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Research Article

Topological Quasilinear Spaces

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We introduce, in this work, the notion of topological quasilinear spaces as a generalization of the notion of normed quasilinear spaces defined by Aseev (1986). He introduced a kind of the concept of a quasilinear spaces both including a classical linear spaces and also nonlinear spaces of subsets and multivalued mappings. Further, Aseev presented some basic quasilinear counterpart of linear functional analysis by introducing the notions of norm and bounded quasilinear operators and functionals. Our investigations show that translation may destroy the property of being a neighborhood of a set in topological quasilinear spaces in contrast to the situation in topological vector spaces. Thus, we prove that any topological quasilinear space may not satisfy the localization principle of topological vector spaces.

1. Introduction

In [1], Aseev introduced the concept of quasilinear spaces both including classical linear spaces and modelling nonlinear spaces of subsets and multivalued mappings. Then, he proceeds a similar way to linear functional analysis on quasilinear spaces by introducing notions of the norm and quasilinear operators and functionals. Further, he presented some results which are quasilinear counterparts of fundamental definitions and theorems in linear functional analysis and differential calculus in Banach spaces. This pioneering work has motivated a lot of authors to introduce new results on multivalued mappings, fuzzy quasilinear operators, and set-valued analysis [2–4].

One of the most useful example of a quasilinear space is the set $K_C(E)$ of all convex compact subsets of a normed space E. The investigation of this class involves convex and interval analysis. Intervals are excellent tools for handling global optimization problems and for supplementing standard techniques. This is because an interval is an infinite set and is thus a carrier of an infinite amount of information which means global information. We refer the reader to [5] for detailed information about global optimization related to interval analysis. Further, the theory of set differential equations also needs the analysis of $K_C(E)$ [3].

There are various ways introducing and handling quasilinear spaces. Another important treatment is those of Markow's approach (see [6, 7]). However, we think that Aseev's treatment provides the most suitable base and necessary tools to proceed a similar analysis on quasilinear spaces to those of classical linear functional analysis. Further, it reflects more aspects of set-valued algebra and analysis by the advantages of the ordering relation. After the introduction of normed quasilinear spaces and bounded quasilinear operators in [1], we think that the investigation of quasilinear topologies on a quasilinear spaces and the introduction of some new results may provide important contributions to the improvement of the quasilinear functional analysis.

2. Preliminaries and Some New Results on Quasilinear Spaces

Let us start this section by giving some notation and preliminary results. We mainly follow the terminology of [1, 8]. For some topological space X, the notation $\mathcal{N}_x$ stands for the family of all neighborhoods of an $x \in X$. Let X be a topological vector space (TVS, for short), $x \in X$ and $G \subset X$. Then $G \in \mathcal{N}_x$ if and only if $G - x \in \mathcal{N}_0$ and $x - G \in \mathcal{N}_0$. This is the localization principle of TVSs.

A set X is called a *quasilinear space* (QLS, for short), [1], if a partial ordering relation "$\leq$", an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such way that the following conditions hold for any elements $x, y, z, u \in X$, and any real scalars α, β:

$$
\begin{gathered}
x \leq x, \\
x \leq z \quad \text{if } x \leq y,\ y \leq z, \\
x = y \quad \text{if } x \leq y,\ y \leq x, \\
x + y = y + x, \\
x + (y + z) = (x + y) + z, \\
\text{there exists an element } 0 \in X \text{ such that } x + 0 = x, \\
\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x, \\
\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y, \\
1 \cdot x = x, \\
0 \cdot x = 0, \\
(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x, \\
x + z \leq y + v \quad \text{if } x \leq y,\ z \leq v, \\
\alpha \cdot x \leq \alpha \cdot y \quad \text{if } x \leq y.
\end{gathered}
\tag{2.1}
$$

A linear space is a QLS with the partial ordering relation "$x \leq y$ if and only if $x = y$".

Perhaps the most popular example of nonlinear QLSs is the set of all closed intervals of real numbers with the inclusion relation "$\subseteq$" , algebraic sum operation

$$
A + B = \{a + b : a \in A,\ b \in B\},
\tag{2.2}
$$

and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}. \tag{2.3}$$

We denote this set by $K_C(\mathbb{R})$. Another one is $K(\mathbb{R})$, the set of all compact subsets of real numbers. In general, $K(E)$ and $K_C(E)$ stand for the space of all nonempty closed bounded and nonempty convex and closed bounded subsets of any normed linear space E, respectively. Both are QLSs with the inclusion relation and with a slight modification of addition as follows:

$$A + B = \overline{\{a + b : a \in A,\ b \in B\}}, \tag{2.4}$$

and with the real-scalar multiplication above.

