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Research Article

On Some Generalizations of Commuting Mappings

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It is shown that occasionally $\mathcal{JH}$ operators as well as occasionally weakly biased mappings reduce to weakly compatible mappings in the presence of a unique point of coincidence (and a unique common fixed point) of the given maps.

1. Introduction and Preliminaries

The study of finding a common fixed point of pair of commuting mappings was initiated by Jungck [1]. Later, on this condition was weakened in various ways. Sessa [2] introduced the notion of weakly commuting maps. Jungck [3] gave the notion of compatible mappings in order to generalize the concept of weak commutativity. One of the conditions that was used most often was the weak compatibility, introduced by Jungck in [4] fixed point results for various classes of mappings on a metric space, utilizing these concepts. Jungck and Pathak [5] defined the concept of a weakly biased maps in order to generalize the concept of weak compatibility. In the paper [6], published in 2008, Al-Thagafi and Shahzad introduced an even weaker condition which they called occasionally weak compatibility (see also [7]). Many authors (see, e.g., [8–13],) used this condition to obtain common fixed point results, sometimes trying to generalize results that were known to use (formally stronger) condition of weak compatibility. Recently, Hussain et al. [14] have introduced two new and different classes of noncommuting self-maps: $\mathcal{JH}$-operators and occasionally weakly biased mappings. These classes contain the occasionally weakly compatible and weakly biased self-maps as proper subclasses. For these new classes, authors have proved common fixed point results on the space (X, d) which is more general than metric space. We will show in this short note that in the presence of a unique point of coincidence (and a unique common fixed point) of the given mappings, occasionally $\mathcal{JH}$-operators as well as occasionally weakly biased mappings reduce to weakly compatible mappings (and so occasionally weakly compatible mappings). For more details on the subject, we refer the reader to [15–18].

Let X be a nonempty set and let f and g be two self-mappings on X. The set of fixed points of f (resp., g) is denoted by $F(f)$ (resp., $F(g)$). A point $x \in X$ is called a coincidence point (CP) of the pair (f, g) if $fx = gx(= w)$. The point w is then called a point of coincidence (POC) for (f, g). The set of coincidence points of (f, g) will be denoted as $C(f, g)$. Let $\mathrm{PC}(f, g)$ represent the set of points of coincidence of the pair (f, g). A point $x \in X$ is a common fixed point of f and g if $fx = gx = x$. The self-maps f and g on X are called

(1) commuting if $fgx = gfx$ for all $x \in X$;

(2) weakly compatible (WC) if they commute at their coincidence points, that is, if $fgx = gfx$ whenever $fx = gx$ [4];

(3) occasionally weakly compatible (OWC) if $fgx = gfx$ for some $x \in X$ with $fx = gx$ [6].

Let d be symmetric on X. Then f and g are called

(4) $\mathcal{D}$-operators if there is a point $x \in X$ such that $x \in C(f, g)$ and $d(x, fx) \leq \delta(C(f, g))$, where $\delta(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}$ [19];

(5) $\mathcal{JH}$-operators if there is a point $w = fx = gx$ in $\mathrm{PC}(f, g)$ such that $d(w, x) \leq \delta(\mathrm{PC}(f, g))$ [14];

(6) weakly g-biased, if $d(gfx, gx) \leq d(fgx, fx)$ whenever $fx = gx$ [5];

(7) occasionally weakly g-biased, if there exists some $x \in X$ such that $fx = gx$ and $d(gfx, gx) \leq d(fgx, fx)$ [14].

Let $d : X \times X \to [0, +\infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Then f and g are called

(8) $\mathcal{JH}$-operators if there is a point $w = fx = gx$ in $\mathrm{PC}(f, g)$ such that $d(w, x) \leq \delta(\mathrm{PC}(f, g))$ and $d(x, w) \leq \delta(\mathrm{PC}(f, g))$, where

$$\delta(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}. \tag{1.1}$$

2. Results

We begin with the following results.

Lemma 2.1 (see [20]). *If a WC pair (f, g) of self-maps on X has a unique POC, then it has a unique common fixed point.*

The following lemma is according to Jungck and Rhoades [12].

