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Research Article

Application of Mawhin's Coincidence Degree and Matrix Spectral Theory to a Delayed System

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This paper gives an application of Mawhin's coincidence degree and matrix spectral theory to a predator-prey model with M-predators and N-preys. The method is different from that used in the previous work. Some new sufficient conditions are obtained for the existence and global asymptotic stability of the periodic solution. The existence and stability conditions are given in terms of spectral radius of explicit matrices which are much different from the conditions given by the algebraic inequalities. Finally, an example is given to show the feasibility of our results.

1. Introduction and Motivation

1.1. History and Motivations

Mawhin's coincidence degree theory has been applied extensively to study the existence of periodic solutions for nonlinear differential systems (e.g. see [1–16] and references therein). The most important step of applying Mawhin's degree theory to nonlinear differential equations is to obtain the priori bounds of unknown solutions to the operator equation $Lx = \lambda Nx$. However, different estimation techniques for the priori bounds of unknown solutions to the equation $Lx = \lambda Nx$ may lead to different results. Most of papers obtained the priori bounds by employing the inequalities:

$$x(t) \le x(\xi) + \int_0^{\omega} |\dot{x}(t)|dt, \qquad x(t) \ge x(\eta) - \int_0^{\omega} |\dot{x}(t)|dt,$$
$$x(\xi) = \min_{t\in[0,\omega]} x(t), \qquad x(\eta) = \max_{t\in[0,\omega]} x(t). \tag{1.1}$$

These inequalities lead to a relatively strong condition given in terms of algebraic inequality or classic norms (see e.g., [3–16]). Different from standard consideration, in this paper, we employ matrix spectral theory to obtain the priori bounds, *not* the above inequalities. So in this paper, the existence and stability of periodic solution for a multispecies predator-prey model is studied by jointly employing Mawhin's coincidence degree and matrix spectral theory.

1.2. Model Formulation

One of classical Lotka-Vlterra system is predator-prey models which have been investigated extensively by mathematicians and ecologist. Many good results have been obtained for stability, bifurcations, chaos, uniform persistence, periodic solution, almost periodic solutions. It has been observed that most of works focus on either two or three species model. There are few paper considering the multispecies model. To model the dynamic behavior of multispecie predator-prey system, Yang and Rui [17] proposed a predator-prey model with M-predators and N-preys of the form:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[b_i(t) - \sum_{k=1}^{N} a_{ik}(t)x_k(t) - \sum_{l=1}^{M} c_{il}(t)y_l(t)\right], \quad i = 1,2,\ldots,N, \\ \dot{y}_j(t) &= y_j(t)\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t) - \sum_{l=1}^{M} e_{jl}(t)y_l(t)\right], \quad j = 1,2,\ldots,M, \end{aligned} \tag{1.2}$$

where $x_i(t)$ denotes the density of prey species X_i at time t, $y_j(t)$ denotes the density of predator species Y_j at time t. The coefficients $b_i(t)$, $r_j(t)$, $a_{ik}(t)$, $c_{il}(t)$, $d_{jk}(t)$, and $e_{jl}(t)$, $(i,k = 1,\ldots,N;\ j,l = 1,\ldots,M)$ are nonnegative continuous periodic functions defined on $t \in (-\infty,+\infty)$. The coefficient b_i is the intrinsic growth rate of prey species X_i, r_j is the death rate of the predator species Y_j, a_{ik} measures the amount of competition between the prey species X_i and X_k $(k \neq i,\ i,k = 1,\ldots,N)$, e_{jl} measures the amount of competition between the predator species Y_j and Y_k $(k \neq j, j,k = 1,\ldots,M)$, and the constant $\widetilde{k}_{ij} \triangleq d_{ij}/c_{ij}$ denotes the coefficient in conversing prey species X_i into new individual of predator species Y_j $(i = 1,\ldots,N;\ j = 1,\ldots,M)$. By using the differential inequality, Zhao and Chen [18] improved the results of Yang and Rui [17]. Recently, Xia et al. [19] obtained some sufficient conditions for the existence and global attractivity of a unique almost periodic solution of the system (1.2).

It is more natural to consider the delay model because most of the species start interacting after reaching a maturity period. Hence many scholars think that the delayed models are more realistic and appropriate to be studied than ordinary model. Delayed system is important also because sometimes time delays may lead to oscillation, bifurcation, chaos, instability which may be harmful to a system. Inspired by the above argument, Wen [20] considered a periodic delayed multispecies predator-prey system as follows:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\left[b_i(t) - a_{ii}(t)x_i(t) - \sum_{k=1,k\neq i}^{N} a_{ik}(t)x_k(t-\tau_{ik}) - \sum_{l=1}^{M} c_{il}(t)y_l(t-\eta_{il})\right], \\ \dot{y}_j(t) &= y_j(t)\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)x_k(t-\delta_{jk}) - e_{jj}(t)y_j(t) - \sum_{l=1,l\neq j}^{M} e_{jl}(t)y_l(t-\xi_{jl})\right], \end{aligned} \tag{1.3}$$

where $b_i(t)$, $r_j(t)$, $a_{ik}(t)$, $c_{il}(t)$, $d_{jk}(t)$, $e_{jl}(t)$, $e_{jl}(t)$ $(i,k = 1,2,\ldots,N;\ j,l = 1,2,\ldots,M)$ are assumed to be continuous ω-periodic functions and the delays τ_{ik}, δ_{jk}, η_{il}, ξ_{jl} are assumed to be positive constants. The system (1.3) is supplemented with the initial condition:

$$x_i(\theta) = \phi_i(\theta), \quad y_j(\theta) = \psi_i(\theta), \quad \theta \in [-\tau, 0],\ \phi_i(0) > 0,\ \psi_i(0) > 0, \tag{1.4}$$

where

$$\tau = \max\left\{ \max_{1\le i,k\le n} \tau_{ik}, \max_{1\le i\le N, 1\le l\le M} \eta_{il}, \max_{1\le j,l\le M} \xi_{jl}, \max_{1\le j\le M, 1\le k\le N} \delta_{jk} \right\} > 0. \tag{1.5}$$

It is easy to see that for such given initial conditions, the corresponding solution of the system (1.3) remains positive for all $t \ge 0$. The purpose of this paper is to obtain some new and interesting criteria for the existence and global asymptotic stability of periodic solution of the system (1.3).

1.3. Comparison with Previous Work

To obtain the periodic solutions of the system (1.3), the method used in [20] is based on employing the differential inequality and Brower fixed point theorem. Different from consideration taken by [20], our method is based on combining matrix spectral theory with Mawhin's degree theory. In our method, we study the global asymptotic stability by combining matrix's spectral theory with Lyapunov functional method. The existence and stability conditions are given in terms of spectral radius of explicit matrices. These conditions are much different from the sufficient conditions obtained in [20].

1.4. Outline of This Work

The structure of this paper is as follows. In Section 2, some new and interesting sufficient conditions for the existence of periodic solution of system (1.3) are obtained. Section 3 is devoted to examining the stability of the periodic solution obtained in the previous section. In Section 4, some corollaries are presented to show the effectiveness of our results. Finally, an example is given to show the feasibility of our results.

