal{K}_{\overline{x}}^{s} \equiv \left\{ \{\lambda_n\}_{n\in N} \subset C : \sum_{n=1}^{\infty} \lambda_n x_n \text{ converges in } X_s \right\}. \tag{3.1}$$

It is not difficult to see that $\mathcal{K}_{\overline{x}}^{w}$ and $\mathcal{K}_{\overline{x}}^{s}$ are linear spaces with regard to component-specific summation and component-specific multiplication by a scalar. Take $\forall \lambda \equiv \{\lambda_n\}_{n\in N} \in \mathcal{K}_{\overline{x}}^{w}$, and assume that

$$\mu_K\left(\overline{\lambda};t\right) = \inf_m \mu\left(\sum_{n=1}^{m} \lambda_n x_n; t\right); \qquad \nu_K\left(\overline{\lambda};t\right) = \sup_m \nu\left(\sum_{n=1}^{m} \lambda_n x_n; t\right). \tag{3.2}$$

Let us show that μ_K and ν_K satisfy the conditions (1)–(11).

(1) It is clear that $\mu_K(\overline{\lambda};t) = 0$, $\forall t \leq 0$.

(2) Let $\mu_K(\overline{\lambda};t) = 1$, $\forall t > 0$. Hence, $\mu(\sum_{n=1}^{m} \lambda_n x_n;t) = 1$, $\forall m \in N$, $\forall t > 0$. Suppose that the system $\{x_n\}_{n\in N}$ is nondegenerate. It follows from the above-stated relations that, for $m = 1$, we have $\mu(\lambda_1 x_1;t) = 1$, $\forall t > 0$. Hence, $\lambda_1 x_1 = 0 \Rightarrow \lambda_1 = 0$. Continuing this way, we get at the end of this process that $\lambda_n = 0$, $\forall n \in N$, that is, $\overline{\lambda} = 0$.

(3) The validity of relation $\mu_K(A\overline{\lambda};t) = \mu_K(\overline{\lambda};t/|c|)$, $\forall c \neq 0$ is beyond any doubt.

(4) As $\mu(x;\cdot)$ is a nondecreasing function on R it is not difficult to see that $\mu_K(\overline{\lambda};\cdot)$ has the same property. Let us show that $\lim_{t\to\infty}\mu_K(\overline{\lambda};t) = 1$. Take $\forall\varepsilon > 0$. Let $S_m = \sum_{n=1}^{m} \lambda_n x_n$ and $w\text{-}\lim_{m\to\infty} S_m = S \in X_w$. It is clear that $\exists t_0 > 0 : \mu(S;t_0) \geq 1-\varepsilon$. Then it follows from the definition of $w\text{-}\lim_m$ that $\exists m_0(\varepsilon;t_0) : \mu(S_m - S;t_0) \geq 1-\varepsilon$, $\forall m \geq m_0(\varepsilon;t_0)$. Property (4) implies

$$\mu(S_m;2t_0) = \mu(S_m - S + S;t_0 + t_0) \geq \mu(S_m - S;t_0) * \mu(S;t_0). \tag{3.3}$$

As a result, we get

$$\mu(S_m;t_0) \geq 1-\varepsilon, \quad \forall m \geq m_0(\varepsilon;t_0). \tag{3.4}$$

As $\mu(x;\cdot)$ is a nondecreasing function of t, it follows from (3.4) that

$$\mu(S_m;t) \geq 1-\varepsilon, \quad \forall m \geq m_0(\varepsilon;t_0),\ \forall t \geq t_0. \tag{3.5}$$

We have

$$\mu_K\left(\overline{\lambda};t\right) = \inf_m \mu(S_m;t) = \min\left\{\mu(S_1;t);\ldots;\mu(S_{m_0-1};t);\ \inf_{m\geq m_0}\mu(S_m;t)\right\}, \tag{3.6}$$

where $m_0 = m_0(\varepsilon;t_0)$. As $\lim_{t\to\infty}\mu(S_k;t) = 1$ for $\forall k \in N$, we have $\exists t_k(\varepsilon)$; $\forall t \geq t_k(\varepsilon) : \mu(S_k;t) \geq 1-\varepsilon$, $k = \overline{1,m_0-1}$. Let $t_\varepsilon^0 = \max\{t_k(\varepsilon), k = \overline{1,m_0-1}\}$, then it is clear that

$$\mu(S_k;t) \geq 1-\varepsilon, \quad \forall t \geq t_\varepsilon^0. \tag{3.7}$$

