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Research Article

New Classes of Spatial Central Configurations for $N + N + 2$-Body Problem

Liu Xuefei, Zhang Chuntao, Luo Jianmei, and Zhang Gan

Department of Mathematics and Statistics, Chongqing Three Gorges University, Chongqing 404100, China

Correspondence should be addressed to Liu Xuefei, liu_xf666@163.com

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Under arbitrary masses, in this paper, we discuss the existence of new families of spatial central configurations for the $N + N + 2$-body problem, $N \geq 2$. We study some necessary conditions and sufficient conditions for a families of spatial double pyramidical central configurations (d.p.c.c.), where $2N$ bodies are at the vertices of a nested regular N-gons $\Gamma_1 \cup \Gamma_2$, and the other two bodies are symmetrically located on the straight line that is perpendicular to the plane that contains $\Gamma_1 \cup \Gamma_2$ and passes through the geometric center of $\Gamma_1 \cup \Gamma_2$. We prove that if the bodies are in a d.p.c.c., then the masses on each N-gon are equal, and the other two are also equal. And also we prove the existence and uniqueness of the central configurations for any given ratios of masses.

1. Main Results

The Newtonian n-body problem (see [1–7]) concerns the motion of n point particles with masses $m_j \in R^+$ and positions $\overline{q}_j \in R^3 (j = 1, \ldots, n)$. The motion is governed by Newton's law:

$$m_j \ddot{\overline{q}}_j = \frac{\partial U(\overline{q})}{\partial \overline{q}_j}, \tag{1.1}$$

where $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)$ and $U(\overline{q})$ is the Newtonian potential:

$$U(\overline{q}) = \sum_{1 \leq k < j \leq n} \frac{m_k m_j}{\left|\overline{q}_k - \overline{q}_j\right|}. \tag{1.2}$$

Consider the space

$$X = \left\{ \overline{q} = (\overline{q}_1, \dots, \overline{q}_n) \in R^{3n} : \sum_{k=1}^{n} m_k \overline{q}_k = 0 \right\}, \tag{1.3}$$

that is, suppose that the center of mass is fixed at the origin of the space. Because the potential is singular when two particles have the same position, it is natural to assume that the configuration avoids the set $\Delta = \{\overline{q} : \overline{q}_k = \overline{q}_j \text{ for some } k \neq j\}$. The set $X \setminus \Delta$ is called the configuration space.

Definition 1.1 (see [4]). A configuration $\overline{q} = (\overline{q}_1, \dots, \overline{q}_n) \in X \setminus \Delta$ is called a central configuration (c.c.) if there exists a constant λ such that $\ddot{\overline{q}}_k = -\lambda \overline{q}_k$, $k = 1, 2, \dots, n$, or

$$\sum_{j=1, j \neq k}^{n} \frac{m_j m_k}{\left| \overline{q}_j - \overline{q}_k \right|^3} \left(\overline{q}_j - \overline{q}_k \right) = -\lambda m_k \overline{q}_k, \quad 1 \leq k \leq n, \tag{1.4}$$

and the value of the constant λ in (1.4) is uniquely determined by

$$\lambda = \frac{U}{I}, \tag{1.5}$$

where

$$I = \sum_{k=1}^{n} m_k \left| \overline{q}_k \right|^2. \tag{1.6}$$

For any coordinate system, we have that, if the center of masses $m_1, m_2, \dots, m_n$ with position vectors $r_1, r_2, \dots, r_n$ is not at the origin, central configuration equations (1.4) are equivalent to the following:

$$\sum_{j=1, j \neq k}^{n} \frac{m_j m_k}{\left| r_j - r_k \right|^3} \left(r_j - r_k \right) = -\lambda m_k (r_k - r_0), \quad 1 \leq k \leq n, \tag{1.7}$$

where $r_0 = \sum_{k=1}^{n} m_k r_k / \sum_{k=1}^{n} m_k$ is the center of masses $m_1, m_2, \dots, m_n$.

The knowledge of central configurations allows us to compute homographic solutions (see [8]); there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum (see [9]); if the N bodies are moving towards a simultaneous collision, then the bodies tend to a central configuration (see [10]). See also [11, 12].

Some examples of spatial central configurations are a regular tetrahedron with arbitrary positive masses at the vertices [13] and a regular octahedron with six equal masses at the vertices [12]. Double nested spatial central configurations for $2N$ bodies were studied for two nested regular polyhedra in [14]. More recently, the same authors studied central configurations of three regular polyhedra for the spatial $3N$-body problem in [15]. See also [16], where nested regular tetrahedrons are studied.

Recently, Hampton and Santoprete [17] provided new examples of stacked spatial central configurations—central configurations for the N-body problem where a proper subset of the N bodies are already on a central configuration—for the 7-body problem where the bodies are arranged as concentric three- and two-dimensional simplex. New classes of stacked spatial central configurations for the 6-body problem which have four bodies at the vertices of a regular tetrahedron and the other two bodies on a straight line connecting one vertex of the tetrahedron with the center of the opposite face were studied in [18].

In this paper, we study new classes of spatial double pyramidical central configurations (d.p.c.c) for the $N+N+2$-body that satisfy the following.

(1) The position vectors $r_1, r_2, \ldots, r_N$ of masses $m_1, m_2, \ldots, m_N$ are at the vertices of a regular N-gon Γ_1, whose sides have length ι_1. The position vectors $r_{N+1}, r_{N+2}, \ldots, r_{2N}$ of masses $m_{N+1}, m_{N+2}, \ldots, m_{2N}$ are at the vertices of another regular N-gon Γ_2, whose sides have length ι_2. Two N-gons have a same geometric center and form a affine nested N-gons (base plane Π).

(2) Let L be the straight line perpendicular to the base plane Π that contains Γ_1 and Γ_2 and passes through the geometric center of $\Gamma_1 \cup \Gamma_2$. The position vectors r_{2N+1} and r_{2N+2} of masses m_{2N+1} and m_{2N+2} are on L and on opposite sides with respect to the plane Π.