2. Existence of Periodic Solutions

In this section, we will obtain some sufficient conditions for the existence of periodic solution of the system (1.3).

2.1. Preliminaries on the Matrix Theory and Degree Theory

For convenience, we introduce some notations, definitions, and lemmas. Throughout this paper, we use the following notations.

(i) We always use $i, k = 1,\ldots,N;\ j, l = 1,\ldots,M$, unless otherwise stated.

(ii) If $f(t)$ is a continuous ω-periodic function defined on R, then we denote

$$\underline{f} = \min_{t\in[0,\omega]} |f(t)|, \qquad \overline{f} = \max_{t\in[0,\omega]} |f(t)|, \qquad m(f) = \frac{1}{\omega}\int_0^{\omega} f(t)dt. \tag{2.1}$$

We use $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ to denote a column vector, $\mathfrak{D} = (d_{ij})_{n\times n}$ is an $n \times n$ matrix, $\mathfrak{D}^T$ denotes the transpose of $\mathfrak{D}$, and E_n is the identity matrix of size n. A matrix or vector $\mathfrak{D} > 0$ (resp., $\mathfrak{D} \geq 0$) means that all entries of $\mathfrak{D}$ are positive (resp., nonnegative). For matrices or vectors $\mathfrak{D}$ and E, $\mathfrak{D} > E$ (resp., $\mathfrak{D} \geq E$) means that $\mathfrak{D} - E > 0$ (resp., $\mathfrak{D} - E \geq 0$). We denote the spectral radius of the matrix $\mathfrak{D}$ by $\rho(\mathfrak{D})$.

If $v = (v_1, v_2, \dots, v_n)^T \in \mathbb{R}^n$, then we have a choice of vector norms in $\mathbb{R}^n$, for instance $\|v\|_1$, $\|v\|_2$, and $\|v\|_\infty$ are the commonly used norms, where

$$\|v\|_1 = \sum_{j=1}^{n} |v_i|, \qquad \|v\|_2 = \left\{\sum_{j=1}^{n} |v_i|^2\right\}^{1/2}, \qquad \|v\|_\infty = \max_{1\leq i\leq n} |v_i|. \tag{2.2}$$

We recall the following norms of matrices induced by respective vector norms. For instance if $\mathcal{A} = (a_{ij})_{n\times n}$, the norm of the matrix $\|\mathcal{A}\|$ induced by a vector norm $\|\cdot\|$ is defined by

$$\|\mathcal{A}\|_p = \sup_{v\in\mathbb{R}^n, v\neq 0} \frac{\|\mathcal{A}v\|_p}{\|v\|_p} = \sup_{\|v\|_p=1} \|\mathcal{A}v\|_p = \sup_{\|v\|_p\leq 1} \|\mathcal{A}v\|_p. \tag{2.3}$$

In particular one can show that $\|\mathcal{A}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ (column norm), $\|\mathcal{A}\|_2 = [\lambda_{\max}(\mathcal{A}^T\mathcal{A})]^{1/2} = [\text{max. eigenvalue of } (\mathcal{A}^T\mathcal{A})]^{1/2}$ and $\|\mathcal{A}\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$ (row norm).

Definition 2.1 (see [1, 21]). Let X, Z be normed real Banach spaces, let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, and $N : X \to Z$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero, if $\dim\operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then $L \mid \operatorname{dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Definition 2.2 (see [1, 22]). Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n / f(\partial\Omega \cup N_f)$, that is, y is a regular value of f. Here, $N_f = \{x \in \Omega : J_f(x) = 0\}$, the critical set of f and J_f is the Jacobian of f at x. Then the *degree* $\deg\{f, \Omega, y\}$ is defined by

$$\deg\{f, \Omega, y\} = \sum_{x\in f^{-1}(y)} \operatorname{sgn} J_f(x), \tag{2.4}$$

with the agreement that $\sum \phi = 0$. For more details about Degree Theory, the reader may consult Deimling [22].

Lemma 2.3 (Continuation Theorem [1]). *Let $\Omega \subset X$ be an open and bounded set and L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$ (i.e., $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact). Assume*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Definition 2.4 (see [23, 24]). A real $n \times n$ matrix $\mathcal{A} = (a_{ij})$ is said to be an M-matrix if $a_{ij} \leq 0$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and $\mathcal{A}^{-1} \geq 0$.

Lemma 2.5 (see [23, 24]). *Let $\mathcal{A} \geq 0$ be an $n \times n$ matrix and $\rho(\mathcal{A}) < 1$, then $(E_n - \mathcal{A})^{-1} \geq 0$, where E_n denotes the identity matrix of size n.*

Now we introduce some function spaces and their norms, which will be valid throughout this paper. Denote

$$
\begin{aligned}
X &= \left\{U(t) = (u(t), v(t))^T \in C^1\left(\mathbb{R}, \mathbb{R}^{N+M}\right) \mid U(t+\omega) = U(t)\ \forall t \in \mathbb{R}\right\}, \\
Z &= \left\{U(t) = (u(t), v(t))^T \in C\left(\mathbb{R}, \mathbb{R}^{N+M}\right) \mid U(t+\omega) = U(t)\ \forall t \in \mathbb{R}\right\}.
\end{aligned}
\tag{2.5}
$$

The norms are given by

$$
\begin{gathered}
|U_n(t)|_0 = \max_{t \in [0,\omega]} |U_n(t)|, \quad |U_n(t)|_1 = |U_n(t)|_0 + \left|\dot{U}_n(t)\right|_0, \quad i = 1, 2, \ldots, N+M, \\
\|U_n(t)\|_0 = \max_{1 \leq n \leq N+M} \{|U_n(t)|_0\}, \qquad \|U_n(t)\|_1 = \max_{1 \leq n \leq N+M} \{|U_n(t)|_1\}.
\end{gathered}
\tag{2.6}
$$

Obviously, X and Z, respectively, endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_0$ are Banach spaces.

2.2. Result on the Existence of Periodic Solutions

Theorem 2.6. *Assume that the following conditions hold:*

(H_1): *the system of algebraic equations:*

$$
\begin{gathered}
m(b_i) - m(a_{ii})u_i - \sum_{k=1, k \neq i}^{N} m(a_{ik})u_k - \sum_{l=1}^{M} m(c_{il})w_l = 0, \quad i = 1, 2, \ldots, N, \\
m(-r_j) + \sum_{k=1}^{N} m(d_{jk})u_k - m(e_{jj})w_j - \sum_{l=1, l \neq j}^{M} m(e_{jl})w_l = 0, \quad j = 1, 2, \ldots, M,
\end{gathered}
\tag{2.7}
$$

has finite solution $(u_1^*, \ldots, u_N^*, w_1^*, \ldots, w_M^*)^T \in \mathbb{R}_+^{N+M}$ with $u^* > 0$, $w^* > 0$;

