Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 948050, 6 pages
doi:10.1155/2012/948050

Research Article

On the Modified q-Bernoulli Numbers of Higher Order with Weight

T. Kim,[1] J. Choi,[2] Y.-H. Kim,[2] and S.-H. Rim[3]

[1] *Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea*
[2] *Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea*
[3] *Department of Mathematics Education, Kyungpook National University, Daegu 702-701, Republic of Korea*

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

Received 11 August 2011; Revised 24 November 2011; Accepted 13 December 2011

Academic Editor: Ferhan M. Atici

The purpose of this paper is to give some properties of the modified q-Bernoulli numbers and polynomials of higher order with weight. In particular, by using the bosonic p-adic q-integral on $\mathbb{Z}_p$, we derive new identities of q-Bernoulli numbers and polynomials with weight.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic norm of $\mathbb{C}_p$ is defined by $|p|_p = 1/p$. When one talks of a q-extension, q can be considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim (see [1–3]) as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1.1}$$

where $[x]_q$ is the q-number of x which is defined by $[x]_q = (1-q^x)/(1-q)$.

From (1.1), we have

$$q^n I_q(f_n) - I_q(f) = (q-1)\sum_{l=0}^{n-1} q^l f(l) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} q^l f'(l), \tag{1.2}$$

where $f_n(x) = f(x+n)$ (see [2–4]).

As is well known, Bernoulli numbers are inductively defined by

$$B_0 = 1, \qquad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.3}$$

with the usual convention about replacing B^n by B_n (see [3, 5]).

In [2, 5, 6], the q-Bernoulli numbers are defined by

$$B_{0,q} = \frac{q-1}{\log q}, \qquad (qB_q+1)^n - B_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{1.4}$$

with the usual convention about replacing B_q^n by $B_{n,q}$. Note that $\lim_{q\to 1} B_{n,q} = B_n$. In the viewpoint of (1.4), we consider the modified q-Bernoulli numbers with weight.

In this paper we study families of the modified q-Bernoulli numbers and polynomials of higher order with weight. In particular, by using the multivariate p-adic q-integral on $\mathbb{Z}_p$, we give new identities of the higher-order q-Bernoulli numbers and polynomials with weight.

2. Modified q-Bernoulli Numbers with Weight of Higher Order

For $n \in \mathbb{Z}_+$, let us consider the following modified q-Bernoulli numbers with weight α (see [1, 3]):

$$\begin{aligned} \widetilde{B}_{n,q}^{(\alpha)} &= \int_{\mathbb{Z}_p} [x]_{q^\alpha}^n q^{-x} d\mu_q(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{\alpha l}{[\alpha l]_q}, \\ \widetilde{B}_{n,q}^{(\alpha)}(x) &= \int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n q^{-y} d\mu_q(y) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{\alpha l x} \frac{\alpha l}{[\alpha l]_q}. \end{aligned} \tag{2.1}$$

From (2.1), we note that

$$\widetilde{B}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \widetilde{B}_{l,q}^{(\alpha)} \tag{2.2}$$

(see [1, 3]).

For $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by making use of the multivariate p-adic q-integral on $\mathbb{Z}_p$, we consider the following modified q-Bernoulli numbers with weight α of order k, $\widetilde{B}_{n,q}^{(k,\alpha)}$:

$$\widetilde{B}_{n,q}^{(k,\alpha)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_{q^\alpha}^n q^{-x_1-\cdots-x_k} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.3}$$

Note that $\widetilde{B}_{n,q}^{(1,\alpha)} = \widetilde{B}_{n,q}^{(\alpha)}$ and $\lim_{q\to 1}\widetilde{B}_{n,q}^{(k,\alpha)} = B_n^{(k)}$, where $B_n^{(k)}$ are the nth ordinary Bernoulli numbers of order k.

For $k, N \in \mathbb{N}$, we have

$$\begin{aligned}
&\left(\frac{1-q}{1-q^{p^N}}\right)^k \sum_{i_1=0}^{p^N-1} \cdots \sum_{i_k=0}^{p^N-1} [i_1 + \cdots + i_k]_{q^\alpha}^n \\
&= \left(\frac{1-q}{1-q^{p^N}}\right)^k \left(\frac{1}{1-q^\alpha}\right)^n \sum_{i_1,\ldots,i_k=0}^{p^N-1} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{\alpha(i_1+\cdots+i_k)j} \\
&= \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{(1-q)^k}{\left(1-q^{p^N}\right)^k} \underbrace{\left(\frac{1-q^{\alpha p^N j}}{1-q^{\alpha j}} \cdots \frac{1-q^{\alpha p^N j}}{1-q^{\alpha j}}\right)}_{k-\text{times}}.
\end{aligned} \tag{2.4}$$