The central configurations studied in this paper are in some measure related to the double pyramidal central configurations (d.p.c.c) studied in [19] and the paper in [20]. The configuration in [19] consists of n masses on a plane that are located at the vertices of a regular n-gon and two equal masses located on the line perpendicular to passing through the geometric center of the N-gon. And the authors in [20] assumed that the center of the N-gon is at the origin, that is, $\Sigma_{j=1}^{n} m_j r_j = 0$, and more that the origin of the inertial system is the center of mass of the system, that is, $\Sigma_{j=1}^{n+3} m_j r_j = 0$ and $r_{n+3} = 0$. In fact the origin is the geometric center of the N-gon. Hence the configuration in [20] is only to append a mass at the geometric center of the N-gon in [19].

As far as we know, the spatial central configurations studied here are very new. The number of bodies (masses) is increased to $2N+2$, it is not to suppose the origin of the inertial system, and the proofs are more difficult than those in [19, 20].

The main results of this paper are the following.

Theorem 1.2. *Consider $N + N + 2$ bodies with masses $m_1, m_2, \ldots, m_{2N}, m_{2N+1}, m_{2N+2}$ located according to the following.*

(i) *$r_1, r_2, \ldots, r_N$ are at the vertices of a regular n-gon Γ_1 inscribed on a circle of radius α.*

(ii) *$r_{N+1}, r_{N+2}, \ldots, r_{2N}$ are at the vertices of another regular n-gon Γ_2 inscribed on a circle of radius a.*

(iii) *r_{2N+1} and r_{2N+2} are on the straight line L, on opposite sides with respect to the plane Π, where L is the straight line that is perpendicular to Π and passes through the geometric center of $\Gamma_1 \cup \Gamma_2$. Let h_1 = distance (r_{2N+1}, Π) and h_2 = distance (r_{2N+2}, Π).*

In order that the $N + N + 2$ bodies can be in a central configuration (c.c.), the following statements hold.

(1) *If $h_1 = h_2 =: h$, then there is $m_{2N+1} = m_{2N+2}$.*

(2) *If* $h_1 = h_2 =: h$, *then not only* $m_{2N+1} = m_{2N+2}$, *but also*

$$\begin{aligned} m_1 = m_2 = \cdots = m_N =: m, \\ m_{N+1} = m_{N+2} = \cdots = m_{2N} =: \tilde{m}. \end{aligned} \tag{1.8}$$

(3) *The origin is the mass center of* $m_1, m_2, \ldots, m_{2N}$ *and also the mass center of* $m_1, m_2, \ldots, m_{2N+2}$, *that is,*

$$\begin{aligned} \sum_{j=1}^{2N} m_j r_j = 0, \\ \sum_{j=1}^{2N+2} m_j r_j = 0. \end{aligned} \tag{1.9}$$

(4) *Get rid of masses* m_{2N+1} *and* m_{2N+2}, *when ratio of masses* $b = m_{N+1}/m_1$ *in some interval and the masses* $m_1, m_2, \ldots, m_{2N}$ *may form a central configuration.*

Remark 1.3. Let $m_{2N+1} = m_{2N+2}$, and the origin is the mass center of $m_1, m_2, \ldots, m_{2N}$, also it is the mass center of $m_1, m_2, \ldots, m_{2N+2}$, and then we have that $h_1 = h_2$ and (1.8) hold. The conclusions are the oppose problem of some items in Theorem 1.2, which is similar to that in [20]. We may similarly prove $h_1 = h_2$. The proof of (1.8) still see Theorem 1.2 in this paper.

Theorem 1.4. *Under the suppositions of the positions for masses, and* $m_1 = m_2 = \cdots = m_N =: m$, $m_{N+1} = m_{N+2} = \cdots = m_{2N} =: \tilde{m}$, $m_{2N+1} = m_{2N+2}$ *and* $h_1 = h_2 =: h$, *then* $m_1, \ldots, m_N, m_{N+1}, \ldots, m_{2N}, m_{2N+1}, m_{2N+2}$ *are in a c.c., if and only if the parameters* b, a, c *and* h *satisfy the following relationships:*

$$\begin{aligned} \frac{\lambda}{M} &= \frac{1}{(N + Nb + 2c)} \left(\sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3} + \sum_j \frac{b(1 - a\rho_j)}{|1 - a\rho_j|^3} + \frac{2c}{(1 + h^2)^{3/2}} \right), \\ \frac{\lambda}{M} &= \frac{1}{h(N + Nb + 2c)} \left[\frac{Nh}{(1 + h^2)^{3/2}} + \frac{Nbh}{(a^2 + h^2)^{3/2}} + \frac{c}{4h^2} \right], \\ b &= \frac{\sum_j \left((a - \rho_j)/|a - \rho_j|^3 \right) - a \sum_{j \neq N} \left((1 - \rho_j)/|1 - \rho_j|^3 \right) + \left(2c/(a^2 + h^2)^{3/2} \right) - \left(2ca/(1 + h^2)^{3/2} \right)}{a \sum_j \left((1 - a\rho_j)/|1 - a\rho_j|^3 \right) - a^{-2} \sum_{j \neq N} \left((1 - \rho_j)/|1 - \rho_j|^3 \right)}, \end{aligned} \tag{1.10}$$

where $b = \tilde{m}/m, c = m_{2N+1}/m$ *and* $\rho_j = \exp(i(2\pi j/N))$.

Theorem 1.5. *Under the suppositions of the positions for masses, and* $m_1 = m_2 = \cdots = m_N =: m$, $m_{N+1} = m_{N+2} = \cdots = m_{2N} = \tilde{m}$, $m_{2N+1} = m_{2N+2}$ *and* $h_1 = h_2 =: h$, *then for any ratios of masses* $b = \tilde{m}/m$ *and* $c = m_{2N+1}/m, m_1, \ldots, m_N, m_{N+1}, \ldots, m_{2N}, m_{2N+1}, m_{2N+2}$ *may form a unique c.c. such that* $a \in (0, 1)$ *and* $h \in (0, +\infty)$.

2. Some Lemmas

Definition 2.1 (see [21]). If $N \times N$ matrix $A = (a_{i,j})$ satisfies

$$a_{i,j} = a_{i-1,j-1}, \quad 1 \le i,\ j \le N, \tag{2.1}$$

where we assume $a_{i,0} = a_{i,N}$ and $a_{0,j} = a_{N,j}$, then one calls that A is a circular matrix.