(H_2): $\rho(\mathcal{K}) < 1$, *where* $\mathcal{K} = \begin{pmatrix} \mathcal{P}_{N\times N} & \mathcal{Q}_{N\times M} \\ \mathcal{M}_{M\times N} & \mathcal{N}_{M\times M} \end{pmatrix}_{(N+M)\times(N+M)}$,

$$\begin{gathered} \mathcal{P}_{N\times N} = (p_{ik})_{N\times N}, \qquad p_{ik} = \begin{cases} 0, & i = k, \\ \overline{a}_{ik}\underline{a}_{kk}^{-1}, & i \neq k. \end{cases} \\ \mathcal{Q}_{N\times M} = (q_{il})_{N\times M}, \qquad q_{il} = \overline{c}_{il}\underline{e}_{ll}^{-1}, \\ \mathcal{M}_{M\times N} = (m_{jk})_{M\times N}, \qquad m_{jk} = \overline{d}_{jk}\underline{a}_{kk}^{-1}, \\ \mathcal{N}_{M\times M} = (n_{jl})_{M\times M}, \qquad n_{jl} = \begin{cases} 0, & j = l, \\ \overline{e}_{jl}\underline{e}_{ll}^{-1}, & j \neq l. \end{cases} \end{gathered} \tag{2.8}$$

Then system (1.3) *has at least one positive ω-periodic solution.*

Proof. Note that every solution

$$U(t) = (u(t), v(t))^T = (u_1(t), \ldots, u_N(t), v_1(t), \ldots, v_M(t))^T \in X \tag{2.9}$$

of the system (1.3) with the initial condition is positive. By using the following changes of variables:

$$u_i(t) = \ln x_i(t), \quad v_j(t) = \ln y_j(t), \quad i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, M, \tag{2.10}$$

the system (1.3) can be rewritten as

$$\begin{aligned} \dot{u}_i(t) &= b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}, \quad i = 1, 2, \ldots, N, \\ \dot{v}_j(t) &= -r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1,l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}, \quad j = 1, 2, \ldots, M. \end{aligned} \tag{2.11}$$

Obviously, system (1.3) has at least one ω-periodic solution which is equivalent to the system (2.11) having at least one ω-periodic solution. To prove Theorem 2.6, our main tasks are to construct the operators (i.e., L, N, P, and Q) appearing in Lemma 2.3 and to find an appropriate open set Ω satisfying conditions (i), (ii) in Lemma 2.3.

For any $U(t) \in X$, in view of the periodicity, it is easy to check that

$$\begin{aligned} \Delta_i(U, t) &= b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})} \in Z, \\ \Delta_j(U, t) &= -r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1,l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})} \in Z. \end{aligned} \tag{2.12}$$

Now, we define the operators L, N as follows:

$$L : \operatorname{Dom} L \subset X \longrightarrow Z, \quad L(u(t), v(t)) = \left(\frac{du_i(t)}{dt}, \frac{dv_j(t)}{dt} \right) \in Z,$$
$$N : X \longrightarrow Z \text{ is defined by } NU = \begin{pmatrix} \Delta_i(U,t) \\ \Delta_j(U,t) \end{pmatrix}. \tag{2.13}$$

Define, respectively, the projectors $P : X \to X$ and $Q : Z \to Z$ by

$$PU = \frac{1}{\omega} \int_0^\omega U(t)dt, \quad U \in X,$$
$$QU = \frac{1}{\omega} \int_0^\omega U(t)dt, \quad U \in Z. \tag{2.14}$$

It is obvious that the domain of L in X is actually the whole space, and

$$\operatorname{Ker} L = \{x(t) \in X \mid Lx(t) = 0, \text{ i.e. } \dot{x}(t) = 0\} = \mathbb{R}^{N+M},$$
$$\operatorname{Im} L = \left\{ z(t) \in Z \mid \int_0^\omega z(t)dt = 0 \right\} \text{ is closed in } Z. \tag{2.15}$$

Moreover, P, Q are continuous operators such that

$$\operatorname{Im} P = \mathbb{R}^N = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q),$$
$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = N + M < +\infty. \tag{2.16}$$

It follows that L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists, which is given by

$$K_P(y) = \int_0^t y(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t y(s)dsdt. \tag{2.17}$$

Then $QN : X \to Z$ and $K_P(I - Q)N : X \to X$ are defined by

$$QNU = \begin{pmatrix} \dfrac{1}{\omega} \displaystyle\int_0^\omega \Delta_i(U,t)dt \\ \dfrac{1}{\omega} \displaystyle\int_0^\omega \Delta_j(U,t)dt \end{pmatrix},$$
$$K_P(I - Q)Nx = \int_0^t NU(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t NU(s)dsdt - \left(\frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega NU(s)ds. \tag{2.18}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. By using the generalized Arzela-Ascoli theorem, it is not difficult to prove that $(K_P(I-Q)N)(\overline{\Omega})$ is relatively compact in the space $(X, \|\cdot\|_1)$. The proof of this step is complete.

Then, in order to apply condition (i) of Lemma 2.3, we need to search for an appropriate open bounded subset Ω, denoted by

$$\Omega = U_n(t) \in X \mid |U_n(t)|_1 = |U_n(t)|_0 + \left|\dot{U}_n(t)\right|_0 < h_n. \tag{2.19}$$

Specifically, our aim is to find an appropriate h_n. Corresponding to the operator equation $Lx = \lambda Nx$ for each $\lambda \in (0,1)$, we have

$$\begin{aligned} \dot{u}_i(t) &= \lambda\left[b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right], \\ \dot{v}_j(t) &= \lambda\left[-r_j(t) + \sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})} - e_{jj}(t)e^{v_j(t)} - \sum_{l=1,l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right]. \end{aligned} \tag{2.20}$$

Since $U(t) \in X$, each $U_n(t), n = 1,2,\ldots,N+M$, as components of $U(t)$, is continuously differentiable and ω-periodic. In view of continuity and periodicity, there exists $t_i \in [0,\omega]$ such that $u_i(t_i) = \max_{t\in[0,\omega]}|u_i(t)|$, $i = 1,2,\ldots,N$, and there also exists $t_{N+j} \in [0,\omega]$ such that $v_j(t_{N+j}) = \max_{t\in[0,\omega]}|v_j(t)|$, $j = 1,2,\ldots,M$. Accordingly, $\dot{u}_i(t_i) = 0$, $\dot{v}_j(t_{N+j}) = 0$, and we get

$$\begin{aligned} b_i(t_i) - a_{ii}(t_i)e^{u_i(t_i)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})} &= 0, \\ -r_j(t_{N+j}) + \sum_{k=1}^{N} d_{jk}(t_{N+j})e^{u_k(t_i-\delta_{jk})} - e_{jj}(t_{N+j})e^{v_j(t_{N+j})} - \sum_{l=1,l\neq j}^{M} e_{jl}(t_{N+j})e^{v_l(t_{N+j}-\xi_{jl})} &= 0. \end{aligned} \tag{2.21}$$

That is,

$$\begin{aligned} a_{ii}(t_i)e^{u_i(t_i)} &= b_i(t_i) - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})}, \\ e_{jj}(t_{N+j})e^{v_j(t_{N+j})} &= -r_j(t_{N+j}) + \sum_{k=1}^{N} d_{jk}(t_{N+j})e^{u_k(t_i-\delta_{jk})} - \sum_{l=1,l\neq j}^{M} e_{jl}(t_{N+j})e^{v_l(t_{N+j}-\xi_{jl})}. \end{aligned} \tag{2.22}$$