By (1.1), (2.3), and (2.4), we get

$$\widetilde{B}_{n,q}^{(k,\alpha)} = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{(\alpha j)^k}{[\alpha j]_q^k}. \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, one has*

$$\widetilde{B}_{n,q}^{(k,\alpha)} = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \frac{(\alpha j)^k}{[\alpha j]_q^k}. \tag{2.6}$$

Let us consider the modified q-Bernoulli and polynomials with weight α of order k as follows:

$$\widetilde{B}_{n,q}^{(k,\alpha)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k]_{q^\alpha}^n q^{-x_1-\cdots-x_k} d\mu_q(x_1) \cdots d\mu_q(x_k). \tag{2.7}$$

By the same method of (2.5), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,q}^{(k,\alpha)}(x) = \frac{1}{(1-q)^n [\alpha]_q^n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j q^{\alpha x j} \frac{(\alpha j)^k}{[\alpha j]_q^k}. \tag{2.8}$$

By Theorem 2.2, we get

$$\begin{aligned}
\widetilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) &= \frac{1}{(1-q^{-\alpha})^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\frac{(\alpha j)^k}{[\alpha j]_{q^{-1}}^k}q^{-\alpha j(k-x)} \\
&= \frac{(-1)^n q^{\alpha n}}{(1-q^{\alpha})^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(\frac{q^{-1}(q-1)\alpha j}{(q^{\alpha j}-1)q^{-\alpha j}}\right)^k q^{-\alpha j(k-x)} \\
&= \frac{(-1)^n q^{\alpha n}}{(1-q^{\alpha})^n}\sum_{j=0}^{n}\binom{n}{j}(-1)^j q^{\alpha j x}q^{-k}\frac{(\alpha j)^k}{[\alpha j]_q^k} \\
&= (-1)^n q^{\alpha n-k}\widetilde{B}_{n,q}^{(k,\alpha)}(x).
\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) = (-1)^n q^{\alpha n-k}\widetilde{B}_{n,q}^{(k,\alpha)}(x), \qquad \widetilde{B}_{n,q^{-1}}^{(k,\alpha)}(k) = (-1)^n q^{\alpha n-k}\widetilde{B}_{n,q}^{(k,\alpha)}. \tag{2.10}$$

From Theorem 2.3, we note that

$$\lim_{q\to 1}\widetilde{B}_{n,q^{-1}}^{(k,\alpha)}(k-x) = B_n^{(k)}(k-x), \qquad \lim_{q\to 1}\widetilde{B}_{n,q^{-1}}^{(k,\alpha)}(k) = (-1)^n B_n^{(k)}. \tag{2.11}$$

Thus, we have $B_n^{(k)}(k) = (-1)^n B_n^{(k)}$, where $B_n^{(k)}$ are the nth Bernoulli numbers of order k.

From (2.3) and (2.7), we can derive the following equations:

$$\begin{aligned}
\widetilde{B}_{k,q}^{(l,\alpha)}(x) &= \lim_{N\to\infty}\frac{1}{[m]_q^l [p^N]_{q^m}^l}\sum_{i_1,\ldots,i_l=0}^{m-1}\sum_{n_1,\ldots,n_l=0}^{p^N-1}[x+i_1+\cdots+i_l+m(n_1+\cdots+n_l)]_{q^\alpha}^k \\
&= \frac{[m]_{q^\alpha}^k}{[m]_q^l}\sum_{i_1,\ldots,i_l=0}^{m-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left[\frac{x+i_1+\cdots+i_l}{m}+x_1+\cdots+x_l\right]_{q^{\alpha m}}^k \\
&\qquad\qquad \times q^{-mx_1-\cdots-mx_l}d\mu_{q^m}(x_1)\cdots d\mu_{q^m}(x_k) \\
&= \frac{[m]_{q^\alpha}^k}{[m]_q^l}\sum_{i_1,\ldots,i_l=0}^{m-1}\widetilde{B}_{k,q^m}^{(l,\alpha)}\left(\frac{x+i_1+\cdots+i_l}{m}\right).
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.4. *For $k \in \mathbb{Z}_+$ and $l, m \in \mathbb{N}$, one has*

$$\widetilde{B}_{k,q}^{(l,\alpha)}(x) = \frac{[m]_{q^\alpha}^k}{[m]_q^l}\sum_{i_1,\ldots,i_l=0}^{m-1}\widetilde{B}_{k,q^m}^{(l,\alpha)}\left(\frac{x+i_1+\cdots+i_l}{m}\right). \tag{2.13}$$