It follows from (3.5) and (3.6) that

$$\inf_{m\geq m_0}\mu(S_m;t) \geq 1-\varepsilon, \quad \forall t \geq t_0. \tag{3.8}$$

Let $t_\varepsilon = \max\{t_0;t_\varepsilon^0\}$. Hence, we obtain from (3.6) and (3.7) that

$$\mu_K\left(\overline{\lambda};t\right) \geq 1-\varepsilon, \quad \forall t \geq t_\varepsilon. \tag{3.9}$$

Thus, $\lim_{t\to\infty}\mu_K(\overline{\lambda};t) = 1$, $\forall\overline{\lambda} \in \mathcal{K}_{\overline{x}}^{w}$.

(5) Let $\overline{\lambda}, \overline{\mu} \in \mathcal{K}_{\overline{x}}^{w}$ ($\overline{\lambda} \equiv \{\lambda_n\}_{n\in N}$; $\overline{\mu} \equiv \{\mu_n\}_{n\in N}$) and $s, t \in R$. We have

$$\begin{aligned}\mu_K\left(\overline{\lambda}+\overline{\mu}; s+t\right) &= \inf_m \mu\left(\sum_{n=1}^{m}(\lambda_n+\mu_n)x_n; s+t\right)\\ &= \inf_m \mu\left(\sum_{n=1}^{m}\lambda_n x_n + \sum_{n=1}^{m}\mu_n x_n; s+t\right)\\ &\geq \inf_m\left[\mu\left(\sum_{n=1}^{m}\lambda_n x_n; s\right) * \mu\left(\sum_{n=1}^{m}\mu_n x_n; t\right)\right]\\ &= \left[\inf_m \mu\left(\sum_{n=1}^{m}\lambda_n x_n; s\right)\right] * \left[\inf_m \mu\left(\sum_{n=1}^{m}\mu_n x_n; t\right)\right]\\ &= \mu\left(\overline{\lambda}; s\right) * \mu(\overline{\mu}; t).\end{aligned} \tag{3.10}$$

(6) As $\nu(x;t) = 1$, $\forall t \leq 0$, it is clear that $\nu_K(\overline{\lambda};t) = 1$, $\forall t \leq 0$, $\forall \overline{\lambda} \in \mathcal{K}_{\overline{x}}^{w}$.

(7) Let the system $\{x_n\}_{n\in N}$ be nondegenerate. Assume that $\nu_K(\overline{\lambda};t) = 0$, $\forall t > 0$, then $\nu(\sum_{n=1}^{m}\lambda_n x_n; t) = 0$, $\forall t > 0$, $\forall m \in N$. For $m = 1$, we have $\nu(\lambda_1 x_1; t) = 0$, $\forall t > 0 \Rightarrow \lambda_1 x_1 = 0 \Rightarrow \lambda_1 = 0$. Continuing this process, we get $\lambda_n = 0$, $\forall n \in N \Rightarrow \overline{\lambda} = 0$.

(8) Clearly, $\nu_K(c\overline{\lambda};t) = \nu_K(\overline{\lambda}; t/|c|)$, $\forall c \neq 0$.

(9) It follows from the property (9) that $\nu(x;\cdot)$ is a nonincreasing function on R. Therefore, $\nu_K(\overline{\lambda};\cdot)$ is a nonincreasing function on R. Let us show that $\lim_{t\to\infty}\nu_K(\overline{\lambda};t) = 0$. Let $S_m = \sum_{n=1}^{m}\lambda_n x_n$ and w-$\lim_{m\to\infty}S_m = S \in X$. Take $\forall \varepsilon > 0$. It is clear that $\exists t_0 > 0 : \nu(S;t_0) \leq \varepsilon$. Then it follows from the definition of w-$\lim_m$ that $\exists m_0 = m_0(\varepsilon; t_0) : \nu(S_m - S; t_0) \leq \varepsilon$, $\forall m \geq m_0$. We have

$$\nu(S_m;t_0) = \nu(S_m - S + S; t_0 + t_0) \leq \nu(S_m - S; t_0) \diamond \nu(S;t_0) \leq \varepsilon, \quad \forall m \geq m_0 \tag{3.11}$$