Note that $u_k(t_k) = \max_{t\in[0,\omega]}|u_k(t)|$ and $v_l(t_{N+l}) = \max_{t\in[0,\omega]}|v_l(t)|$, which implies

$$\begin{gathered} u_k(t_i) \le u_k(t_k), \qquad u_k(t_i - \tau_{ik}) \le u_k(t_k), \qquad u_k(t_i - \delta_{jk}) \le u_k(t_k); \\ v_l(t_{N+j}) \le v_l(t_{N+l}), \qquad v_l(t_{N+j} - \eta_{il}) \le v_l(t_{N+l}), \qquad v_l(t_{N+j} - \xi_{jl}) \le v_l(t_{N+l}). \end{gathered} \tag{2.23}$$

It follows that

$$\begin{aligned}
\underline{a}_{ii}e^{u_i(t_i)} &\le \left| a_{ii}(t_i)e^{u_i(t_i)} \right| \\
&= \left| b_i(t_i) - \sum_{k=1,k\neq i}^{N} a_{ik}(t_i)e^{u_k(t_i-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t_i)e^{v_l(t_{N+j}-\eta_{il})} \right| \\
&\le \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_i-\tau_{ik})} + \sum_{l=1}^{M}\overline{c}_{il}e^{v_l(t_{N+j}-\eta_{il})} \\
&\le \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_k)} + \sum_{l=1}^{M}\overline{c}_{il}e^{v_l(t_{N+l})}, \\
\underline{e}_{jj}e^{v_j(t_{N+j})} &\le \left| e_{jj}\left(t_{N+j}\right)e^{v_j(t_{N+j})} \right| \\
&= \left| -r_j\left(t_{N+j}\right) + \sum_{k=1}^{N} d_{jk}\left(t_{N+j}\right)e^{u_k(t_i-\delta_{jk})} - \sum_{l=1,l\neq j}^{M} e_{jl}\left(t_{N+j}\right)e^{v_l(t_{N+j}-\xi_{jl})} \right| \\
&\le \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_i-\delta_{jk})} + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}e^{v_l(t_{N+j}-\xi_{jl})} \\
&\le \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_k)} + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}e^{v_l(t_{N+l})}.
\end{aligned} \tag{2.24}$$

Let

$$\underline{a}_{ii}e^{u_i(t_i)} = z_i(t_i), \qquad \underline{e}_{jj}e^{v_j(t_{N+j})} = \tilde{z}_j\left(t_{N+j}\right). \tag{2.25}$$

Using (2.25), the inequalities (2.24) become

$$\begin{aligned}
z_i(t_i) &\le \overline{b}_i + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\tilde{z}_l(t_{N+l}), \\
\tilde{z}_j\left(t_{N+j}\right) &\le \overline{r}_j + \sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}\underline{e}_{ll}^{-1}\tilde{z}_l(t_{N+l}),
\end{aligned} \tag{2.26}$$

or

$$\begin{aligned}
z_i(t_i) - \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) - \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\tilde{z}_l(t_{N+l}) &\le \overline{b}_i, \\
\tilde{z}_j\left(t_{N+j}\right) - \sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k) - \sum_{l=1,l\neq j}^{M} \overline{e}_{jl}\underline{e}_{ll}^{-1}\tilde{z}_l(t_{N+l}) &\le \overline{r}_j,
\end{aligned} \tag{2.27}$$

which implies

$$\begin{pmatrix} E_{N\times N}-\mathcal{P}_{N\times N} & -\mathcal{Q}_{N\times M} \\ -\mathcal{M}_{M\times N} & E_{M\times M}-\mathcal{N}_{M\times M} \end{pmatrix}_{(N+M)\times(N+M)} \times \begin{pmatrix} z_1(t_1), \\ \dots, \\ z_N(t_N), \\ \widetilde{z}_1(t_{N+1}), \\ \dots, \\ \widetilde{z}_M(t_{N+M}) \end{pmatrix} \le \begin{pmatrix} \overline{b}_1, \\ \dots, \\ \overline{b}_N, \\ \overline{r}_1, \\ \dots, \\ \overline{r}_M \end{pmatrix}, \tag{2.28}$$

where

$$\begin{gathered} \mathcal{P}_{N\times N} = (p_{ik})_{N\times N}, \qquad p_{ik} = \begin{cases} 0, & i = k, \\ \overline{a}_{ik}\underline{a}_{kk}^{-1}, & i \ne k, \end{cases} \\ \mathcal{Q}_{N\times M} = (q_{il})_{N\times M}, \qquad q_{il} = \overline{c}_{il}\underline{e}_{ll}^{-1}, \\ \mathcal{M}_{M\times N} = (m_{jk})_{M\times N}, \qquad m_{jk} = \overline{d}_{jk}\underline{a}_{kk}^{-1}, \\ \mathcal{N}_{M\times M} = (n_{jl})_{M\times M}, \qquad n_{jl} = \begin{cases} 0, & j = l, \\ \overline{e}_{jl}\underline{e}_{ll}^{-1}, & j \ne l. \end{cases} \end{gathered} \tag{2.29}$$

Set $D = (\overline{b}_1, \dots, \overline{b}_N, \overline{r}_1, \dots, \overline{r}_M)^T$. It follows from (2.28) and (H_2) that

$$(E-\mathcal{K})(z_1(t_1), \dots, z_N(t_N), \widetilde{z}_1(t_{N+1}), \dots, \widetilde{z}_M(t_{N+M}))^T \le D. \tag{2.30}$$

In view of $\rho(\mathcal{K}) < 1$ and Lemma 2.5, we get $(E_{N+M} - \mathcal{K})^{-1} \ge 0$. Let

$$H = \left(\widetilde{h}_1, \dots, \widetilde{h}_N, \widetilde{h}_{N+1}, \dots, \widetilde{h}_{N+M}\right)^T := (E-\mathcal{K})^{-1}D \ge 0. \tag{2.31}$$

Using (2.30) and (2.31), we get

$$(z_1(t_1), \dots, z_N(t_N), \widetilde{z}_1(t_{N+1}), \dots, \widetilde{z}_M(t_{N+M}))^T \le H, \tag{2.32}$$

or

$$z_i(t_i) \le \widetilde{h}_i, \quad i = 1, 2, \dots, N, \qquad \widetilde{z}_j(t_{N+j}) \le \widetilde{h}_{N+j}, \quad j = 1, 2, \dots, M. \tag{2.33}$$

Then

$$u_i(t_i) \le \ln\frac{\widetilde{h}_i}{\underline{a}_{ii}}, \qquad v_j(t_{N+j}) \le \ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}, \tag{2.34}$$

which implies

$$|u_n(t)|_0 = \max_{t\in[0,\omega]}|u_n(t)| = \max_{t\in[0,\omega]}\left\{u_i(t_i), v_j\left(t_{N+j}\right)\right\} = \max\left\{\ln\frac{\widetilde{h}_i}{\underline{a}_{ii}}, \ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}. \tag{2.35}$$