Hence, $K_C(E) = \{A \in K(E) : A \text{ convex}\}$.

Lemma 2.1 (see [1]). *In a* QLS *X the element* 0 *is minimal, that is, $x = 0$ if $x \le 0$.*

Definition 2.2. An element $x' \in X$ is called an inverse of an $x \in X$ if $x + x' = 0$. If an inverse element exists, then it is unique. An element x having an inverse is called regular; otherwise, it is called singular.

We show later that the minimality is not only a property of 0 but also is shared by the other regular elements.

Lemma 2.3 (see [1]). *Suppose that each element x in the QLS X has an inverse element $x' \in X$. Then the partial ordering in X is determined by equality, the distributivity conditions hold, and, consequently, X is a linear space.*

Corollary 2.4 (see [1]). *In a real linear space, equality is the only way to define a partial ordering such that conditions* (2.1) *hold.*

It will be assumed in what follows that $-x = (-1)x$. An element x in a QLS is regular if and only if $x - x = 0$ if and only if $x' = -x$.

Definition 2.5. Suppose that X is a QLS and $Y \subseteq X$. Y is called a subspace of X whenever Y is a quasilinear space with the same partial ordering and the same operations on X.

Theorem 2.6. *Y is a subspace of a QLS X if and only if for every $x, y \in Y$ and $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y \in Y$.*

Proof of this theorem is quite similar to its classical linear algebraic counterpart.

Let X be a QLS and Y be a subspace of X. Suppose that each element x in Y has an inverse element $x' \in Y$; then by Lemma 2.3 the partial ordering on Y is determined by the equality. In this case the distributivity conditions hold on Y, and Y is a linear subspace of X.

Definition 2.7. Let X be a QLS. An element $x \in X$ is said to be symmetric provided that $-x = x$, and X_b denotes the set of all such elements. Further, X_r and X_s stand for the sets of all regular and singular elements in X, respectively.

Theorem 2.8. *X_r, X_d, and $X_s \cup \{0\}$ are subspaces of X.*

Proof. X_r is a subspace since the element $\lambda x' + y'$ is the inverse of $\lambda x + y$.

$X_s \cup \{0\}$ is a subspace of X. Let $x, y \in X_s \cup \{0\}$ and $\lambda \in \mathbb{R}$. The assertion is clear for $x = y = 0$. Let $x \neq 0$ and suppose that $(x + \lambda y) \notin X_s \cup \{0\}$, that is, $(x + \lambda y) + u = 0$ for some $u \in X$. Then $x + (\lambda y + u) = 0$ and so $x' = \lambda y + u$. This implies that $x \in X_r$. Analogously we obtain $y \in X_r$ if $y \neq 0$. This contradiction shows that $x + \lambda y \in X_s \cup \{0\}$.

The proof for X_d is similar. □

X_r, X_d, and $X_s \cup \{0\}$ are called *regular*, *symmetric*, and *singular subspaces* of X, respectively.

Example 2.9. Let $X = K_C(\mathbb{R})$ and $Z = \{0\} \cup \{[a,b] : a, b \in \mathbb{R}$ and $a \neq b\}$. Z is the singular subspace of X. However, the set $\{\{a\} : a \in \mathbb{R}\}$ of all singletons constitutes X_r and is a linear subspace of X. In fact, for any normed linear space E, each singleton $\{a\}$, $a \in E$ is identified with a, and hence E is considered as the regular subspace of both $K_C(E)$ and $K(E)$.

Proposition 2.10. *In a quasilinear space* X *every regular element is minimal.*

Proof. We must show that $y \leq x$ implies that $y = x$ for each $x \in X_r$. Consider

$$y \leq x \Longrightarrow y + x' \leq x + x' = 0 \Longrightarrow y + x' \leq 0. \tag{2.5}$$

Hence $y + x' = 0$ by the minimality of 0. This implies that $y = x$ by the uniqueness of the inverse element. □

Example 2.11. Consider again the subspace Z of $K_C(\mathbb{R})$ in the former example. $\{0\}$ is the only minimal element in Z, and there is no else minimal element in Z.