As $\nu(x;\cdot)$ is a nonincreasing function, it is clear that

$$\nu(S_m;t) \leq \varepsilon, \quad \forall m \geq m_0,\ \forall t \geq t_0. \tag{3.12}$$

We have

$$\nu_K\left(\overline{\lambda};t\right) = \sup_m \nu\left(S_m;t\right) = \max\left\{\nu(S_1;t);\ldots;\nu(S_{m_0-1};t);\ \sup_{m\geq m_0}\nu(S_m;t)\right\}. \tag{3.13}$$

As $\lim_{t\to\infty}\nu(S_k;t) = 0$ for $\forall k \in N$, we have $\exists t_k(\varepsilon)$; $\forall t \geq t_k(\varepsilon) : \nu(S_k;t) \leq \varepsilon$, $k = \overline{1, m_0 - 1}$. Let $t_\varepsilon^0 = \max\{t_k(\varepsilon),\ k = \overline{1,\ m_0 - 1}\,\}$. It is clear that $\nu(S_k;t) \leq \varepsilon$, $\forall t \geq t_\varepsilon^0$. It follows from (3.12) that $\sup_{m\geq m_0}\nu(S_m;t) \leq \varepsilon$, $\forall t \geq t_0$. Let $t_\varepsilon = \max\{t_0;\ t_\varepsilon^0\}$, then it is clear that $\nu_K(\overline{\lambda};t) \leq \varepsilon$, $\forall t \geq t_\varepsilon \Rightarrow \lim_{t\to\infty}\nu_K(\overline{\lambda};t) = 0$.

(10) Let $\overline{\lambda}, \overline{\mu} \in \mathcal{K}^{w}_{\overline{x}}(\overline{\lambda} \equiv \{\lambda_n\}_{n\in N};\ \overline{\mu} \equiv \{\mu_n\}_{n\in N})$ and $s, t \in R$. We have

$$\begin{aligned}\nu_K\left(\overline{\lambda}+\overline{\mu}; s+t\right) &= \sup_m \nu\left(\sum_{n=1}^{m}(\lambda_n+\mu_n)x_n; s+t\right)\\ &\le \sup_m\left[\nu\left(\sum_{n=1}^{m}\lambda_n x_n; s\right)\diamond\nu\left(\sum_{n=1}^{m}\mu_n x_n; t\right)\right]\\ &= \left[\sup_m \nu\left(\sum_{n=1}^{m}\lambda_n x_n; s\right)\right]\diamond\left[\sup_m \nu\left(\sum_{n=1}^{m}\mu_n x_n; t\right)\right]\\ &= \nu_K\left(\overline{\lambda};\ s\right)\diamond\nu_K(\overline{\mu}; t).\end{aligned} \tag{3.14}$$

(11) Consider the following:

$$\begin{aligned}\mu_K\left(\overline{\lambda}; t\right)+\nu_K\left(\overline{\lambda}; t\right) &= \inf_m \mu\left(\sum_{n=1}^{m}\lambda_n x_n; t\right)+\sup_m \nu\left(\sum_{n=1}^{m}\lambda_n x_n; t\right)\\ &\le \sup_m\left[\mu\left(\sum_{n=1}^{m}\lambda_n x_n; t\right)+\nu\left(\sum_{n=1}^{m}\lambda_n x_n; t\right)\right]\\ &\le 1, \quad \forall\overline{\lambda}\in\mathcal{K}^{w}_{\overline{x}},\ \forall\lambda\in R.\end{aligned} \tag{3.15}$$

Thus, we have proved the validity of the following.

Theorem 3.1. *Let $(X;\mu;\nu)$ be a fuzzy normed space, and let $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system, then the space of coefficients $(\mathcal{K}^{s}_{\overline{x}};\mu_K;\nu_K)$ is also strongly fuzzy normed space.*

The following theorem is proved in absolutely the same way.