On the other hand, it follows from (2.31) that

$$(E-\mathcal{K})H = D, \quad \text{or} \quad H = \mathcal{K}H + D, \tag{2.36}$$

that is

$$\begin{aligned}
\widetilde{h}_i &= \sum_{k=1,k\neq i}^{N} p_{ik}\widetilde{h}_k + \sum_{l=1}^{M} q_{il}\widetilde{h}_{N+l} + \overline{b}_i, \\
\widetilde{h}_{N+j} &= \sum_{k=1}^{N} m_{jk}\widetilde{h}_k + \sum_{l=1,l\neq j}^{M} n_{jl}\widetilde{h}_{N+l} + \overline{r}_j.
\end{aligned} \tag{2.37}$$

Estimating (2.20), by using (2.25), (2.33), and (2.37), we have

$$\begin{aligned}
|\dot{u}_i(t)|_0 &= \lambda\left|b_i(t) - a_{ii}(t)e^{u_i(t)} - \sum_{k=1,k\neq i}^{N} a_{ik}(t)e^{u_k(t-\tau_{ik})} - \sum_{l=1}^{M} c_{il}(t)e^{v_l(t-\eta_{il})}\right|_0 \\
&\leq \overline{b}_i + \overline{a}_{ii}\left|e^{u_i(t)}\right|_0 + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\left|e^{u_k(t-\tau_{ik})}\right|_0 + \sum_{l=1}^{M}\overline{c}_{il}\left|e^{v_l(t-\eta_{il})}\right|_0 \\
&= \overline{b}_i + \overline{a}_{ii}e^{u_i(t_i)} + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}e^{u_k(t_k)} + \sum_{l=1}^{M}\overline{c}_{il}e^{v_l(t_{N+l})} \\
&= \overline{b}_i + \overline{a}_{ii}\underline{a}_{ii}^{-1}z_i(t_i) + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) \\
&= \overline{b}_i + z_i(t_i) + \sum_{k=1,k\neq i}^{N} \overline{a}_{ik}\underline{a}_{kk}^{-1}z_k(t_k) + \sum_{l=1}^{M}\overline{c}_{il}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l}) \\
&\leq \overline{b}_i + \widetilde{h}_i + \sum_{k=1,k\neq i}^{N} p_{ik}\widetilde{h}_k + \sum_{l=1}^{M} q_{il}\widetilde{h}_{N+l} \\
&= \widetilde{h}_i + \widetilde{h}_i = 2\widetilde{h}_i,
\end{aligned}$$

$$
\begin{aligned}
\left|\dot{v}_j(t)\right|_0 &= \lambda\left|-r_j(t)+\sum_{k=1}^{N} d_{jk}(t)e^{u_k(t-\delta_{jk})}-e_{jj}(t)e^{v_j(t)}-\sum_{l=1,l\neq j}^{M} e_{jl}(t)e^{v_l(t-\xi_{jl})}\right|_0 \\
&\le \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\left|e^{u_k(t-\delta_{jk})}\right|_0+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\left|e^{v_l(t-\xi_{jl})}\right|_0+\overline{e}_{jj}\left|e^{v_j(t)}\right|_0 \\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}e^{u_k(t_k)}+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}e^{v_l(t_{N+l})}+\overline{e}_{jj}e^{v_j(t_{N+j})} \\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k)+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l})+\overline{e}_{jj}\underline{e}_{jj}^{-1}z_j\left(t_{N+j}\right) \\
&= \overline{r}_j+\sum_{k=1}^{N}\overline{d}_{jk}\underline{a}_{kk}^{-1}z_k(t_k)+\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\underline{e}_{ll}^{-1}\widetilde{z}_l(t_{N+l})+\widetilde{z}_j\left(t_{N+j}\right) \\
&\le \overline{r}_j+\sum_{k=1}^{N}m_{jk}\widetilde{h}_k+\sum_{l=1,l\neq j}^{M}n_{jl}\widetilde{h}_{N+l}+\widetilde{h}_{N+j} \\
&= \widetilde{h}_{N+j}+\widetilde{h}_{N+j}=2\widetilde{h}_{N+j}.
\end{aligned}
\tag{2.38}
$$

The above relations imply

$$
|\dot{u}_n(t)|_0=\max_{t\in[0,\omega]}|\dot{u}_n(t)|=\max_{t\in[0,\omega]}\left\{\dot{u}_i(t),\dot{v}_j(t)\right\}=\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}. \tag{2.39}
$$

Further, it follows form the definition of norm that

$$
|u_n(t)|_1=|u_n(t)|_0+|\dot{u}_n(t)|_0=\max\left\{\ln\frac{\widetilde{h}_i}{\underline{a}_{ii}},\ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}+\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}. \tag{2.40}
$$

Let us set the following:

$$
h_n=\max\left\{\ln\frac{\widetilde{h}_i}{\underline{a}_{ii}},\ln\frac{\widetilde{h}_{N+j}}{\underline{e}_{jj}}\right\}+\max\left\{2\widetilde{h}_i,2\widetilde{h}_{N+j}\right\}+d, \tag{2.41}
$$

where d is any positive constant.

Then for any solution of $Lx=\lambda Nx$, we have $|u_n(t)|_1=|u_n(t)|_0+|\dot{u}_n(t)|_0<h_n$ for all $n=1,2,\dots,N+M$. Obviously, h_n are independent of λ and the choice of $U(t)$. Consequently, by taking this h_n, the open subset Ω satisfies that $\Omega\cap\operatorname{Dom}L$, that is, the open subset Ω satisfies the assumption (i) of Lemma 2.3.

Now in the last step of the proof, we need to verify that for the given open bounded set Ω obtained in Step 2, the assumption (ii) of Lemma 2.3 also holds. That is, for each $U\in\partial\Omega\cap\operatorname{Ker}L$, $QNU\neq 0$ and $\deg\{JQN,\Omega\cap\operatorname{Ker}L,0\}\neq 0$.

Take $U \in \partial\Omega \cap \operatorname{Ker} L$. Then, in view of $\operatorname{Ker} L = \mathbb{R}^{N+M}$, U is a constant vector in $\mathbb{R}^{N+M}$, denoted by $U = (u_1, \ldots, u_N, v_1, \ldots, v_M)^T$ and with the property

$$|U_n| = |U_n|_0 = |U_n|_1 = h_n. \tag{2.42}$$

By operating U by QN gives

$$(QNU)_n = \begin{pmatrix} m(b_i) - m(a_{ii})e^{u_i} - \sum\limits_{k=1,k\neq i}^{N} m(a_{ik})e^{u_k} - \sum\limits_{l=1}^{M} m(c_{il})e^{v_l} \\ m(-r_j) + \sum\limits_{k=1}^{N} m(d_{jk})e^{u_k} - m(e_{jj})e^{v_j} - \sum\limits_{l=1,l\neq j}^{M} m(e_{jl})e^{v_l} \end{pmatrix}. \tag{2.43}$$

It is easy to obtain that $(QNU)_n$ and $\deg\{JQN, \Omega\cap\operatorname{Ker} L, 0\} \neq 0$, where $\deg(\cdot)$ is the Brouwer degree and J is the identity mapping since $\operatorname{Im} Q = \operatorname{Ker} L$. We have shown that the open subset $\Omega \subset X$ satisfies all the assumptions of Lemma 2.3. Hence, by Lemma 2.3, the system (2.11) has at least one positive ω-periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. By (2.10), the system (1.3) has at least one positive ω-periodic solution. This completes the proof of Theorem 2.6. □

3. Globally Asymptotic Stability

Under the assumption of Theorem 2.6, we know that system (1.3) has at least one positive ω-periodic solution, denoted by $X^*(t) = (x_1^*(t), \ldots, x_N^*(t), y_1^*(t), \ldots, y_M^*(t))^T$. The aim of this section is to derive a set of sufficient conditions which guarantee the existence and global asymptotic stability of the positive ω-periodic solution $X^*(t)$.