Let X be a quasilinear space. A real function $\|\cdot\|_X : X \to \mathbb{R}$ is called a *norm*, [1], if the The following conditions are satisfied:

$$\|x\|_X > 0 \text{ if } x \neq 0; \tag{2.6}$$

$$\|x + y\|_X \leq \|x\|_X + \|y\|_X; \tag{2.7}$$

$$\|\alpha \cdot x\|_X = |\alpha| \cdot \|x\|_X; \tag{2.8}$$

$$\text{if } x \leq y, \quad \text{then } \|x\|_X \leq \|y\|_X; \tag{2.9}$$

$$\text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that,} \tag{2.10}$$

$$x \leq y + x_\varepsilon, \quad \|x_\varepsilon\|_X \leq \varepsilon \quad \text{then } x \leq y. \tag{2.11}$$

A quasilinear space X with a norm defined on it is called *normed quasilinear* space. It follows from Lemma 2.3 that if any $x \in X$ has an inverse, then the concept of a normed quasilinear space coincides with the concept of a real normed linear space.

Hausdorff metric or norm metric on a normed QLS X is defined by the following equality:

$$h_X(x, y) = \inf\left\{r \geq 0 : x \leq y + a_1^r,\ y \leq x + a_2^r, \left\|a_i^r\right\| \leq r,\ i = 1, 2\right\}. \tag{2.12}$$

Since $x \le y + (x - y)$ and $y \le x + (y - x)$, the quantity $h_X(x, y)$ is well defined for any elements $x, y \in X$, further $h_X(x, y) \le \|x - y\|_X$ [1]. It is not hard to see that $h_X(x, y)$ satisfies all of the metric axioms.

Lemma 2.12 (see [1]). *The operations of algebraic operations of addition and scalar multiplication are continuous with respect to the Hausdorff metric. The norm is continuous function with respect to the Hausdorff metric.*

Lemma 2.13 (see [1]). *(a) Suppose that $x_n \to x_0$ and $y_n \to y_0$, and that $x_n \le y_n$ for any positive integer n. Then $x_0 \le y_0$. (b) Suppose that $x_n \to x_0$ and $z_n \to x_0$. If $x_n \le y_n \le z_n$ for any n, then $y_n \to x_0$. (c) Suppose that $x_n + y_n \to x_0$ and $y_n \to 0$; then $x_n \to x_0$.*

Example 2.14 (see [1]). Let X be a real complete normed linear space (a real Banach space). Then X is a complete normed quasilinear space with partial ordering given by equality. Conversely, if X is complete normed quasilinear space and any $x \in X$ has an inverse element $x' \in X$, then X is a real Banach space, and the partial ordering on X is the equality. In this case $h_x(x, y) = \|x - y\|_X$. Note that $h_X(x, y) \ne \|x - y\|_X$ for nonlinear QLSs, in general.

Example 2.15 (see [1]). For example, if E is a Banach space, then a norm on $K(E)$ is defined by $\|A\|_{K(E)} = \sup \|a\|_E$. Then $K(E)$ and $K_C(E)$ are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:

$$h(A, B) = \inf\{r \ge 0 : A \subseteq B + S_r(0),\ B \subseteq A + S_r(0)\}, \tag{2.13}$$

where $S_r(0)$ stands for 0-centered closed ball with radius r in E.

3. Topological Quasilinear Spaces

Definition 3.1. A topological quasilinear space (TQLS, for short) X is a topological space and a quasilinear space such that the algebraic operation of addition and scalar multiplication are continuous, and, following conditions are satisfied for any $x, y \in X$:

$$\text{for any } U \in \mathcal{N}_0,\ x \le y \text{ and } y \in U \text{ implies } x \in U, \tag{3.1}$$

$$\begin{gathered}\text{for any } U \in \mathcal{N}_x, y \in U \Longleftrightarrow \text{there exists some } V \in \mathcal{N}_0 \text{ satisfying } x + V \subseteq U, \\ \text{such that } x \le y + a \text{ for some } a \in V \text{ or } y \le x + b \text{ for some } b \in V.\end{gathered} \tag{3.2}$$

Any topology τ, which makes (X, τ) be a topological quasilinear space, will be called a *quasilinear topology*. The conditions (3.1) and (3.2) provide necessary harmony of the topology with the ordering structure on X.

Example 3.2. Let X be a TVS. Then, for any $x, y \in X$ and for any $U \in \mathcal{N}_x$, $y \in U$ if and only if there exists a neighborhood V of 0 satisfying $x + V \subseteq U$ such that $x = y + a$ for some $a \in V$ or $y = x + b$ for some $b \in V$. In fact, this is true by the localization principle of TVSs since $U - x$ and $x - U$ are neighborhood of 0. So we can obtain desired V by taking $V = U - x$ or $V = x - U$. This provides the condition (3.2). Further, the condition (3.1) obviously holds. Hence, X is a TQLS.