Theorem 3.2. *Let $(X;\mu;\nu)$ be a fuzzy normed space, and let $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system, then the space of coefficients $(\mathcal{K}^{w}_{\overline{x}};\mu_K;\nu_K)$ is also weakly fuzzy normed space.*

3.2. Completeness of the Space of Coefficients

Subsequently, we assume that $(X;\mu;\nu)$ is *IFBS*. Let us show that $(\mathcal{K}^{s}_{\overline{x}};\mu_K;\nu_K)$ is a strongly fuzzy complete normed space. First, we prove the following.

Lemma 3.3. *Let $x_0 \neq 0$, $x_0\in X$, and let $\{\lambda_n\}_{n\in N}\subset R$ be some sequence. If* s-$\lim_{n\to\infty}(\lambda_n x_0) = 0$, *that is, $\forall\varepsilon>0$, $\exists n_0 = n_0(\varepsilon): \mu(\lambda_n x_0; t) > 1-\varepsilon$, $\nu(\lambda_n x_0; t) < \varepsilon$, $\forall t\in R_+$, and $\forall n\ge n_0$, then $\lambda_n \to 0$, $n\to\infty$.*

Proof. As $x_0\neq 0$, it is clear that $\exists t_0 > 0 : \mu(x_0; t_0) < 1$. We have $\mu(\lambda_n x_0; t) = \mu(x_0; (t)/|\lambda_n|)$ for $\lambda_n\neq 0$. Assume that the relation $\lim_{n\to\infty}\lambda_n = 0$ is not true, then $\exists\{\lambda_{n_k}\}_{k\in N}$ and $\exists\delta > 0$: $|\lambda_{n_k}|\ge\delta$, $\forall k\in N$. It is clear that $\lim_{k\to\infty}\mu(\lambda_{n_k}x_0; t) = 1$ uniformly in t. On the other hand, for $t_k = |\lambda_{n_k}|\ t_0$, we have $\mu(\lambda_{n_k}x_0; t_k) = \mu(x_0; t_0) < 1$. So we came upon a contradiction which proves the lemma. □

Further, we assume that the following condition is also fulfilled.

(12) The functions $\mu(x;\cdot), \nu(x;\cdot) : R \to [0,1]$ are continuous for $\forall x \in X$.

Take *s-fundamental* sequence $\{\overline{\lambda}_n\}_{n\in N} \subset \mathcal{K}^s_{\overline{x}}$, $\overline{\lambda}_n \equiv \{\lambda_k^{(n)}\}_{k\in N}$. Then $\lim_{n,m\to\infty}\mu_K(\overline{\lambda}_n - \overline{\lambda}_m; t) = 1$ uniformly in $t \in R$, that is,

$$\lim_{n,m\to\infty}\inf_r \mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k\right) = 1, \tag{3.16}$$

uniformly in $t \in R$. Take $\forall k_0 \in N$ and fix it. We have

$$\left(\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right)x_{k_0} = \sum_{k=1}^{k_0}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k - \sum_{k=1}^{k_0-1}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k. \tag{3.17}$$

Then from property (5), we get

$$\mu\left(\left(\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right)x_{k_0}; t\right) \geq \mu\left(\sum_{k=1}^{k_0}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k; \frac{t}{2}\right) * \mu\left(\sum_{k=1}^{k_0-1}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right)x_k; \frac{t}{2}\right). \tag{3.18}$$

It follows directly from this relation that $\lim_{n,m\to\infty}\mu((\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)})x_{k_0}; t) = 1$ uniformly in t. As $x_{k_0} \neq 0$, Lemma 3.3 implies $\lim_{n,m\to\infty}|\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}| = 0$, that is, the sequence $\{\lambda_{k_0}^{(n)}\}_{n\in N}$ is fundamental in R. Let $\lambda_{k_0}^{(n)} \to \lambda_{k_0}$, as $n \to \infty$. Denote $\overline{\lambda} \equiv \{\lambda_n\}_{n\in N}$. Let us show that $\lim_{n\to\infty}\mu_K(\overline{\lambda}_n - \overline{\lambda}; t) = 1$ uniformly in t. Take $\forall \varepsilon > 0$. It is clear that $\exists n_0$, $\forall n \geq n_0$, $\forall p \in N : \mu_K(\overline{\lambda}_n - \overline{\lambda}_{n+p}; t) > 1 - \varepsilon$, $\forall t \in R$. Consequently,

$$\inf_r \mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(n+p)}\right)x_k; t\right) > 1 - \varepsilon, \quad \forall n \geq n_0,\ \forall p \in N,\ \forall t \in R_+. \tag{3.19}$$