Before the formal analysis, we recall some facts which will be used in the proof.

Lemma 3.1 (see [25]). *Let f be a nonnegative function defined on $[0, +\infty]$ such that f is integrable on $[0, +\infty]$ and is uniformly continuous on $[0, +\infty]$. Then $\lim_{t\to+\infty} f(t) = 0$.*

Lemma 3.2 (see [23, 24]). *Let $\mathcal{K} = (\Gamma_{ij})_{n\times n}$ be a matrix with nonpositive off-diagonal elements. $\mathcal{K}$ is an M-matrix if and only if there exists a positive diagonal matrix $\xi = \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ such that*

$$\xi_i \underline{a}_{ii} > \sum_{j\neq i} \xi_j \overline{a}_{ij}, \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Theorem 3.3. *Assume that all the assumptions in Theorem 2.6 hold. Then system (1.3) has a unique positive ω-periodic solution $X^*(t)$ which is globally asymptotically stable.*

Proof. Let $X(t) = (x(t), y(t))^T = (x_1(t), \ldots, x_N(t), y_1(t), \ldots, y_M(t))^T$ be any positive solution of system (1.3). It is easy to see that $\rho(\mathcal{K}^T) = \rho(\mathcal{K}) < 1$. Thus, in view of Lemma 2.5 and Definition 2.4, $(E - \mathcal{K}^T)$ is an M-matrix, where E denotes an identity matrix of size $N + M$. Therefore, by Lemma 3.2, there exists a diagonal matrix $L = \operatorname{diag}(\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M)$ with positive diagonal elements such that the product $(E - \mathcal{K}^T)L$ is strictly diagonally dominant with positive diagonal entries, namely,

$$\begin{aligned} \alpha_i \underline{a}_{ii} &> \sum_{k=1,k\neq i}^{N} \alpha_k \overline{a}_{ki} + \sum_{l=1}^{M} \beta_l \overline{d}_{li}, \quad i = 1, \ldots, N, \\ \beta_j \underline{e}_{jj} &> \sum_{k=1}^{N} \alpha_k \overline{c}_{kj} + \sum_{l=1,l\neq j}^{M} \beta_l \overline{e}_{lj}, \quad j = 1, \ldots, M. \end{aligned} \tag{3.2}$$

Now, we define a Lyapunov function $V(t)$ as follows:

$$\begin{aligned} V(t) = &\sum_{i=1}^{N} \alpha_i \Bigg[\left| \ln x_i(t) - \ln x_i^*(t) \right| + \sum_{K=1,K\neq i}^{N} \int_{t-\tau_{ik}}^{t} a_{ik}(s + \tau_{ik}) \left| x_k(t) - x_k^*(t) \right| ds \\ &+ \sum_{l=1}^{M} \int_{t-\eta_{il}}^{t} c_{il}(s + \eta_{il}) \left| y_l(t) - y_l^*(t) \right| ds \Bigg] \\ &+ \sum_{j=1}^{M} \beta_j \Bigg[\left| \ln y_j(t) - \ln y_j^*(t) \right| + \sum_{K=1}^{N} \int_{t-\delta_{jk}}^{t} d_{jk}(s + \delta_{jk}) \left| x_k(t) - x_k^*(t) \right| ds \\ &+ \sum_{l=1,l\neq j}^{M} \int_{t-\xi_{jl}}^{t} e_{jl}(s + \xi_{jl}) \left| y_l(t) - y_l^*(t) \right| ds \Bigg], \quad t \geq t_0. \end{aligned} \tag{3.3}$$

Calculating the upper right derivative of $V(t)$ and using (3.2), we get

$$\begin{aligned} D^+ V(t) \leq &\sum_{i=1}^{N} \alpha_i \Bigg[- a_{ii}(t) \left| x_i(t) - x_i^*(t) \right| + \sum_{k=1,k\neq i}^{N} a_{ik}(t + \tau_{ik}) \left| x_k(t) - x_k^*(t) \right| \\ &+ \sum_{l=1}^{M} c_{il}(t + \eta_{il}) \left| y_l(t) - y_l^*(t) \right| \Bigg] \\ &+ \sum_{j=1}^{M} \beta_j \Bigg[- e_{jj}(t) \left| y_j(t) - y_j^*(t) \right| + \sum_{k=1}^{N} d_{jk}(t + \delta_{jk}) \left| x_k(t) - x_k^*(t) \right| \\ &+ \sum_{l=1,l\neq j}^{M} e_{jl}(t + \xi_{jl}) \left| y_l(t) - y_l^*(t) \right| \Bigg] \end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{i=1}^{N}\alpha_i\Bigg[-\underline{a}_{ii}\left|x_i(t)-x_i^*(t)\right|+\sum_{k=1,k\neq i}^{N}\overline{a}_{ik}\left|x_k(t)-x_k^*(t)\right| \\
&\qquad\qquad +\sum_{l=1}^{M}\overline{c}_{il}\left|y_l(t)-y_l^*(t)\right|\Bigg] \\
&\quad +\sum_{j=1}^{M}\beta_j\Bigg[-\underline{e}_{jj}\left|y_j(t)-y_j^*(t)\right|+\sum_{k=1}^{N}\overline{d}_{jk}\left|x_k(t)-x_k^*(t)\right| \\
&\qquad\qquad +\sum_{l=1,l\neq j}^{M}\overline{e}_{jl}\left|y_l(t)-y_l^*(t)\right|\Bigg] \\
&= -\sum_{i=1}^{N}\left(\alpha_i\underline{a}_{ii}-\sum_{k=1,k\neq i}^{N}\alpha_k\overline{a}_{ki}-\sum_{l=1}^{M}\beta_l\overline{d}_{li}\right)\left|x_i(t)-x_i^*(t)\right| \\
&\quad -\sum_{j=1}^{M}\left(\beta_j\underline{e}_{jj}-\sum_{k=1}^{N}\alpha_k\overline{c}_{kj}-\sum_{l=1,l\neq j}^{M}\beta_l\overline{e}_{lj}\right)\left|y_j(t)-y_j^*(t)\right| \\
&= -c\left\{\sum_{i=1}^{N}\left|x_i(t)-x_i^*(t)\right|+\sum_{j=1}^{M}\left|y_j(t)-y_j^*(t)\right|\right\},
\end{aligned}
\tag{3.4}
$$

where

$$
c=\min\left\{\alpha_i\underline{a}_{ii}-\sum_{k=1,k\neq i}^{N}\alpha_k\overline{a}_{ki}-\sum_{l=1}^{M}\beta_l\overline{d}_{li},\beta_j\underline{e}_{jj}-\sum_{k=1}^{N}\alpha_k\overline{c}_{kj}-\sum_{l=1,l\neq j}^{M}\beta_l\overline{e}_{lj}\right\}>0. \tag{3.5}
$$