We later show that some TQLSs may *not* satisfy the localization principle.

Remark 3.3. In condition (3.2), for some $U \in \mathcal{N}_x$, we may find a $V \in \mathcal{N}_0$ satisfying $x + V \subseteq U$ such that both $x \leq y + a$ and $y \leq x + b$ for some $a, b \in V$. This comfortable situation depends on the selection of U. However, we may not find such a suitable $V \in \mathcal{N}_0$ for some $U \in \mathcal{N}_x$ even in TVSs.

Example 3.4. Consider real numbers with usual metric. Take $x = 3$, $y = 5$, and $U = [2, 7] \in \mathcal{N}_x$. Then any $V \in \mathcal{N}_0$ satisfying $3 + V \subseteq U$ must be a subset of $[-1, 4]$. Further $3 = 5 + a$ and $5 = 3 + b$ gives $a = -2$, $b = 2$, and hence V can only include b.

Remark 3.5. In a semimetrizable TQLS the condition (3.1) and the condition (3.2) can be reformulated by balls as follows:

$$\text{for any } \varepsilon > 0, x \leq y \text{ and } y \in S_\varepsilon(0) \text{ implies } x \in S_\varepsilon(0); \tag{3.3}$$

equivalently,

$$\begin{gathered} x \leq y \text{ implies } d(x, 0) \leq d(y, 0), \\ \text{for any } \varepsilon > 0, y \in S_\varepsilon(x) \Longleftrightarrow \text{there exists some } S_\varepsilon(0), \\ \text{with } x + S_\varepsilon(0) \subseteq S_\varepsilon(x) \text{ such that } x \leq y + a \text{ for some } a \in S_\varepsilon(0), \\ \text{or } y \leq x + b \text{ for some } b \in S_\varepsilon(0). \end{gathered} \tag{3.1$'$}$$

A TQLS with a (semi)metrizable quasilinear topology will be called a *(semi)metric QLS*.

Example 3.6. Any normed QLS is a Hausdorff TQLS. By the definition $y \in S_\varepsilon(x) \Leftrightarrow h(x, y) \leq \varepsilon \Leftrightarrow x \leq y + a$ and $y \leq x + b$ for some $a, b \in S_\varepsilon(0)$, whence the condition (3.2) holds.

Proposition 3.7. *Let X be a Hausdorff TQLS and $x, y \in X$. If for any $V \in \mathcal{N}_0$ there exists some $b \in V$ such that $x \leq y + b$, then $x \leq y$.*

Proof. Suppose that there exists some $b \in V$ satisfying $x \leq y + b$ for every $V \in \mathcal{N}_0$, but $x \leq y$ is not true. Then there exists distinct open neighborhoods U_x and U_y of x and y, respectively. Since $x \notin U_y$, this implies by the condition (3.2) that for any $V \in \mathcal{N}_0$ with the property $y + V \subseteq U_y$ we cannot find $b \in V$ satisfying $x \leq y + b$. This is a contradiction to the hypothesis. □

Example 3.8. In Proposition 3.7, the condition "X is Hausdorff" is indispensable. Let us consider $K_C(\mathbb{R}^2)$ and the function

$$p(A) = \sup\{|x_1| : (x_1, x_2) \in A\} \tag{3.4}$$

for some $A \in K_C(\mathbb{R}^2)$. We can construct a topology τ on $K_C(\mathbb{R}^2)$ by p in such a way that $U \in \tau$ if and only if $U \supseteq \{A : p(A) < \varepsilon\}$ for some $\varepsilon > 0$ (we later call p as a seminorm on $K_C(\mathbb{R}^2)$). τ is a semimetrizable topology by the semimetric

$$d(A, B) = \inf\{r \geq 0 : A \subseteq B + C_1^r,\ B \subseteq A + C_2^r; p(C_i^r) \leq r,\ i = 1, 2\}. \tag{3.5}$$

Let

$$A = \left\{(0,t) \in \mathbb{R}^2 : 0 \le t \le 1\right\}, \qquad B = \left\{(0,t) \in \mathbb{R}^2 : 0 \le t \le 2\right\}. \tag{3.6}$$

Then, there is not separate neighborhoods of the points A and B of $K_C(\mathbb{R}^2)$ in this topology. So, τ cannot be a Hausdorff topology. Now let $\varepsilon > 0$ be arbitrary and define

$$B_\varepsilon = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le \varepsilon,\ 0 \le x_2 \le 1\right\}. \tag{3.7}$$

Then for every $V \in \mathcal{N}_0$ there exists some $B_\varepsilon \in V$ such that $B \subseteq A + B_\varepsilon$. But, $B \nsubseteq A$.