Hence,

$$\mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(n+p)}\right)x_k; t\right) > 1 - \varepsilon, \quad \forall n \geq n_0,\ \forall r, p \in N,\ \forall t \in R_+. \tag{3.20}$$

As shown above, $\lim_{n,m\to\infty}\mu((\lambda_k^{(n)} - \lambda_k^{(m)})x_k; t) = 1$ uniformly in $t \in R_+$. Now let us take into account the fact that $\lim_{m\to\infty}\mu(\lambda_k^{(m)}x_k; t) = \mu(\lambda_k x_k; t)$, $\forall t \in R_+$. Indeed, if $\lambda_k = 0$, then $\mu(0; t) = 1$, $\forall t \in R_+$, and clearly, $\lim_{m\to\infty}\mu(\lambda_k^{(m)}x_k; t) = 1$ for $\forall t \in R_+$. If $\lambda_k \neq 0$, then for sufficiently large values of m we have $\lambda_k^{(m)} \neq 0$, and as a result,

$$\mu\left(\lambda_k^{(m)}x_k; t\right) = \mu\left(x_k; \frac{t}{\left|\lambda_k^{(m)}\right|}\right) \overset{m\to\infty}{\longrightarrow} \mu\left(x_k; \frac{t}{|\lambda_k|}\right) = \mu(\lambda_k x_k; t), \quad \forall t \in R_+. \tag{3.21}$$

Passage to the limit in the inequality (3.20) as $p \to \infty$ yields

$$\mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;t\right) \geq 1-\varepsilon, \quad \forall n \geq n_0,\ \forall r \in N,\ \forall t \in R_+. \tag{3.22}$$

We have

$$\begin{aligned}
\mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;t\right) &= \mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k - \sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\ t\right) \\
&\geq \mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\ \frac{t}{2}\right) * \mu\left(\sum_{k=1}^{r-1}\left(\lambda_k^{(n)}-\lambda_k\right)x_k;\ \frac{t}{2}\right) \\
&\geq 1-\varepsilon, \quad \forall n \geq n_0, \forall r,p \in N,\ \forall t \in R_+.
\end{aligned} \tag{3.23}$$

As $\overline{\lambda}_n \in \mathcal{K}_{\overline{x}}^{s}$, it is clear that $\exists m_0^{(n)} : \forall m \geq m_0^{(n)}, \forall p \in N$:

$$\mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;t\right) > 1-\varepsilon, \quad \forall t \in R_+. \tag{3.24}$$

We have

$$\begin{aligned}
\mu\left(\sum_{k=m}^{m+p}\lambda_k x_k;t\right) &= \mu\left(\sum_{k=m}^{m+p}\left(\lambda_k-\lambda_k^{(n)}\right)x_k + \sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;\ t\right) \\
&\geq \mu\left(\sum_{k=m}^{m+p}\left(\lambda_k-\lambda_k^{(n)}\right)x_k;\ \frac{t}{2}\right) * \mu\left(\sum_{k=m}^{m+p}\lambda_k^{(n)}x_k;\ \frac{t}{2}\right) \\
&\geq 1-\varepsilon, \quad \forall m \geq m_0^{(n)}, \quad \forall p \in N,\ \forall t \in R_+.
\end{aligned} \tag{3.25}$$

It follows that the series $\sum_{k=1}^{\infty}\lambda_k x_k$ is strongly fuzzy convergent, that is, $\exists$ s-$\lim_{m\to\infty}\sum_{\mathrm{k}=1}^{\mathrm{m}}\lambda_k x_k$. Consequently, $\overline{\lambda} \in \mathcal{K}_{\overline{x}}^{s}$, and the relation (3.22) implies that $\lim_{n\to\infty}\mu_K(\overline{\lambda}_n - \overline{\lambda};t) = 1$ uniformly in, $\forall t \in R_+$. It can be proved in a similar way that $\lim_{n\to\infty}\nu_K(\overline{\lambda}_n - \overline{\lambda};\mathrm{t}) = 0$ uniformly in $\forall t \in R_+$. As a result, we obtain that the space $(\mathcal{K}_{\overline{x}}^{s};\mu_K;\nu_K)$ is strongly fuzzy complete. Thus, we have proved the following.