Lemma 2.2 (see [21]). (i) *If A and B are $N \times N$ circular matrices, for any numbers γ and β, then $A + B$, $A - B$, AB and $\gamma A + \beta B$ are also circular matrices, and $AB = BA$.*

(ii) *Let $A = (a_{i,j})$ be an $N \times N$ circular matrix; the eigenvalues λ_k and the eigenvectors $\vec{v}_k$ of A are*

$$\begin{aligned} \lambda_k(A) &= \sum_j a_{1,j}\rho_{k-1}^{j-1}, \\ \vec{v}_k &= \left(1, \rho_{k-1}, \rho_{k-1}^2, \dots, \rho_{k-1}^{N-1}\right)^T. \end{aligned} \tag{2.2}$$

(iii) *Let A and B be circular matrices, and let $\lambda_k(A)$ and $\lambda_k(B)$ be eigenvalues of A and B. Then the eigenvalues of $A + B$, $A - B$ and $A \cdot B$ are $\lambda_k(A) + \lambda_k(B)$, $\lambda_k(A) - \lambda_k(B)$ and $\lambda_k(A) \cdot \lambda_k(B)$.*

It is clear that.

Lemma 2.3. *If $A = (a_{i,j})$ is an $N \times N$ circular matrices, and $AX = 0$, where $X = (x_1, \dots, x_n)^T$, $x_i > 0\ (i = 1, \dots, N)$, then*

$$\begin{aligned} a_{1,j} + \cdots + a_{N,j} &= 0, \quad 1 \le j \le N, \\ a_{i,1} + \cdots + a_{i,N} &= 0, \quad 1 \le i \le N. \end{aligned} \tag{2.3}$$

Lemma 2.4. *Let A and B be $N \times N$ Hermite circular matrices; then $A + B$, $A - B$, AB and $\gamma A + \beta B (\gamma, \beta \in R)$ are also Hermite circular matrices.*

Lemma 2.5. *Let A be a Hermite circular matrix; then the eigenvalues of A are real number and*

(i) *when $n = 2m + 1 (m \ge 1)$, A can be denoted by $A = A_{2m+1} = cir(a, b_1, b_2, \dots, b_m, \overline{b}_m, \dots, \overline{b}_2, \overline{b}_1)$, where $a \in R$ and $\overline{b}_l$ is a conjugate complex number of b_l. It has eigenvalues*

$$\begin{aligned} \lambda_0 &= a + 2\sum_{l=1}^{m} \operatorname{Re} b_l, \\ \lambda_k &= a + 2\sum_{l=1}^{m}\left[\operatorname{Re} b_l \cos\frac{2k\pi l}{2m+1} - \operatorname{Im} b_l \sin\frac{2k\pi l}{2m+1}\right], \quad 1 \le k \le 2m, \end{aligned} \tag{2.4}$$

(ii) *when* $n = 2m(m \geq 1)$, *A can be denoted by* $A = A_{2m} = \operatorname{cir}(a, b_1, b_2, \ldots, b_{m-1}, b_m, \overline{b}_{m-1}, \ldots, \overline{b}_2, \overline{b}_1,)$. *It has eigenvalues*

$$\lambda_0 = a + 2\sum_{l=1}^{m-1} \operatorname{Re} b_l + b_m,$$

$$\lambda_m = a + 2\sum_{l=1}^{m-1} (-1)^l \operatorname{Re} b_l + (-1)^m b_m, \tag{2.5}$$

$$\lambda_k = a + 2\sum_{l=1}^{m-1} \left[\operatorname{Re} b_l \cos\frac{2k\pi l}{2m} - \operatorname{Im} b_l \sin\frac{2k\pi l}{2m}\right] + (-1)^k b_m, \quad 1 \leq k \leq 2m-1, k \neq m.$$

Lemma 2.6. *The complex subspace* L *of* C^N *generated by* $X_1 = (1, 1, \ldots, 1)$ *and* $X_2 = (1, \rho, \ldots, \rho^{N-1})$, *where* $N = 2k > 2(\rho = \exp 2\pi I/N)$, *and the complex subspace* $\widetilde{L}$ *generated by* X_1, X_2, *and* $X_3 = (1, \rho^{k+1}, \ldots, \rho^{(N-1)(k+1)})$, *where* $N = 2k + 1 > 3$, *all contain no real vectors other than the multiples of* $(1, 1, \ldots, 1)$.

Remark 2.7. Lemmas 2.3–2.5 can be simply proved by properties of circular matrix and Hermite matrix, and after some algebraic computations, Lemma 2.6 can be also simply proved.

Lemma 2.8 (see [5]). *Let* $A = (1/4)\sum_{j \neq N} \csc(\pi j/N)$; *then* $A(N)$ *has the following asymptotic expansion for* N *large:*

$$A(N) \sim \frac{N}{2\pi}\left(\gamma + \log\frac{2N}{\pi}\right) + \sum_{k \geq 0} \frac{(-1)^k (2^{2k-1} - 1) B_{2k}^2 \pi^{2k-1}}{(2k)(2k)!} \frac{1}{N^{2k-1}}, \tag{2.6}$$

where γ *stands for the Euler-Mascheroni constant and* B_{2k} *stands for the Bernoulli numbers.*

Lemma 2.9 (see [5]). *Let* $\Phi_\nu(x) = \sum_j 1/d_j^\nu$, *where* $\nu > 0$ *and* $d_j = 1 + x^2 - 2x\cos(2\pi j/N)$; *then, for* $0 < x < 1$, $\Phi_\nu(x)$ *and all of its any order derivatives are positive. Moreover, the same is thus for* $\Psi_\nu(x) = \sum_j \cos(2\pi j/N)/d_j^\nu$.

3. The Proof of Theorem

Proposition 3.1. *The central configuration equations* (1.4) *or* (1.5) *is equivalent to the following:*

$$\sum_{j=1, j \neq k}^{n} m_j (r_j - r_k)\left(R_{j,k} - \frac{\lambda}{M}\right) = 0, \quad 1 \leq k \leq n, \tag{3.1}$$

where $M = \sum_{k=1}^{n} m_k$, $R_{j,k} = |r_j - r_k|^{-3}$.