It follows from (3.4) that $D^+V(t)\leq 0$. Obviously, the zero solution of (1.3) is Lyapunov stable. On the other hand, integrating (3.4) over $[t_0,t]$ leads to

$$
V(t)-V(t_0)\leqslant -c\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds,\quad t\geqslant 0, \tag{3.6}
$$

or

$$
V(t)+c\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds\leqslant V(t_0)<+\infty,\quad t\geqslant t_0. \tag{3.7}
$$

Noting that $V(t)\geqslant 0$, it follows that

$$
\int_{t_0}^{t}\left[\sum_{i=1}^{N}\left|x_i(s)-x_i^*(s)\right|+\sum_{j=1}^{M}\left|y_j(s)-y_j^*(s)\right|\right]ds\leqslant\frac{V(t_0)}{c}<+\infty,\quad t\geqslant t_0. \tag{3.8}
$$

Therefore, by Lemma 3.1, it is not difficult to conclude that

$$\lim_{t\to+\infty}\left|X_i(t)-X_i^*(t)\right|=0. \tag{3.9}$$

Theorem 3.3 follows. □

4. Corollaries and Remarks

In order to illustrate some features of our main results, we will present some corollaries and remarks in this section. From the proofs of Theorems 2.6 and 3.3, one can easily deduce the following corollary.

Corollary 4.1. *In addition to* (H_1)*, further suppose that* $E-\mathcal{K}$ *or* $E-\mathcal{K}^T$ *is an* M*-matrix. Then system* (1.3) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

Now recall that for a given matrix $\mathcal{K}$, its spectral radius $\rho(\mathcal{K})$ is equal to the minimum of all matrix norms of $\mathcal{K}$, that is, for any matrix norm $\|\cdot\|$, $\rho(\mathcal{K})\le\|\mathcal{K}\|$. Therefore, we have the following corollary.

Corollary 4.2. *In addition to* (H_1)*, if one further supposes that there exist positive constants* ξ_i*,* $i=1,2,\dots,n$*,* η_j*,* $j=1,2,\dots,m$ *such that one of the following inequalities holds.*

(1) $\max\{\max_{1\le k\le n}\{\underline{a}_{kk}^{-1}\xi_k^{-1}[\sum_{i=1,i\ne k}^{n}\xi_i\overline{a}_{ik}+\sum_{l=1}^{m}\eta_j\overline{d}_{jk}]\},\max_{1\le l\le m}\{\underline{e}_{ll}^{-1}\eta_l^{-1}[\sum_{i=1}^{n}\xi_i\overline{c}_{li}+\sum_{j=1,j\ne l}^{m}\eta_j\overline{e}_{jl}]\}\}<1$*, or equivalently, for all* $k=1,\dots,n$*,* $l=1,2,\dots,n$*,*

$$\begin{aligned}\xi_k\underline{a}_{kk}&>\sum_{i=1,i\ne k}^{n}\xi_i\overline{a}_{ik}+\sum_{l=1}^{m}\eta_j\overline{d}_{jk},\\ \eta_l\underline{e}_{ll}&>\sum_{i=1}^{n}\xi_i\overline{c}_{li}+\sum_{j=1,j\ne l}^{m}\eta_j\overline{e}_{jl}.\end{aligned} \tag{4.1}$$

(2) $\sum_{i=1}^{n+m}\sum_{j=1}^{n+m}(\xi_i^{-1}\xi_j k_{ij})^2<1$*, where* $\mathcal{K}=(k_{ij})_{(n+m)\times(n+m)}$ *has been defined in Theorem 2.6.*

(3) $\max\{\max_{1\le i\le n}\{\underline{a}_{ii}^{-1}\xi_i^{-1}[\sum_{k=1,k\ne i}^{n}\xi_k\overline{a}_{ki}+\sum_{l=1}^{m}\eta_l\overline{d}_{li}]\},\max_{1\le j\le m}\{\underline{e}_{jj}^{-1}\eta_j^{-1}[\sum_{k=1}^{n}\xi_k\overline{c}_{kj}+\sum_{l=1,l\ne j}^{m}\eta_l\overline{e}_{lj}]\}\}<1$*, or equivalently, for all* $i=1,\dots,n$*,* $j=1,2,\dots,n$*,*

$$\begin{aligned}\xi_i a_{ii}&>\sum_{k=1,k\ne i}^{n}\xi_k|a_{ki}|+\sum_{l=1}^{m}\eta_l|d_{li}|,\\ \eta_j e_{jj}&>\sum_{k=1}^{n}\xi_k\left|c_{kj}\right|+\sum_{l=1,l\ne j}^{m}\eta_l\left|e_{lj}\right|.\end{aligned} \tag{4.2}$$

Then system (1.3) *has a unique positive* ω*-periodic solution which is globally asymptotically stable.*

Proof. For any matrix norm $\|\cdot\|$ and any nonsingular matrix S, $\|\mathcal{K}\|_S=\|S^{-1}\mathcal{K}S\|$ also defines a matrix norm. Let us denote $D=\operatorname{diag}(\xi_1,\xi_2,\dots,\xi_n)$, then the conditions (1.2) and (1.3) correspond to the column norms and Frobenius norm of matrix $D\mathcal{K}D^{-1}$, respectively.

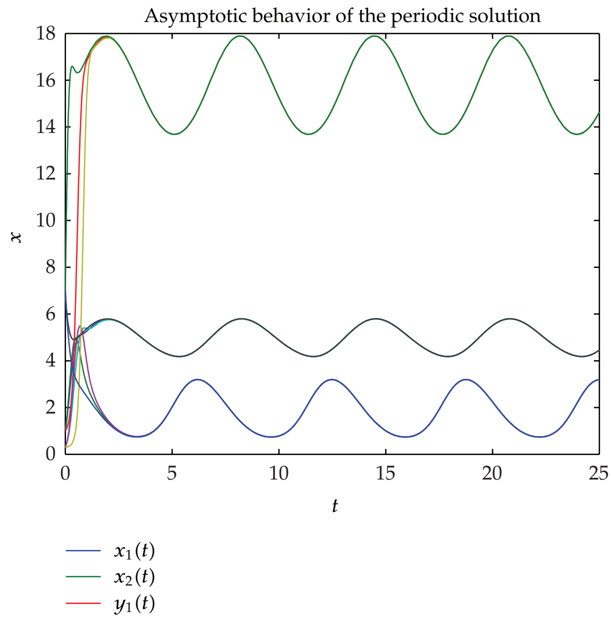

Figure 1: Asymptotic behavior of system (5.1) with initial values $(x_1(0), x_2(0), y_1(0)) = (1,1,1)$, $(0.3, 0.3, 0.3)$, $(7,7,7)$, respectively, $t \in [0, 25]$.