Theorem 3.9. *Let X be a* TQLS*. Then* X_r *and* X_d *are closed in* X*.*

Proof. $\{x_i\}$ is a net in X_r converging to an $x \in X$. By the continuity of algebraic operations $-x_i \to -x$ and $x_i - x_i \to x - x$. This means $x - x = 0$ since $x_i - x_i = 0$ for each i, whence $x \in X_r$. The proof is easier for X_d. □

The result of this theorem may not be true for $X_s \cup \{0\}$. Let $X = K_C(\mathbb{R})$ and define $x_n = [1, 1 + 1/n] \in X_s \cup \{0\}$ for each $n \in \mathbb{N}$. Then $x_n \to \{1\} \notin X_s \cup \{0\}$.

Definition 3.10. Let X be a quasilinear space. A paranorm on X is a function $p : X \to \mathbb{R}$ satisfying the following conditions. For every $x, y \in X$,

(1) $p(0) = 0$,

(2) $p(x) \ge 0$,

(3) $p(-x) = p(x)$,

(4) $p(x + y) \le p(x) + p(y)$,

(5) if $\{t_n\}$ is a sequence of scalars with $t_n \to t$ and $\{x_n\} \subset X$ with $p(x_n) \to p(x)$, then $p(t_n x_n) \to p(tx)$ and (continuity of scalar multiplication),

(6) if $x \le y$, then $p(x) \le p(y)$.

The pair (X, p) with the function p satisfying the conditions (1)–(6) is called a paranormed QLS.

It follows from Lemma 2.3 that if any $x \in X$ has an inverse element $x' \in X$, then the concept of paranormed quasilinear space coincides with the concept of a real paranormed linear space.

The paranorm is called *total* if, in addition, we have

$$\begin{gathered} p(x) = 0 \Longleftrightarrow x = 0, \\ \text{if for any } \varepsilon > 0 \text{ there exists an element } x_\varepsilon \in X \text{ such that,} \\ x \le y + x_\varepsilon \text{ and } p(x_\varepsilon) \le \varepsilon, \text{then } x \le y. \end{gathered} \tag{3.8}$$

The equality

$$d(x, y) = \inf\{r \ge 0 : x \le y + a_1^r,\ y \le x + a_2^r,\ p(a_i^r) \le r\} \tag{3.9}$$

defines a semimetric on a paranormed quasilinear space X. d is metric whenever p is total.

Example 3.11. Let us prove the above assertion. First of all we should note that d is well defined since $x \le y + (x - y)$ and $y \le x + (y - x)$ at least. Now $x \le y$ and $y \le x$ if $x = y$, and so $x \le y + a_1$, $y \le x + a_2$ for $a_1 = a_2 = 0$ that implies $d(x, y) = 0$ since $p(a_1) = p(a_2) = 0$. Obviously d is symmetric. Further, for every element a_1^r and b_2^r such that $x \le z + a_1^r$ and $z \le y + b_2^r$, observe that $x \le y + b_2^r + a_1^r$. Similarly, $y \le x + a_2^r + b_1^r$ for every a_2^r and b_1^r such that $y \le z + b_1^r$ and $z \le x + a_2^r$. Since $p(a_1^r + b_2^r) \le p(a_1^r) + p(b_2^r)$ and $p(a_2^r + b_1^r) \le p(a_2^r) + p(b_1^r)$, we get $d(x, y) \le d(x, z) + d(z, y)$ by the definition.

Let p be total and $d(x, y) = 0$. Then for any $\varepsilon > 0$ there exist elements $x_\varepsilon^1, x_\varepsilon^2 \in X$ such that $x \le y + x_\varepsilon^1$, and $y \le x + x_\varepsilon^2$, for $p(x_\varepsilon^i) \le \varepsilon$, $i = 1, 2$. Hence the totality conditions imply that $x \le y$ and $y \le x$, that is, $x = y$.

Further, we have the inequality $d(x, y) \le p(x - y)$.

Note that these definitions are inspired from the definitions in [1] about normed quasilinear spaces. The proofs of some facts given here are quite similar to that of Aseev's corresponding results.