Proof. From central configuration equation (1.4) or (1.5), we easily prove. □

Denote $r_k = (x_k, y_k, z_k) \in R^3$, and $\vec{z} = (0,0,1)$. Observing (1.5), one could have a free choice of the origin for a configuration. Without loss of generality, consider that the origin is at the geometric center of $\Gamma_1 \cup \Gamma_2$, and let $z_k = 0,\ 1 \leq k \leq 2N$.

Proposition 3.2. *Under the hypotheses of Theorem 1.2, if* $h_1 = h_2 =: h$, *then the following equations are verified:*

$$m_{2N+1} = m_{2N+2}, \tag{3.2}$$

$$R_{2N+1,j} = R_{2N+1,k} = R_{2N+2,j} = R_{2N+2,k}, \quad 1 \leq j \neq k \leq N, \tag{3.3}$$

$$R_{2N+1,N+j} = R_{2N+1,N+k} = R_{2N+2,N+j} = R_{2N+2,N+k}, \quad 1 \leq j \neq k \leq N, \tag{3.4}$$

where $R_{j,k} = |r_j - r_k|^{-3}$.

Proof. In (3.1), considering the equations along the direction $\vec{z}$, and $k = 1,\ N+1$, we have

$$\begin{gathered} m_{2N+1}(z_{2N+1} - z_1)\left[R_{1,2N+1} - \frac{\lambda}{M}\right] + m_{2N+2}(z_{2N+2} - z_1)\left[R_{1,2N+2} - \frac{\lambda}{M}\right] = 0, \\ m_{2N+1}(z_{2N+1} - z_{N+1})\left[R_{N+1,2N+1} - \frac{\lambda}{M}\right] + m_{2N+2}(z_{2N+2} - z_{N+1})\left[R_{N+1,2N+2} - \frac{\lambda}{M}\right] = 0, \end{gathered} \tag{3.5}$$

where

$$\begin{gathered} R_{1,2N+1} = \left(|r_1|^2 + |z_{2N+1}|^2\right)^{-3/2}, \\ R_{1,2N+2} = \left(|r_1|^2 + |z_{2N+2}|^2\right)^{-3/2}, \\ R_{N+1,2N+1} = \left(|r_{N+1}|^2 + |z_{2N+1}|^2\right)^{-3/2}, \\ R_{N+1,2N+2} = \left(|r_{N+1}|^2 + |z_{2N+2}|^2\right)^{-3/2}, \\ z_1 = z_{N+1} = 0. \end{gathered} \tag{3.6}$$

By the $h_1 = h_2$, we have

$$z_{2N+1} = -z_{2N+2}, \qquad R_{1,2N+1} = R_{1,2N+2}, \qquad R_{N+1,2N+1} = R_{N+1,2N+2}. \tag{3.7}$$

Hence

$$\begin{gathered} (m_{2N+1} - m_{2N+2})\left[R_{1,2N+1} - \frac{\lambda}{M}\right] = 0, \\ (m_{2N+1} - m_{2N+2})\left[R_{N+1,2N+1} - \frac{\lambda}{M}\right] = 0. \end{gathered} \tag{3.8}$$

Obviously $R_{1,2N+1} \neq R_{N+1,2N+1}$, so

$$m_{2N+1} = m_{2N+2}. \tag{3.9}$$

By our assumptions for the origin, (3.3), (3.4) clearly holds. □

Equation (3.9) of Proposition 3.2 proves item (1) of Theorem 1.2.

Proposition 3.3. *Under the hypotheses of Theorem 1.2, if $m_{2N+1} = m_{2N+2}$, the origin is the mass center of $m_1, m_2, \ldots, m_{2N}$, and it also is the mass center of $m_1, m_2, \ldots, m_{2N+2}$, then*

$$h_1 = h_2 =: h \tag{3.10}$$

and (3.3), (3.4) *hold.*

Proof. The proof is very similar to that in [20]. □

Proposition 3.4. *Under the hypotheses of Theorem 1.2, if $h_1 = h_2 =: h$, then the masses at the vertices of circle Γ_1 are equal, and also the masses at the vertices of circle Γ_2 are equal, that is,*

$$\begin{aligned} m_1 &= m_2 = \cdots = m_N, \\ m_{N+1} &= m_{N+2} = \cdots = m_{2N}. \end{aligned} \tag{3.11}$$

Proof. Because if $r \to \varepsilon r$ is a transformation in a central configuration, then $\lambda \to (1/\varepsilon^2)\lambda$ can be a new parameter of a central configuration. We say that the old and the new are equivalent. Hence without loss of generality, we may let $\alpha = 1,\ 0 < a < 1$. Then the vectors of positions based on the previous assumptions can be interpreted by the following:

$$\begin{aligned} r_k &= \left(\rho_k, 0\right) \quad (1 \leq k \leq N), \\ r_{N+k} &= \left(\tilde{\rho}_k, 0\right) \quad (1 \leq k \leq N), \\ r_{2N+1} &= \left(0 \cdot \rho_0, h\right) \quad (h > 0), \\ r_{2N+2} &= \left(0 \cdot \rho_0, -h\right) \quad (h > 0), \end{aligned} \tag{3.12}$$

where $\rho_k = \exp((2\pi k/N)i), \tilde{\rho}_k = a\rho_k,\ a > 0$, and $i = \sqrt{-1}, \rho_k$ denote the N complex kth roots of unity, that is, that $m_k (1 \leq k \leq N)$ each locates at the vertices r_k of the one regular N-gon $\Gamma_1, m_{N+k} (1 \leq k \leq N)$ each locates at the vertices r_{N+k} of the other regular N-gon Γ_2, and m_{2N+1} and m_{2N+2} lie on the vertices of r_{2N+1}, r_{2N+2}. Then the center of masses is

$$r_0 = \frac{\sum_j \left(m_j r_j + m_{N+j} r_{N+j}\right) + m_{2N+1} r_{2N+1} + m_{2N+2} r_{2N+2}}{M}, \tag{3.13}$$

where

$$M=\sum_{j}\left(m_j+m_{N+j}\right)+m_{2N+1}+m_{2N+2}. \tag{3.14}$$

In (3.13)-(3.14) and throughout this paper, unless other restricted, all indices and summations will range from 1 to N.

