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Research Article

A New Fractional Integral Inequality with Singularity and Its Application

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We prove an integral inequality with singularity, which complements some known results. This inequality enables us to study the dependence of the solution on the initial condition to a fractional differential equation in the weighted space.

1. Introduction

Integral inequalities provide an excellent tool for the properties of solutions to differential equations, such as boundedness, existence, uniqueness, and stability (e.g., see [1–10]). For this reason, the study of integral inequalities has been emphasized by many authors. For example, in 1919, Gronwall in [11] proved a remarkable inequality which can be described by the following.

Suppose that $x(t)$ satisfies the relation

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \quad t_0 \leq t < T, \tag{1.1}$$

where all the functions involved are continuous on the interval $[t_0, T)$, $T \leq \infty$, and $k(t) \geq 0$. Consider

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\exp\left(\int_{s}^{t} k(\tau)d\tau\right)ds, \quad t_0 \leq t < T. \tag{1.2}$$

The inequality has attracted and continues to attract considerable attention in the literature. In 2007, Ye et al. [12] reported an integral inequality with singular kernel. The inequality can be stated as follows.

If $\beta > 0$, $a(t)$ is a nonnegative and locally integrable on $0 \leq t < T$, $g(t)$ is a nonnegative, nondecreasing continuous function on $0 \leq t < T$, and $g(t) \leq M$, where $T \leq \infty$, M is a positive constant. Further suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds, \quad 0 \leq t < T. \tag{1.3}$$

Then

$$u(t) \leq a(t) + \sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} \int_0^t (t-s)^{n\beta-1} a(s) ds, \quad 0 \leq t < T. \tag{1.4}$$

Besides the above-mentioned inequalities, there are still many inequalities (e.g., see [13–15]).

But in the analysis of the dependence of the solution on the initial condition of a fractional differential equation in the weighted space, the bounds provided by the existing inequalities are not adequate. So it is natural and necessary to seek new inequality in order to obtain our desired results. In this paper, we present a new integral inequality, and then apply our inequality to investigate the dependence of the solution on the initial condition of a fractional differential equations in the weighted space.

2. An Integral Inequality

In this section, our main aim is to establish an integral inequality with singularity. Before proceeding, we give some useful definitions and lemmas.

Definition 2.1 (see [14, 16]). The gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt,\ z > 0$.

Definition 2.2 (see [14, 16]). The beta function is defined by $B(z, w) = \int_0^1 (1-t)^{z-1} t^{w-1} dt,\ z, w > 0$.

The beta function is connected with gamma function by the following relation [3, 14]:

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad z, w > 0. \tag{2.1}$$

Lemma 2.3 (see [14]). *Let $z > 0$, $a, b \in R$. Then the quotient expansion of two gamma functions at infinity can be represented as follows:*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + O\left(\frac{1}{z}\right)\right), \quad z \to \infty. \tag{2.2}$$

Lemma 2.4. *Let $z > 0$, $a, b \in R$. Then one has*

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = O\left(z^{a-b}\right), \quad z \longrightarrow \infty. \tag{2.3}$$

Proof. By Lemma 2.3, we have $\lim_{z\to\infty}(\Gamma(z+a)/\Gamma(z+b))/z^{a-b} = \lim_{z\to\infty}(1+O(1/z)) = 1$, which proves that $\Gamma(z+a)/\Gamma(z+b) = O(z^{a-b})$ as $z \to \infty$. The proof of this lemma is completed. □

Based on Lemma 2.4, we can define a function.

Definition 2.5. Let $b > a > 0$, $\rho > 0$. Then the following definition:

$$F_{\rho,a,b}(z) := \sum_{k=0}^{\infty} c_k z^k, \quad z \in R \tag{2.4}$$

is well defined, where c_0 is a positive constant, and $c_{k+1} = (\Gamma(k\rho+a)/\Gamma(k\rho+b))c_k$.

Proof. We only need to show that the series in (2.4) is uniformly convergent for $z \in R$. By Lemma 2.4, we know that $c_{k+1}/c_k = \Gamma(k\rho+a)/\Gamma(k\rho+b) = O((k\rho)^{a-b})$ as $k \to \infty$. Since $b > a > 0$, $c_{k+1}/c_k \to 0$ as $k \to \infty$. This implies that the series in (2.4) is uniformly convergent for $z \in R$. It follows that the definition is well defined. □

Lemma 2.6. *Let $z, w > 0$, $t, s \in R$ and $t \neq s$. Then one has*

$$\int_s^t (t-\tau)^{z-1}(\tau-s)^{w-1}d\tau = (t-s)^{z+w-1}\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{2.5}$$

Proof. Making the substitution $\tau = s + \xi(t-s)$ and combining the relation (2.1), we obtain

$$\begin{aligned}\int_s^t (t-\tau)^{z-1}(\tau-s)^{w-1}d\tau &= (t-s)^{z+w-1}\int_0^1 (1-\xi)^{z-1}\xi^{w-1}d\xi \\ &= (t-s)^{z+w-1}B(z,w) = (t-s)^{z+w-1}\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.\end{aligned} \tag{2.6}$$

The proof of this lemma is completed. □

Now we can state the integral inequality.