Lemma 2.2 (see [12]). *If an OWC pair (f, g) of self-maps on X has a unique POC, then it has a unique common fixed point.*

Proof. Since (f, g) is an OWC, there exists $x \in C(f, g)$ such that $fx = gx =: w$ and $fw = gfx = gfx = gw$. Hence, $fw = gw$ is also a POC for (f, g), and since it must be unique, we have that $w = fw = gw$, that is, w is a common fixed point for (f, g). If z is any common fixed point for (f, g) (i.e., $fz = gz = z$), then, again by the uniqueness of POC, it must be $z = w$. □

The following result is due to Đorić et al. [21]. It shows that the results of Jungck and Rhoades are not generalizations of results obtained from Lemma 2.1.

Proposition 2.3 (see [21]). *Let a pair of mappings (f, g) have a unique POC. Then it is WC if and only if it is OWC.*

Proof. In this case, we have only to prove that OWC implies WC. Let $w_1 = fx = gx$ be the given POC, and let (f, g) be OWC. Let $y \in C(f, g), y \neq x$. We have to prove that $fgy = gfy$. Now $w_2 = fy = gy$ is a POC for the pair (f, g). By the assumption, $w_2 = w_1$, that is, $fy = gy = fx = gx$. Since, by Lemma 2.2, w_1 is a unique common fixed point of the pair (f, g), it follows that $w_1 = fw_1 = fgy$ and $w_1 = gw_1 = gfy$, hence $fgy = gfy$. The pair (f, g) is WC. □

Proposition 2.4. *Let $d : X \times X \to [0, +\infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Let a pair of mappings (f, g) have a unique POC. If it is a pair of $\mathcal{JH}$-operators, then it is WC.*

Proof. Let (f, g) be a pair of $\mathcal{JH}$-operators. Then there is a point $w = fx = gx$ in $\mathrm{PC}(f, g)$ such that $d(x, w) \leq \delta(\mathrm{PC}(f, g))$ and $d(w, x) \leq \delta(\mathrm{PC}(f, g))$. Clearly PC (f, g) is a singleton. If not, then $w_1 = fy = gy$ is a POC for the pair (f, g). By the assumption, $w = w_1$. As a result, we have $\delta(\mathrm{PC}(f, g)) = 0$, which implies $d(x, w) = d(w, x) = 0$, that is, $x = fx = gx$. Consequently, we have $gfx = gx = fx = fgx$ and thus (f, g) is OWC. By Proposition 2.3, (f, g) is WC. □

It is worth mentioning that if i_X is the identity mapping, then the pair (f, i_X) is always WC, but it is a pair of $\mathcal{JH}$-operators if and only if f has a fixed point.

Proposition 2.5. *Let $d : X \times X \to [0, +\infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Suppose (f, g) is a pair of $\mathcal{JH}$-operators satisfying*

$$\begin{aligned} d(fx, fy) \leq{}& ad(gx, gy) + b \max\{d(fx, gx), d(fy, gy)\} \\ &+ c \max\{d(gx, gy), d(gx, fx), d(gy, fy)\} \end{aligned} \tag{2.1}$$

for each $x, y \in X$, where a, b, c are real numbers such that $0 < a + c < 1$.Then (f, g) is WC.

Proof. By hypothesis, there exists some $x \in X$ such that $w = fx = gx$. It remains to show that (f, g) has a unique POC. Suppose there exists another point $w_1 = fy = gy$ with $w \neq w_1$. Then, we have

$$d(w, w_1) = d(fx, fy) \leq (a + c)d(fx, fy) = (a + c)d(w, w_1), \tag{2.2}$$

which is a contradiction since $a + c < 1$. Thus (f, g) has a unique POC. By Proposition 2.4, (f, g) is WC. □

Proposition 2.6. *Let d be symmetric on X. Let a pair of mappings (f, g) have a unique POC which belongs to $F(f)$. If it is a pair of occasionally weakly g-biased mappings, then it is WC.*

Proof. Let (f, g) be a pair of occasionally weakly g-biased mappings. Then there exists some $x \in X$ such that $fx = gx$ and $d(gfx, gx) \le d(fgx, fx)$. Since $w = fx = gx$ belong to $F(f)$, then $fw = w$, that is, $fgx = fx = gx$. Thus $d(gfx, gx) \le d(fgx, fx) = 0$ and thus $gfx = gx = fx = fgx$. Hence (f, g) is OWC. By Proposition 2.3, (f, g) is WC. □

Let $\phi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function satisfying the condition $\phi(t) < t$ for each $t > 0$.