Let $h_1=h_2=:h$; then $m_{2N+1}=m_{2N+2}$. Now we discuss all equations for the $2N$ masses on the base plane Π. According to (3.1), then

$$\begin{gathered}
\sum_{j\neq k}m_j\left(\frac{1}{\left|\rho_k-\rho_j\right|^3}-\frac{\lambda}{M}\right)\left(\rho_k-\rho_j\right)+\sum_{j}m_{N+j}\left(\frac{1}{\left|\rho_k-\tilde{\rho}_j\right|^3}-\frac{\lambda}{M}\right)\left(\rho_k-\tilde{\rho}_j\right)\\
+2m_{2N+1}\left(\frac{1}{\left(1+h^2\right)^{3/2}}-\frac{\lambda}{M}\right)\rho_k=0,\\
\sum_{j}m_j\left(\frac{1}{\left|\tilde{\rho}_k-\rho_j\right|^3}-\frac{\lambda}{M}\right)\left(\tilde{\rho}_k-\rho_j\right)+\sum_{j\neq k}m_{N+j}\left(\frac{1}{\left|\tilde{\rho}_k-\tilde{\rho}_j\right|^3}-\frac{\lambda}{M}\right)\left(\tilde{\rho}_k-\tilde{\rho}_j\right)\\
+2m_{2N+1}\left(\frac{1}{\left(a^2+h^2\right)^{3/2}}-\frac{\lambda}{M}\right)a\rho_k=0.
\end{gathered} \tag{3.15}$$

Multiplying both sides by ρ_{N-k}, noting that $|\rho_k-\rho_j|=|\rho_k||1-\rho_{j-k}|=|1-\rho_{j-k}|$ and $\tilde{\rho}_k=a\rho_k$, we see that (3.15) may be written as

$$\begin{gathered}
\sum_{j\neq k}m_j\left(\frac{1}{\left|1-\rho_{j-k}\right|^3}-\frac{\lambda}{M}\right)\left(1-\rho_{j-k}\right)+\sum_{j}m_{N+j}\left(\frac{1}{\left|1-a\rho_{j-k}\right|^3}-\frac{\lambda}{M}\right)\left(1-a\rho_{j-k}\right)\\
+2m_{2N+1}\left(\frac{1}{\left(1+h^2\right)^{3/2}}-\frac{\lambda}{M}\right)=0,\\
\sum_{j}m_j\left(\frac{1}{\left|a-\rho_{j-k}\right|^3}-\frac{\lambda}{M}\right)\left(a-\rho_{j-k}\right)+\sum_{j\neq k}m_{N+j}\left(\frac{1}{\left|a-a\rho_{j-k}\right|^3}-\frac{\lambda}{M}\right)\left(a-a\rho_{j-k}\right)\\
+2m_{2N+1}\left(\frac{1}{\left(a^2+h^2\right)^{3/2}}-\frac{\lambda}{M}\right)a=0,
\end{gathered} \tag{3.16}$$

where $k=1,2,\ldots,N$.

Now we define the $N \times N$ circular matrices $C = [c_{k,j}]$, $A = [a_{k,j}]$, $B = [b_{k,j}]$, and $D = [d_{k,j}]$ as follows:

$$
\begin{gathered}
c_{k,j} = 0, \quad \text{for } k = j, \\
c_{k,j} = \left(\frac{1}{\left|1 - \rho_{j-k}\right|^3} - \frac{\lambda}{M} \right) (1 - \rho_{j-k}), \quad \text{for } k \neq j, \\
a_{k,j} = \left(\frac{1}{\left|1 - a\rho_{j-k}\right|^3} - \frac{\lambda}{M} \right) (1 - a\rho_{j-k}), \\
b_{k,j} = \left(\frac{1}{\left|a - \rho_{j-k}\right|^3} - \frac{\lambda}{M} \right) (a - \rho_{j-k}), \\
d_{k,j} = 0, \quad \text{for } k = j, \\
d_{k,j} = \left(\frac{1}{\left|a - a\rho_{j-k}\right|^3} - \frac{\lambda}{M} \right) (a - a\rho_{j-k}) \quad \text{for } k \neq j.
\end{gathered}
\tag{3.17}
$$

Also define

$$
\begin{gathered}
\vec{1} = (1, \dots, 1)^T, \\
E = 2\left(\frac{1}{(1 + h^2)^{3/2}} - \frac{\lambda}{M} \right) \cdot \vec{1}, \\
F = 2a\left(\frac{1}{(a^2 + h^2)^{3/2}} - \frac{\lambda}{M} \right) \cdot \vec{1}.
\end{gathered}
\tag{3.18}
$$

We see that (3.16) holds if and only if the matrix equation

$$
\begin{pmatrix} C & A & E \\ B & D & F \end{pmatrix}
\begin{pmatrix} m_1 \\ \vdots \\ m_N \\ m_{N+1} \\ \vdots \\ m_{2N} \\ m_{2N+1} \end{pmatrix} = 0
\tag{3.19}
$$

has a positive solution. Let

$$
\vec{m} = (m_1, \dots, m_N)^T, \qquad \vec{\tilde{m}} = (m_{N+1}, \dots, m_{2N})^T.
\tag{3.20}
$$

Then (3.19) is equivalent to

$$C\vec{m} + A\vec{\vec{m}} + m_{2N+1}E = \vec{0}, \tag{3.21}$$

$$B\vec{m} + D\vec{\vec{m}} + m_{2N+1}F = 0. \tag{3.22}$$

Noticing that A, B, C, and D are $N \times N$ circular matrix, using the properties of circular matrix, we know that they must have positive real eigenvector $\vec{1}$. Each of (3.21) and (3.22) left multiplies $1^{\vec{T}} = (1, 1, \ldots, 1)$; there are

$$\begin{aligned}
&\left(\sum_k m_k\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\lambda}{M}\right)(1-\rho_j)+\left(\sum_k m_{N+k}\right)\sum_j\left(\frac{1}{|1-a\rho_j|^3}-\frac{\lambda}{M}\right)(1-a\rho_j)\\
&\quad+2m_{2N+1}\cdot N\left(\frac{1}{(1+h^2)^{3/2}}-\frac{\lambda}{M}\right)=0,\\
&\left(\sum_k m_k\right)\sum_{j}\left(\frac{1}{|a-\rho_j|^3}-\frac{\lambda}{M}\right)(a-\rho_j)+\left(\sum_k m_{N+k}\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\lambda}{M}\right)(a-a\rho_j)\\
&\quad+2m_{2N+1}\cdot N\left(\frac{1}{(a^2+h^2)^{3/2}}-\frac{\lambda}{M}\right)a=0.
\end{aligned} \tag{3.23}$$