Theorem 2.7. *Let $\alpha, \beta, \gamma > 0$, $\delta = \alpha+\gamma-1 > 0$, $\nu = \beta+\gamma-1 > 0$, $a > 0$, and let $b(t)$ be a nonnegative, nondecreasing continuous function on $0 \le t < T$, $b(t) \le M$, where $T \le \infty$, M is a positive constant. Further suppose that $u(t)$ is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \le t < T$ with*

$$u(t) \le at^{\alpha-1} + b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}u(s)ds, \quad 0 \le t < T. \tag{2.7}$$

Then one has

$$u(t) \leq at^{\alpha-1} F_{\nu,\delta,\delta+\beta}\left(\Gamma(\beta) b(t) t^{\beta}\right), \quad 0 \leq t < T. \tag{2.8}$$

Proof. For convenience, we define an operator

$$(\mathcal{R}u)(t) = b(t) \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds. \tag{2.9}$$

Then (2.7) can be rewritten in the form

$$u(t) \leq at^{\alpha-1} + (\mathcal{R}u)(t). \tag{2.10}$$

Since $b(t)$ and $u(t)$ are nonnegative, it is easy to induce that

$$u(t) \leq \sum_{k=0}^{n} \left(\mathcal{R}^k at^{\alpha-1}\right)(t) + \left(\mathcal{R}^{n+1} u\right)(t), \quad n \in N. \tag{2.11}$$

Let us prove that the following relation

$$(\mathcal{R}^n u)(t) \leq \begin{cases} b(t)(\Gamma(\beta) b(t))^{n-1} \displaystyle\prod_{i=1}^{n-1} \frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)} \int_0^t (t-s)^{n\nu-\gamma} s^{\gamma-1} u(s) ds, & 0 < \gamma < 1, \\ \dfrac{(\Gamma(\beta) b(t))^n t^{(n-1)(\gamma-1)}}{\Gamma(n\beta)} \displaystyle\int_0^t (t-s)^{n\beta-1} s^{\gamma-1} u(s) ds, & \gamma \geq 1, \end{cases} \tag{2.12}$$

holds for any $n \in N^+$, where $\prod_{i=1}^{0} 1 = 1$, and $(\mathcal{R}^n u)(t) \to 0$ as $n \to \infty$ for each t in $0 \leq t < T$.

Obviously, inequality (2.12) is valid for $n = 1$, due to $\prod_{i=1}^{0} 1 = 1$. Suppose that the inequality is satisfied for any fixed $n \in N^+$. Let us verify that it is also satisfied for $n+1$. We first prove the case $0 < \gamma < 1$. According to the induction hypothesis and Lemma 2.6, we have

$$\begin{aligned}
\left(\mathcal{R}^{n+1}u\right)(t) &= b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}(\mathcal{R}^n u)(s)ds \\
&\le b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\int_0^s (s-\tau)^{n\nu-\gamma}\tau^{\gamma-1}u(\tau)d\tau ds \\
&= b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}s^{\gamma-1}(s-\tau)^{n\nu-\gamma}ds \\
&\le b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{n\nu-1}ds \\
&= b^2(t)\left(\Gamma(\beta)b(t)\right)^{n-1}\prod_{i=1}^{n-1}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{n\nu+\beta-1}\frac{\Gamma(\beta)\Gamma(n\nu)}{\Gamma(n\nu+\beta)}d\tau \\
&= b(t)\left(\Gamma(\beta)b(t)\right)^{n}\prod_{i=1}^{n}\frac{\Gamma(i\nu)}{\Gamma(i\nu+\beta)}\int_0^t (t-\tau)^{(n+1)\nu-\gamma}\tau^{\gamma-1}u(\tau)d\tau,
\end{aligned} \tag{2.13}$$

which is estimated with the help of

$$s^{\gamma-1} \le (s-\tau)^{\gamma-1}, \quad 0 \le \tau \le s,\ 0 < \gamma < 1. \tag{2.14}$$

So, for the case $0 < \gamma < 1$, inequality (2.12) is true for any $n \in N^+$. Now we prove the case $\gamma \ge 1$. Similarly, according to the induction hypothesis and Lemma 2.6, we get

$$\begin{aligned}
\left(\mathcal{R}^{n+1}u\right)(t) &= b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}(\mathcal{R}^n u)(s)ds \\
&\le b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\frac{\left(\Gamma(\beta)b(s)\right)^n s^{(n-1)(\gamma-1)}}{\Gamma(n\beta)}\int_0^s (s-\tau)^{n\beta-1}\tau^{\gamma-1}u(\tau)d\tau ds \\
&= b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}s^{n(\gamma-1)}(s-\tau)^{n\beta-1}ds \\
&\le b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n t^{n(\gamma-1)}}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)d\tau\int_\tau^t (t-s)^{\beta-1}(s-\tau)^{n\beta-1}ds \\
&= b(t)\frac{\left(\Gamma(\beta)b(t)\right)^n t^{n(\gamma-1)}}{\Gamma(n\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{n\beta+\beta-1}\frac{\Gamma(\beta)\Gamma(n\beta)}{\Gamma(n\beta+\beta)}d\tau \\
&= \frac{\left(\Gamma(\beta)b(t)\right)^{n+1}t^{n(\gamma-1)}}{\Gamma((n+1)\beta)}\int_0^t \tau^{\gamma-1}u(\tau)(t-\tau)^{(n+1)\beta-1}d\tau,
\end{aligned} \tag{2.15}$$