In particular,

$$\widetilde{B}_{k,q}^{(l,\alpha)}(mx) = \frac{[m]_{q^\alpha}^k}{[m]_q^l} \sum_{i_1,\ldots,i_l=0}^{m-1} \widetilde{B}_{k,q^m}^{(l,\alpha)}\left(x + \frac{i_1 + \cdots + i_l}{m}\right). \tag{2.14}$$

From (1.2), we can derive the following integral:

$$\begin{aligned} \int_{\mathbb{Z}_p} f(x+1)q^{-x}d\mu_q(x) &= \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}f'(0), \\ \int_{\mathbb{Z}_p} f(x+2)q^{-x}d\mu_q(x) &= \int_{\mathbb{Z}_p} f_1(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}f'(1) \\ &= \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}\left(f'(0) + f'(1)\right). \end{aligned} \tag{2.15}$$

Continuing this process, we obtain

$$\int_{\mathbb{Z}_p} f(x+n)q^{-x}d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)q^{-x}d\mu_q(x) + \frac{q-1}{\log q}\sum_{l=0}^{n-1} f'(l). \tag{2.16}$$

By (2.16), we get

$$\int_{\mathbb{Z}_p} [x+n]_{q^\alpha}^m q^{-x}d\mu_q(x) = \int_{\mathbb{Z}_p} [x]_{q^\alpha}^m q^{-x}d\mu_q(x) + \frac{m\alpha}{[\alpha]_q}\sum_{l=0}^{n-1}[l]_{q^\alpha}^{m-1}q^{\alpha l}. \tag{2.17}$$

Therefore, by (2.1) and (2.17), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$, one has*

$$\widetilde{B}_{m,q}^{(\alpha)}(n) - \widetilde{B}_{m,q}^{(\alpha)} = m\frac{\alpha}{[\alpha]_q}\sum_{l=0}^{n-1}[l]_{q^\alpha}^m q^{\alpha l}. \tag{2.18}$$

In an analogues manner as the previous investigation [7–10], we can define a further generalization of modified q-Bernoulli numbers with weight. Let χ be the Dirichlet character with conductor $d \in \mathbb{N}$. Then the generalized q-Bernoulli numbers with weight attached to χ can be defined as follows:

$$\begin{aligned} \widetilde{B}_{n,\chi,q}^{(\alpha)} &= \int_X \chi(x)[x]_{q^\alpha}^n q^{-x}d\mu_q(x) \\ &= \frac{[d]_{q^\alpha}^n}{[d]_q}\sum_{a=0}^{d-1}\chi(a)\widetilde{B}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \end{aligned} \tag{2.19}$$

We expect to investigate these objects in future papers. This definition $\widetilde{B}_{n,q}^{(\alpha)}$ was also given in a previous paper (see [9]).

Acknowledgments

The authors express their sincere gratitude to the referees for their valuable suggestions and comments. This paper is supported in part by the Research Grant of Kwangwoon University in 2011.

References

[1] S. Araci, D. Erdal, and J. J. Seo, "A study on the fermionic p-adic q-integral representation on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials," *Abstract and Applied Analysis*, vol. 2011, Article ID 649248, 7 pages, 2011.

[2] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.

[3] T. Kim, "On the weighted q-Bernoulli numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 207–215, 2011.

[4] C. S. Ryoo and Y. H. Kim, "A numerical investigation on the structure of the roots of the twisted q-Euler polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 19, no. 1, pp. 131–141, 2009.

[5] L. Carlitz, "q-Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, pp. 987–1000, 1948.

[6] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.

[7] T. Kim, "Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 186–196, 2005.

[8] T. Kim, D. V. Dolgy, S. H. Lee, B. Lee, and S. H. Rim, "A note on the modified q-Bernoulli numbers and polynomials with weight α," *Abstract and Applied Analysis*, vol. 2011, Article ID 545314, 8 pages, 2011.

[9] T. Kim, D. V. Dolgy, B. Lee, and S.-H. Rim, "Identities on the Weighted q-Bernoulli numbers of higher order," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 918364, 6 pages, 2011.

[10] H. M. Srivastava, T. Kim, and Y. Simsek, "q-Bernoulli numbers and polynomials associated with multiple q-zeta functions and basic L-series," *Russian Journal of Mathematical Physics*, vol. 12, no. 2, pp. 241–268, 2005.