By (3.21) and (3.22) we have

$$\begin{aligned}
&(CD-AB)\vec{m}+2m_{2N+1}\left[\left(\frac{1}{(1+h^2)^{3/2}}-\frac{\lambda}{M}\right)D-a\left(\frac{1}{(a^2+h^2)^{3/2}}-\frac{\lambda}{M}\right)A\right]\vec{1}=\vec{0},\\
&(AB-CD)\vec{\vec{m}}+2m_{2N+1}\left[\left(\frac{1}{(1+h^2)^{3/2}}-\frac{\lambda}{M}\right)B-a\left(\frac{1}{(a^2+h^2)^{3/2}}-\frac{\lambda}{M}\right)C\right]\vec{1}=\vec{0}.
\end{aligned} \tag{3.24}$$

From Lemma 2.2 we see that $((1/(1+h^2)^{3/2}) - (\lambda/M))D - a((1/(a^2+h^2)^{3/2}) - (\lambda/M))A$, $((1/(1+h^2)^{3/2}) - (\lambda/M))B - a((1/(a^2+h^2)^{3/2}) - (\lambda/M))C$, and $CD - AB$ are circular matrix, and we know that they must have positive real eigenvector $\vec{1}$. By the properties of circular matrix, (3.24) can be written as

$$\begin{aligned}
&(CD-AB)\cdot\vec{m}+\gamma_1\cdot\vec{1}=\vec{0},\\
&(AB-CD)\cdot\vec{\vec{m}}+\gamma_2\cdot\vec{1}=\vec{0},
\end{aligned} \tag{3.25}$$

where

$$\gamma_1 \cdot \vec{1} = 2m_{2N+1}\left[\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)D - a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)A\right]\vec{1},$$

$$\gamma_2 \cdot \vec{1} = 2m_{2N+1}\left[\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)B - a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)C\right]\vec{1},$$

$$\begin{aligned}\gamma_1 = 2m_{2N+1}\Bigg[&\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3} - \frac{\lambda}{M}\right)(a-a\rho_j)\\ &-a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j}\left(\frac{1}{|1-a\rho_j|^3} - \frac{\lambda}{M}\right)(1-a\rho_j)\Bigg],\\ \gamma_2 = 2m_{2N+1}\Bigg[&\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j}\left(\frac{1}{|a-\rho_j|^3} - \frac{\lambda}{M}\right)(a-\rho_j)\\ &-a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3} - \frac{\lambda}{M}\right)(1-\rho_j)\Bigg].\end{aligned} \tag{3.26}$$

We easily prove $\gamma_1, \gamma_2 \in R$, and from (3.23), we have

$$\begin{aligned}&\left(\sum_k m_{N+k}\right)\Bigg\{\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3} - \frac{\lambda}{M}\right)(a-a\rho_j)\\ &\qquad -a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j}\left(\frac{1}{|1-a\rho_j|^3} - \frac{\lambda}{M}\right)(1-a\rho_j)\Bigg\}\\ &+\left(\sum_k m_k\right)\Bigg\{\left(\frac{1}{(1+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j}\left(\frac{1}{|a-\rho_j|^3} - \frac{\lambda}{M}\right)(a-\rho_j)\\ &\qquad -a\left(\frac{1}{(a^2+h^2)^{3/2}} - \frac{\lambda}{M}\right)\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3} - \frac{\lambda}{M}\right)(1-\rho_j)\Bigg\} = 0,\end{aligned} \tag{3.27}$$

that is,

$$\gamma_1\sum_k m_{N+k} + \gamma_2\sum_k m_k = 0, \quad m_j > 0,\ \tilde{m}_j > 0. \tag{3.28}$$

Hence one has the following.

(1) If $\gamma_1 = 0$, then $\gamma_2 = 0$. By (3.25), there are

$$(CD - AB) \cdot \vec{m} = \vec{0}, \tag{3.29}$$

$$(CD - AB) \cdot \vec{\tilde{m}} = \vec{0}. \tag{3.30}$$

We notice that (3.29) or (3.30) must have positive real solutions, which is equivalent to that $CD - AB$ has positive real eigenvectors corresponding to eigenvalue 0.

But we notice that (3.29) and (3.30) must hold, and for $k = 1$, we have an eigenvalue $\lambda_1 = 0$, and an matching eigenvector $\vec{v}_1 = (1, 1, \ldots, 1)^T$ of $CD - AB$. Noticing that A, B, C, and D are Hermite circular matrices, from the properties of circular and Hermite matrix in Lemmas 2.2 and 2.4, then $CD - AB = G$ is also a Hermite circular matrix. We may denote G by $\operatorname{cir}(a_0, g_1, g_2, \ldots, g_{m-1}, g_m, \overline{g}_{m-1}, \ldots, \overline{g}_2, \overline{g}_1)$ when $N = 2m$, where $a_0, g_m \in R$. We also denote G by $\operatorname{cir}(a_0, g_1, g_2, \ldots, g_{m-1}, g_m, \overline{g}_m, \ldots, \overline{g}_2, \overline{g}_1)$ when $N = 2m + 1$, where $a_0 \in R$. Using Lemmas 2.4 and 2.5, after complex computation, we may prove that the kernel of G is at most a subspace in L when $N = 2m > 2$, and a subspace in $\widetilde{L}$ when $N = 2m + 1 > 3$ (see [22]), where the meanings of L and $\widetilde{L}$ are in Lemma 2.6. Hence the kernel of G does not contain any positive real vectors other than multiplication of $\vec{1} = (1, 1, \ldots, 1)$.

When $N = 2, 3$, we may easily prove the conclusion.