If the first condition in the definition of norm in a QLS is relaxed into the condition

$$\|x\| \ge 0 \quad \text{if } x \ne 0 \tag{3.10}$$

and if the condition (2.10) of norm is removed, we then obtain the definition of a seminorm. A quasilinear space with a seminorm is called a seminormed QLS. By the same way in linear spaces one can prove that a seminorm on a QLS is a paranorm. Thus we have the following implication chain among the kinds of QLSs:

(normed)seminormed QLS $\Rightarrow$ (total paranormed)paranormed QLS $\Rightarrow$ (metric)semimetric QLS $\Rightarrow$ (Hausdorff)TQLS.

Example 3.12. Let X be a QLS. The discrete topology on X is not a quasilinear topology since the continuity of the scalar multiplication is not satisfied.

Example 3.13. The function

$$p(A) = \sup\{|x_1| : (x_1, x_2) \in A\} \tag{3.11}$$

in Example 3.8 is a seminorm on $K_C(\mathbb{R}^2)$. Further, the function

$$q(A) = \frac{p(A)}{1 + p(A)} \tag{3.12}$$

is a paranorm on $K_C(\mathbb{R}^2)$ but is not a seminorm.

Definition 3.14. Let (X, d) be a semimetric QLS and x be an element of X. Then, the nonnegative number

$$\rho(x) = d(x - x, 0) \tag{3.13}$$

is called diameter of x.

For each regular element x, $\rho(x) = 0$ since $x - x = 0$. Hence this definition is redundant in linear spaces. Further it should not be confused with the classical notion of the diameter of a subset in a semimetric space for which it is defined by $\delta(U) = \sup_{x,y\in U} d(x,y)$ for any $U \subset X$.

For example, in $K_C(\mathbb{R})$, $[-1,3] \in K_C(\mathbb{R})$ and

$$\begin{aligned}\rho([-1,3]) &= h([-1,3]-[-1,3],0)\\ &= h([-4,4],0) = \|[-4,4]\|\\ &= \sup_{a\in[-4,4]} |a| = 4.\end{aligned} \tag{3.14}$$

However, for the (singleton) subset $U = \{[-1,3]\}$ of $K_C(\mathbb{R})$, $\delta(U) = 0$.

following result is half of the localization principle of TVSs

Theorem 3.15. *Let X be a TQLS, $x \in X$, and U is a set containing 0. If $x + U \in \mathcal{N}_x$, then $U \in \mathcal{N}_0$.*

Proof. The proof is only an application of the fact that the translation operator $f_x : X \to X$, $f_x(v) = v + x$, is continuous by the continuity of the algebraic sum operation. □

Although the converse of this theorem is true in TVSs, it may not be true in some TQLSs.

Example 3.16. Consider $K_C(\mathbb{R})$ again and its closed unit ball $S_1(0)$. Now, for $x = [2,3] \in K_C(\mathbb{R})$, we show that $x + S_1(0)$ is not a neighborhood of x. A careful observation shows that $x + S_1(0)$ doesnot contain elements (intervals) for which the diameter is smaller than 1. However, every x-centered ball $S_r(x)$ with radius r contains a singleton if $r \geq (\rho(x)/2) = 1/2$ and contains an interval such as $[2 + (r/2), 3 - (r/2)]$ if $r < 1/2$ since

$$h\left([2,3], \left[2 + \frac{r}{2}, 3 - \frac{r}{2}\right]\right) = \frac{r}{2} < r. \tag{3.15}$$

That is, $S_r(x)$ contains elements with diameter smaller than 1. However, neither a singleton nor such an element belongs to $x + S_r(0)$. This implies that $S_r(x) \not\subseteq x + S_r(0)$ for every $r > 0$. Eventually, the set $x + S_1(0)$ cannot contain an x-centered ball.

Thus, the localization principle may not be satisfied about a singular element in $K_C(\mathbb{R})$. The example alludes that translation by a singular element destroys the property of being a neighborhood in a TQLS. The following theorem states that the translation by a regular element preserves neighborhoods, and so the localization principle holds for these elements.

Theorem 3.17. *Let X be a TQLS and $x \in X_r$. Then $U \in \mathcal{N}_0 \Leftrightarrow x + U \in \mathcal{N}_x$.*

Proof. Consider again the operator f_x in the proof of Theorem 3.15. In this case the inverse f_x^{-1} exists and just is the continuous operator f_{-x}. Hence f_x is a homeomorphism and so preserves the neighborhoods. □

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