Theorem 3.4. *Let $(X;\mu;\nu)$ be a fuzzy Banach space with condition (12), and let $\{x_n\}_{n\in N} \subset X$ be a nondegenerate system, then the space of coefficients $(\mathcal{K}_{\overline{x}}^{s};\mu_K;\nu_K)$ is a strongly fuzzy complete normed space.*

Consider operator $T : \mathcal{K}_{\overline{x}}^{s} \to X$ defined by

$$T\overline{\lambda} = \sum_{n=1}^{\infty}\lambda_n x_n, \quad \overline{\lambda} \equiv \{\lambda_n\}_{n\in N} \in \mathcal{K}_{\overline{x}}^{s}. \tag{3.26}$$

Let $s\text{-}\lim_{n\to\infty}\overline{\lambda}_n = \overline{\lambda}$ in $\mathcal{K}^s_{\overline{x}}$, where $\overline{\lambda}_n \equiv \{\lambda^{(n)}_k\}_{k\in N} \in \mathcal{K}^s_{\overline{x}}$. We have

$$\mu\Big(T\overline{\lambda}_n - T\overline{\lambda};\ t\Big) = \mu\left(\sum_{k=1}^{\infty}\Big(\lambda^{(n)}_k - \lambda_k\Big)x_k;\ t\right) \geq \inf_m \mu\left(\sum_{k=1}^{m}\Big(\lambda^{(n)}_k - \lambda_k\Big)x_k;\ t\right)$$
$$= \mu\Big(\overline{\lambda}_n - \overline{\lambda};\ t\Big). \tag{3.27}$$

It follows directly that $s\text{-}\lim_{n\to\infty}T\overline{\lambda}_n = T\overline{\lambda}$, that is, the operator T is strongly fuzzy continuous. Let $\overline{\lambda} \in \operatorname{Ker} T$, that is, $T\overline{\lambda} = 0 \Rightarrow \sum_{n=1}^{\infty}\lambda_n x_n = 0$, where $\overline{\lambda} \equiv \{\lambda_n\}_{n\in N} \in \mathcal{K}^s_{\overline{x}}$. It is clear that if the system $\{x_n\}_{n\in N}$ is *s-linearly independent*, then $\lambda_n = 0$, $\forall n \in N$, and as a result, $\operatorname{Ker} T = \{0\}$. In this case, $\exists T^{-1} : \mathrm{ImT} \to \mathcal{K}^s_{\overline{x}}$. If, in addition, ImT is *s-closed* in X, then T^{-1} is also continuous.

Denote by $\{\overline{e}_n\}_{n\in N} \subset \mathcal{K}^s_{\overline{x}}$ a canonical system in $\mathcal{K}^s_{\overline{x}}$, where $\overline{e}_n = \{\delta_{nk}\}_{k\in N} \in \mathcal{K}^s_{\overline{x}}$. Obviously, $T\overline{e}_n = x_n$, $\forall n \in N$. Let us prove that $\{\overline{e}_n\}_{n\in N}$ forms an *s-basis* for $\mathcal{K}^s_{\overline{x}}$. Take $\forall\overline{\lambda} \equiv \{\lambda_n\}_{n\in N} \in \mathcal{K}^s_{\overline{x}}$ and show that the series $\sum_{n=1}^{\infty}\lambda_n\overline{e}_n$ is strongly fuzzy convergent in $\mathcal{K}^s_{\overline{x}}$. In fact, the existence of $s\text{-}\lim_{m\to\infty}\sum_{\mathrm{n}=1}^{\mathrm{m}}\lambda_n x_n$ in X_s implies that $\forall\varepsilon > 0$, $\exists m_0 \in N$,

$$\mu\left(\sum_{n=m}^{m+p}\lambda_n x_n; t\right) > 1 - \varepsilon, \quad \forall m \geq m_0,\ \forall p \in N,\ \forall t \in R_+. \tag{3.28}$$