(2) If $\gamma_1 \neq 0$, then $\gamma_2 \neq 0$; from (3.25) and (3.28),we get

$$(AB - CD)\left[\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k m_{N+k}\right)\vec{m}\right] = \vec{0}. \tag{3.31}$$

If

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k m_{N+k}\right)\vec{m} \neq 0, \tag{3.32}$$

let $G = AB - CD = (g_{ij})$, which is not zero circular matrix; by Lemmas 2.2 and 2.3 we see that $\sum_j g_{ij} = 0$ and G has an eigenvalue $\lambda_1 = 0$. Using the properties of circular matrix, we have $(AB - CD)\vec{1} = \vec{0}$ or $\vec{1}^T(AB - CD) = \vec{0}^T$. Let $\vec{1}^T$ left multiplies (3.25); we get $\gamma_1 = \gamma_2 = 0$, which is a contradiction to the supposition. So

$$\left(\sum_k m_k\right)\vec{\tilde{m}} - \left(\sum_k \tilde{m}_k\right)\vec{m} = 0, \tag{3.33}$$

and then $m_{N+j} = b m_j$, where $b = \sum_k \tilde{m}_k / \sum_k m_k$, that is, $\vec{\tilde{m}} = b\vec{m}$. Substituting it into (3.21)-(3.22), we have

$$\begin{aligned} (C + bA)\vec{m} + 2m_{2N+1}\left(\frac{1}{(1 + h^2)^{3/2}} - \frac{\lambda}{M}\right)\vec{1} &= 0, \\ (B + bD)\vec{m} + 2a \cdot m_{2N+1}\left(\frac{1}{(a^2 + h^2)^{3/2}} - \frac{\lambda}{M}\right)\vec{1} &= 0. \end{aligned} \tag{3.34}$$

Then

$$\left[a\cdot\left(\frac{1}{(a^2+h^2)^{3/2}}-\frac{\lambda}{M}\right)(C+bA)-\left(\frac{1}{(1+h^2)^{3/2}}-\frac{\lambda}{M}\right)(B+bD)\right]\vec{m}=0. \tag{3.35}$$

Noticing that $a\cdot((1/(a^2+h^2)^{3/2})-(\lambda/M))(C+bA)-((1/(1+h^2)^{3/2})-(\lambda/M))(B+bD)$ is a Hermite circular matrix, similarly we also have $\vec{m}=(m_1,m_1,\ldots,m_1)^T$ and $\vec{\vec{m}}=(m_{N+1},\ldots,m_{N+1})$, where $m_1,m_{N+1}>0$. □

Remark 3.5. Proposition 3.4 proves item (2) of Theorem 1.2.

Proposition 3.6. *Under the hypothesis of Theorem 1.2, if $h_1=h_2$, then the origin is the mass center of $m_1,m_2,\ldots,m_{2N}$ and also is the mass center of $m_1,m_2,\ldots,m_{2N+2}$.*

Proof. We have already

$$m_{2N+1}=m_{2N+2}, \tag{3.36}$$

$$m_1=m_2=\cdots=m_N, \tag{3.37}$$

$$m_{N+1}=m_{N+2}=\cdots=m_{2N}. \tag{3.38}$$

So there are

$$\begin{gathered}\sum_{j=1}^{2N}m_jr_j=0,\\ \sum_{j=1}^{2N+2}m_jr_j=0,\end{gathered} \tag{3.39}$$

by the positions of $r_j,\ j=1,2,\ldots,2N+2$. □

Remark 3.7. Proposition 3.6 proves item (3) of Theorem 1.2. By (3.37) and (3.38), item (4) of Theorem 1.2 may be proved.

Proposition 3.8. *Under the hypothesis of Theorem 1.4, the conclusion of Theorem 1.4 holds.*

Proof (The Proof of the Necessary). By the hypothesis of Theorem 1.4, let $\widetilde{m} = bm, m_{2N+1} = cm$; from (3.23), there are

$$\begin{aligned}
&\left[\sum_{j\neq N}\left(\frac{1}{|1-\rho_j|^3}-\frac{\lambda}{M}\right)(1-\rho_j)+b\sum_{j}\left(\frac{1}{|1-a\rho_j|^3}-\frac{\lambda}{M}\right)(1-a\rho_j)\right]\\
&\quad+2c\left(\frac{1}{(1+h^2)^{3/2}}-\frac{\lambda}{M}\right)=0,\\
&\left[\sum_{j}\left(\frac{1}{|a-\rho_j|^3}-\frac{\lambda}{M}\right)(a-\rho_j)+b\sum_{j\neq N}\left(\frac{1}{|a-a\rho_j|^3}-\frac{\lambda}{M}\right)(a-a\rho_j)\right]\\
&\quad+2c\left(\frac{1}{(a^2+h^2)^{3/2}}-\frac{\lambda}{M}\right)a=0.
\end{aligned}\tag{3.40}$$

We know that

$$\begin{aligned}
\sum_j(1-\rho_j)&=N,\\
\sum_j b(1-a\rho_j)&=bN,\\
\sum_j(a-\rho_j)&=aN,\\
\sum_j b(a-a\rho_j)&=abN.
\end{aligned}\tag{3.41}$$

From (3.40) we have

$$\begin{aligned}
\frac{\lambda}{M}&=\frac{1}{N+Nb+2c}\left[\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-a\rho_j}{|1-a\rho_j|^3}+\frac{2c}{(1+h^2)^{3/2}}\right],\\
\frac{\lambda}{M}&=\frac{1}{a(N+Nb+2c)}\left[\sum_j\frac{a-\rho_j}{|a-\rho_j|^3}+b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}+\frac{2c}{(a^2+h^2)^{3/2}}\right],
\end{aligned}\tag{3.42}$$

and similarly we also have

$$\frac{\lambda}{M}=\frac{1}{h(N+Nb+2c)}\left[\frac{Nh}{(1+h^2)^{3/2}}+\frac{Nbh}{(a^2+h^2)^{3/2}}+\frac{ch}{4h^3}\right]\tag{3.43}$$

by (3.1). Then

$$a\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-a\rho_j}{|1-a\rho_j|^3}+\frac{2c}{(1+h^2)^{3/2}}\right)=\sum_j\frac{a-\rho_j}{|a-\rho_j|^3}+b\sum_{j\neq N}\frac{a-a\rho_j}{|a-a\rho_j|^3}+\frac{2c}{(a^2+h^2)^{3/2}}, \tag{3.44}$$

$$b=\frac{\sum_j\left((a-\rho_j)/|a-\rho_j|^3\right)-a\sum_{j\neq N}\left((1-\rho_j)/|1-\rho_j|^3\right)+\left(2c/(a^2+h^2)^{3/2}\right)-\left(2ca/(1+h^2)^{3/2}\right)}{a\sum_j\left((1-a\rho_j)/|1-a\rho_j|^3\right)-a^{-2}\sum_{j\neq N}\left((1-\rho_j)/|1-\rho_j|^3\right)}, \tag{3.45}$$

where (3.42), (3.43), (3.45) imply (1.10). □

Proof of Sufficient. For $N\geq 2$, under the suppositions of Theorem 1.4, then $m_1,m_2,\ldots,m_{2N+2}$ in a c.c. if and only if (3.42)–(3.43) hold, which are equivalent to (1.10). Hence the proof of the sufficient was finished. □