Proposition 2.7. *Let f, g be self-maps of symmetric space X and let the pair (f, g) be occasionally weakly g-biased. If for the control function ϕ, we have*

$$d(fx, fy) \le \phi(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}), \tag{2.3}$$

for each $x, y \in X$, then (f, g) is WC.

Proof. It remains to show that (f, g) has a unique POC which belongs to $F(f)$. Since (f, g) are occasionally weakly g-biased mappings, there exists some $x \in X$ such that $w = fx = gx$ and $d(gfx, gx) \le d(fgx, fx)$. If $w_1 = fy = gy$ and $w \neq w_1$, then

$$\begin{aligned} d(fx, fy) &\le \phi(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}) \\ &= \phi(d(fx, fy)) < d(fx, fy), \end{aligned} \tag{2.4}$$

which is a contradiction. Also if $ffx \neq fx$, we have

$$\begin{aligned} d(ffx, fx) &\le \phi(\max\{d(gfx, gx), d(gfx, fx), d(gx, ffx), d(gx, fx)\}) \\ &\le \phi(\max\{d(fgx, fx), d(fgx, fx), d(gx, ffx), d(gx, fx)\}) \\ &= \phi(d(ffx, fx)) < d(ffx, fx), \end{aligned} \tag{2.5}$$

which is a contradiction. By Proposition 2.6, (f, g) is WC. □

Remark 2.8. According to Propositions 2.4, 2.5, 2.6, and 2.7, it follows that results from [14]: (Theorems 2.8, 2.9, 2.10, 2.11, 2.12, 3.7, 3.9 and Corollary 3.8) are not generalizations (extensions) of some common fixed point theorems due to Bhatt et al. [9], Jungck and Rhoades [12, 13], and Imdad and Soliman [11]. Moreover, all mappings in these results are WC.

Proposition 2.9. *Let d be symmetric on X, and let a pair of mappings (f, g) have a unique (CP), that is, $C(f, g)$ is a singleton. If (f, g) is $\mathcal{P}$-operator pair, then it is WC.*

Proof. According to (4), there is a point $x \in X$ such that $x \in C(f, g) = \{x\}$ and $d(x, fx) \le \delta(C(f, g)) = \delta(\{x\}) = 0$. Hence, $x = fx = gx$ is a unique POC of pair (f, g) and since $gfx = gx = x = fx = fgx$,(f, g) is OWC. By Proposition 2.3 it is WC. □

Remark 2.10. By Proposition 2.9, it follows that Theorem 2.1 from [19] is not a generalization result of [6], the main result of Jungck [1], and other results.

Proposition 2.11. *Let d be symmetric on X, and let a pair of mappings (f, g) have a unique* POC. *Then it is weakly g-biased if and only if it is occasionally weakly g-biased.*

Proof. In this case, we have only to prove that (7) implies (6). Let $w = fx = gx$ be the given POC. Let $y \in C(f,g), y \neq x$. We have to prove that $d(gfy, gy) \leq d(fgy, fy)$. Now $w_1 = fy = gy$ is a POC for the pair (f,g). By the assumption, $w = w_1$, that is, $fy = gy = fx = gx$. Further, we have $gfy = gfx$ and $gy = gx$, which implies that $d(gfy, gy) = d(gfx, gx) \leq d(fgx, fx) = d(fgy, fy)$, that is, the pair (f,g) satisfies (6). □

The following example shows that the assumption about the uniqueness of POC in Propositions 2.3, 2.4, 2.6, and 2.11 cannot be removed.

Example 2.12. Let $X = [1, +\infty), d(x,y) = |x-y|, fx = 3x-2, gx = x^2$ (see [21]). It is obvious that $C(f,g) = \{1,2\}$, the pair (f,g) is occasionally weakly g-biased, but it is not weakly g-biased. Also, (f,g) is occasionally weakly compatible, but it is not weakly compatible. However, the pair (f,g) has not the unique POC.

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