Condition (2.10) corresponds to the row norms of $D\mathcal{K}^T D^{-1}$ and note that $\rho(D\mathcal{K}^T D^{-1}) = \rho(D\mathcal{K}D^{-1})$. Now Corollary 4.2 follows immediately. □

5. Example

In this section, an example and its simulations are presented to illustrate the feasibility and effectiveness of our results.

Example 5.1. Consider the following periodic predator-prey model with 2-predators and 1-prey:

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\left[7 + \sin t - x_1(t) - \frac{1}{4}x_2(t) - \frac{1}{10}y_1(t-1)\right],\\
\dot{x}_2(t) &= x_2(t)\left[7 + \cos t - \frac{1}{4}x_1(t) - x_2(t) - \frac{1}{4}y_1(t)\right],\\
\dot{y}_1(t) &= y_1(t)\left[-\frac{1}{20}(1+\cos t) + \frac{3}{2}x_1(t-1) + \frac{1}{4}x_2(t) - \frac{1}{2}y_1(t)\right].
\end{aligned} \tag{5.1}$$

Simple computation leads to

$$\mathcal{K} = \begin{pmatrix} 0 & a_{22}^{-1}a_{12} & e_{11}^{-1}c_{11} \\ a_{11}^{-1}a_{21} & 0 & e_{11}^{-1}c_{21} \\ a_{11}^{-1}d_{11} & a_{22}^{-1}d_{12} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & 0 & 0 \\ \frac{3}{2} & \frac{1}{4} & 0 \end{pmatrix}. \tag{5.2}$$

By using mathematica, we see that $\rho(\mathcal{K}) = 0.633982 < 1$. Thus, the system (5.1) has a periodic solution which is globally asymptotically stable. Figure 1 shows the asymptotic behavior of the periodic solution.

Remark 5.2. In this example, one can observe that though the spectral $\rho(\mathcal{K}) < 1$, the matrix norms of the matrix $\mathcal{K}$ are all bigger than 1. For instance, the column norm: is

$$\|\mathcal{K}\|_1 = \max_{1\le j\le 3}\left\{ a_{jj}^{-1} \sum_{i=1, i\neq j}^{3} a_{ij} \right\} = 0 + \frac{3}{2} + \frac{1}{4} > 1. \tag{5.3}$$

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References

[1] R. E. Gaines and J. L. Mawhin, *Coincidence Degree, and Nonlinear Differential Equations*, Springer, Berlin, Germany, 1977.

[2] J. K. Hale and J. Mawhin, "Coincidence degree and periodic solutions of neutral equations," *Journal of Differential Equations*, vol. 15, pp. 295–307, 1974.

[3] J. Mawhin, "Leray-Schauder continuation theorems in the absence of a priori bounds," *Topological Methods in Nonlinear Analysis*, vol. 9, no. 1, pp. 179–200, 1997.

[4] A. Capietto, J. Mawhin, and F. Zanolin, "Continuation theorems for periodic perturbations of autonomous systems," *Transactions of the American Mathematical Society*, vol. 329, no. 1, pp. 41–72, 1992.

[5] Z. Zhang and J. Luo, "Multiple periodic solutions of a delayed predator-prey system with stage structure for the predator," *Nonlinear Analysis*, vol. 11, no. 5, pp. 4109–4120, 2010.

[6] Z. Zhang and Z. Hou, "Existence of four positive periodic solutions for a ratio-dependent predator-prey system with multiple exploited (or harvesting) terms," *Nonlinear Analysis*, vol. 11, no. 3, pp. 1560–1571, 2010.

[7] Y. Li, "Periodic solutions of a periodic delay predator-prey system," *Proceedings of the American Mathematical Society*, vol. 127, no. 5, pp. 1331–1335, 1999.

[8] M. Fan, Q. Wang, and X. Zou, "Dynamics of a non-autonomous ratio-dependent predator-prey system," *Proceedings of the Royal Society of Edinburgh A*, vol. 133, no. 1, pp. 97–118, 2003.

[9] H.-F. Huo, "Periodic solutions for a semi-ratio-dependent predator-prey system with functional responses," *Applied Mathematics Letters*, vol. 18, no. 3, pp. 313–320, 2005.

[10] W. Ding and M. Han, "Dynamic of a non-autonomous predator-prey system with infinite delay and diffusion," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1335–1350, 2008.

[11] Y. Xia, J. Cao, and S. S. Cheng, "Periodic solutions for a Lotka-Volterra mutualism system with several delays," *Applied Mathematical Modelling*, vol. 31, no. 9, pp. 1960–1969, 2007.

[12] Y. Xia and M. Han, "Multiple periodic solutions of a ratio-dependent predator-prey model," *Chaos, Solitons & Fractals*, vol. 39, no. 3, pp. 1100–1108, 2009.

[13] J. Chu and M. Li, "Positive periodic solutions of Hill's equations with singular nonlinear perturbations," *Nonlinear Analysis A*, vol. 69, no. 1, pp. 276–286, 2008.

[14] J. Chu and J. J. Nieto, "Impulsive periodic solutions of first-order singular differential equations," *Bulletin of the London Mathematical Society*, vol. 40, no. 1, pp. 143–150, 2008.

[15] J. Wang, Y. Zhou, and W. Wei, "Impulsive problems for fractional evolution equations and optimal controls in infinite dimensional spaces," *Topological Methods in Nonlinear Analysis*, vol. 38, no. 1, pp. 17–43, 2011.

[16] J. Wang, Y. Zhou, and M. Medved, "Qualitative analysis for nonlinear fractional differential equations via topological degree method," *Topological Methods in Nonlinear Analysis*. In press.

[17] P. Yang and X. Rui, "Global attractivity of the periodic Lotka-Volterra system," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 221–232, 1999.

[18] J. Zhao and W. Chen, "Global asymptotic stability of a periodic ecological model," *Applied Mathematics and Computation*, vol. 147, no. 3, pp. 881–892, 2004.

[19] Y. Xia, F. Chen, A. Chen, and J. Cao, "Existence and global attractivity of an almost periodic ecological model," *Applied Mathematics and Computation*, vol. 157, no. 2, pp. 449–475, 2004.

[20] X. Z. Wen, "Global attractivity of a positive periodic solution of a multispecies ecological competition-predator delay system," *Acta Mathematica Sinica*, vol. 45, no. 1, pp. 83–92, 2002.

[21] D. Guo, J. Sun, and Z. Liu, *Functional Method in Nonlinear Ordinary Differential Equations*, Shangdong Scientific Press, Shandong, China, 2005.

[22] K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, NY, USA, 1985.

[23] J. P. LaSalle, *The Stability of Dynamical System*, SIAM, Philadelphia, Pa, USA, 1976.

[24] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Science*, Academic Press, New York, NY, USA, 1929.

[25] I. Barbălat, "Systems d'equations differentielle d'oscillations nonlineaires," *Revue Roumaine de Mathématique Pures et Appliquées*, vol. 4, pp. 267–270, 1959.