Proposition 3.9. *Under the hypothesis of Theorem 1.5, for any ratios of masses $b=\tilde{m}/m$ and $c=m_{2N+1}/m$, $m_1,m_2,\ldots,m_{2N+2}$ may form a unique* c.c. *such that $a\in(0,1)$ and $h\in(0,+\infty)$.*

Here is Theorem 1.5.

Proof. Under the suppositions of the positions for masses, and $h_1=h_2=:h$, $m_1=m_2=\cdots=m_N=:m$, $m_{N+1}=m_2=\cdots=\tilde{m}_{2N}=:\tilde{m}$, and $m_{2N+1}=m_{2N+2}$, then for any ratios of masses $b=\tilde{m}/m$ and $c=m_{2N+1}/m$, $m_1,m_2,\ldots,m_{2N+2}$ are in a unique c.c. such that $a\in(0,1), h>0$, if and only if that (1.10)–(1.14) or (3.42)–(3.43) have a unique positive solution on $0<a<1$ and $h>0$ for any given numbers b, c (>0).

Let $a=x$, $h=y$, and

$$\begin{aligned}K(x,y)&=x\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-x\rho_j}{|1-x\rho_j|^3}+\frac{2c}{(1+y^2)^{3/2}}\right)-\sum_j\frac{x-\rho_j}{|x-\rho_j|^3}\\&\quad-b\sum_{j\neq N}\frac{x-x\rho_j}{|x-x\rho_j|^3}-\frac{2c}{(x^2+y^2)^{3/2}},\\L(x,y)&=\left(\sum_{j\neq N}\frac{1-\rho_j}{|1-\rho_j|^3}+b\sum_j\frac{1-x\rho_j}{|1-x\rho_j|^3}+\frac{2c}{(1+y^2)^{3/2}}\right)-\frac{N}{(1+y^2)^{3/2}}\\&\quad-\frac{Nb}{(x^2+y^2)^{3/2}}-\frac{c}{4y^3}.\end{aligned} \tag{3.46}$$

It suffices to prove that $K(x,y)=0$, and $L(x,y)=0$ have a unique positive solution (x,y) for any given ratios b, c (>0) in $x\in(0,1)$, and $y\in(0,+\infty)$.

Now let

$$d_j^2 = 1 + x^2 - 2x\cos\left(\frac{2\pi j}{N}\right),$$
$$P(x) = \sum_j \frac{1}{d_j^3},$$
$$Q(x) = \sum_j \frac{\cos(2\pi j/N)}{d_j^3},$$
$$A = \sum_{j \neq N} \frac{1-\rho_j}{\left|1-\rho_j\right|^3} = \begin{cases} \dfrac{1}{4}\left(2\displaystyle\sum_{j=1}^{(N/2)-1} \csc\left(\frac{\pi j}{N}\right) + 1\right) & \text{when } N \text{ is even}, \\ \dfrac{1}{2}\displaystyle\sum_{j=1}^{(N-1)/2} \csc\left(\frac{\pi j}{N}\right) & \text{when } N \text{ is odd}, \end{cases}$$
$$\Phi(x) = \sum_j \frac{1}{d_j}. \tag{3.47}$$

It follows from the definitions that

$$\Phi(x) = \left(1 + x^2\right)P(x) - 2xQ(x), \tag{3.48}$$

and it implies

$$P(x) - xQ(x) = Q(x) + x(Q(x) - xP(x)). \tag{3.49}$$

Since

$$\frac{d\Phi}{dx} = Q(x) - xP(x), \tag{3.50}$$

then now K and L in (3.46) can be written as follows:

$$\begin{aligned} K(x,y) &= \left(x - \frac{b}{x^2}\right)A + bx(P(x) - xQ(x)) + (Q(x) - xP(x)) + \frac{2cx}{\left(1+y^2\right)^{3/2}} - \frac{2c}{\left(x^2+y^2\right)^{3/2}} \\ &= \left(x - \frac{b}{x^2}\right)A + bx\Phi(x) + \left(1 + bx^2\right)\frac{d\Phi}{dx} + \frac{2cx}{\left(1+y^2\right)^{3/2}} - \frac{2c}{\left(x^2+y^2\right)^{3/2}}, \end{aligned}$$

$$L(x,y) = \left(\sum_{j\neq N}\frac{1-\rho_j}{\left|1-\rho_j\right|^3} + b\sum_{j}\frac{1-x\rho_j}{\left|1-x\rho_j\right|^3} + \frac{2c}{\left(1+y^2\right)^{3/2}}\right) - \frac{N}{\left(1+y^2\right)^{3/2}} - \frac{Nb}{\left(x^2+y^2\right)^{3/2}} - \frac{c}{4y^3}$$
$$= A + b\left[Q(x) + x\frac{d\Phi}{dx}\right] + \frac{2c}{\left(1+y^2\right)^{3/2}} - \frac{N}{\left(1+y^2\right)^{3/2}} - \frac{Nb}{\left(x^2+y^2\right)^{3/2}} - \frac{c}{4y^3}, \tag{3.51}$$

where $x = a \in (0,1)$. From Lemmas 2.8 and 2.9 and their proofs, and with implicit function theory, after some complex calculation (some ideas partially see [23]), we can prove that $K = 0$, and $L = 0$ have only one solution for any given ratios of masses $b(> 0)$ and $c(> 0)$ such that $0 < x < 1$, and $0 < h < +\infty$. □

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