We have

$$\mu_K\left(\sum_{n=m}^{m+p}\lambda_n\overline{e}_n; t\right) = \inf_r\left(\sum_{n=m}^{r}\lambda_n x_n; t\right) \geq 1 - \varepsilon, \quad \forall m \geq m_0,\ \forall p \in N,\ \forall t \in R_+. \tag{3.29}$$

It follows that the series $\sum_{n=1}^{\infty}\lambda_n\overline{e}_n$ is strongly fuzzy convergent in $\mathcal{K}^s_{\overline{x}}$. Moreover,

$$\mu_K\left(\overline{\lambda} - \sum_{n=1}^{m}\lambda_n\overline{e}_n; t\right) = \mu_K(\{\dots;0;\lambda_{m+1};\dots\};t) = \inf_r \mu\left(\sum_{n=m+1}^{r}\lambda_n x_n; t\right)$$
$$\geq 1 - \varepsilon, \quad \forall m \geq m_0,\ \forall t \in R_+. \tag{3.30}$$

Consequently, $s\text{-}\lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n\overline{e}_n = \overline{\lambda}$, that is, $\overline{\lambda} \overset{s}{=} \sum_{n=1}^{m}\lambda_n\overline{e}_n$. Consider the functionals $e^*_n(\overline{\lambda}) = \lambda_n$, $\forall n \in N$. Let us show that they are *s-continuous*. Let $s\text{-}\lim_{n\to\infty}\overline{\lambda}_n = \overline{\lambda}$, where $\overline{\lambda}_n \equiv \{\lambda^{(n)}_k\}_{k\in N} \in \mathcal{K}^s_{\overline{x}}$. As established in the proof of Theorem 3.4, we have $\lambda^{(n)}_k \to \lambda_k$ as $n \to \infty$, that is, $e^*_k(\overline{\lambda}_n) \to e^*_k(\overline{\lambda})$ as $n \to \infty$ for $\forall k \in N$. Thus, e^*_k is *s-continuous* in $\mathcal{K}^s_{\overline{x}}$ for $\forall k \in N$. On the other hand, it is easy to see that $e^*_n(\overline{e}_k) = \delta_{nk}$, $\forall n,k \in N$, that is, $\{e^*_n\}_{n\in N}$ is *s-biorthogonal* to $\{\overline{e}_n\}_{n\in N}$. As a result, we obtain that the system $\{\overline{e}_n\}_{n\in N}$ forms an *s-basis* for $\mathcal{K}^s_{\overline{x}}$. So we get the validity of the following.

Theorem 3.5. *Let $(X;\mu;\nu)$ be a fuzzy Banach space with condition (12), and let $\{x_n\}_{n\in N} \subset X$ be a nondegenerate system. Then the corresponding space of coefficients $(\mathcal{K}^s_{\overline{x}};\mu_K;\nu_K)$ is strongly fuzzy complete with canonical s-basis $\{\overline{e}_n\}_{n\in N}$.*

Suppose that the system $\{x_n\}_{n\in N}$ is *s-linearly independent* and ImT is closed, then it is easily seen that $\{x_n\}_{n\in N}$ forms an *s-basis* for ImT, and in case of its *s-completeness* in X_s, it forms an *s-basis* for X_s. In this case, $\mathcal{K}_{\bar{x}}^{s}$ and X_s are isomorphic, and T is an isomorphism between them. The opposite of it is also true, that is, if the above-defined operator T is an isomorphism between $\mathcal{K}_{\bar{x}}^{s}$ and X_s, then the system $\{x_n\}_{n\in N}$ forms an *s-basis* for X_s. We will call T a coefficient operator. Thus, the following theorem holds.

Theorem 3.6. *Let* $(X;\mu;\nu)$ *be a fuzzy Banach space with condition (12), let* $\{x_n\}_{n\in N} \subset X$ *be a nondegenerate system, let* $(\mathcal{K}_{\bar{x}}^{s};\mu_K;\nu_K)$ *be a corresponding strongly fuzzy complete normed space, and let* $T : \mathcal{K}_{\bar{x}}^{s} \to X_s$ *be a coefficient operator. System* $\{x_n\}_{n\in N}$ *forms an s-basis for* X_s *if and only if the operator* T *is an isomorphism between* $\mathcal{K}_{\bar{x}}^{s}$ *and* X_s.

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