which is calculated with the help of

$$s^{n(\gamma-1)} \leq t^{n(\gamma-1)}, \quad 0 \leq s \leq t, \ \gamma \geq 1, \ n \in N^{+}. \tag{2.16}$$

So, for the case $\gamma \geq 1$, inequality (2.12) is true for any $n \in N^{+}$. Based on this analysis, we conclude that inequality (2.12) holds for any $n \in N^{+}$.

Next, we show that $(\mathcal{R}^{n}u)(t) \rightarrow 0$ as $n \rightarrow \infty$. Now, we go back to inequality (2.12). For the case $0 < \gamma < 1$, we denote $K_n(t,s) = B_n(t-s)^{n\nu-\gamma}$, where $B_n = b(t)(\Gamma(\beta)b(t))^{n-1}\prod_{i=1}^{n-1}(\Gamma(i\nu)/\Gamma(i\nu+\beta))$. Note that

$$B_1 = b(t), \quad \frac{B_{n+1}}{B_n} = \Gamma(\beta)b(t)\frac{\Gamma(n\nu)}{\Gamma(n\nu+\beta)}. \tag{2.17}$$

Since $b(t) \leq M$, by Lemma 2.4, we obtain $B_{n+1}/B_n \rightarrow 0$ as $n \rightarrow \infty$. This implies that $K_n(t,s) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $(R^{n}u)(t) \rightarrow 0$ as $n \rightarrow \infty$ for the case $0 < \gamma < 1$. For the case $\gamma \geq 1$, we denote $\overline{K}_n(t,s) = \overline{B}_n(t-s)^{n\beta-1}$, where $\overline{B}_n = (\Gamma(\beta)b(t))^{n}t^{(n-1)(\gamma-1)}/\Gamma(n\beta)$. Note that

$$\overline{B}_1 = b(t), \quad \frac{\overline{B}_{n+1}}{\overline{B}_n} = \Gamma(\beta)b(t)t^{\gamma-1}\frac{\Gamma(n\beta)}{\Gamma(n\beta+\beta)}. \tag{2.18}$$

Using the same arguments as above, we know that $\overline{K}_n(t,s) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $(\mathcal{R}^{n}u)(t) \rightarrow 0$ as $n \rightarrow \infty$ for the case $\gamma \geq 1$. So, it has $(\mathcal{R}^{n}u)(t) \rightarrow 0$ as $n \rightarrow \infty$ for the two cases $0 < \gamma < 1$ and $\gamma \geq 1$. This, together with (2.11), leads to $u(t) \leq \sum_{k=0}^{\infty}(\mathcal{R}^{k}at^{\alpha-1})(t)$.

Finally, we show that

$$\left(\mathcal{R}^{k}at^{\alpha-1}\right)(t) \leq a\left(\Gamma(\beta)b(t)\right)^{k}c_k t^{\alpha-1}t^{k\nu}, \quad k \in N, \tag{2.19}$$

where $c_0 = 1$, $c_k = \prod_{i=0}^{k-1}(\Gamma(i\nu+\delta)/\Gamma(i\nu+\delta+\beta))$, $k \in N^{+}$.

Obviously, inequality (2.19) is true for $k = 0$. Suppose that the inequality is satisfied for any fixed $k \in N$. Let us verify that it is also satisfied for $k+1$. According to the induction hypothesis and Lemma 2.6, we obtain

$$\begin{aligned}
\left(\mathcal{R}^{k+1}at^{\alpha-1}\right)(t) &\leq b(t)\int_0^t (t-s)^{\beta-1}s^{\gamma-1}\left(\mathcal{R}^{k}as^{\alpha-1}\right)(s)ds \\
&\leq a\left(\Gamma(\beta)b(t)\right)^{k}c_k b(t)\int_0^t (t-s)^{\beta-1}s^{k\nu+\delta-1}ds \\
&= a\left(\Gamma(\beta)b(t)\right)^{k}c_k b(t)t^{k\nu+\delta+\beta-1}\frac{\Gamma(\beta)\Gamma(k\nu+\delta)}{\Gamma(k\nu+\delta+\beta)} \\
&= a\left(\Gamma(\beta)b(t)\right)^{k+1}c_{k+1}t^{\alpha-1}t^{(k+1)\nu}.
\end{aligned} \tag{2.20}$$

This proves that inequality (2.19) is satisfied for any $k \in N$. In other words, we have proved that

$$u(t) \leq \sum_{k=0}^{\infty} a\left(\Gamma(\beta)b(t)\right)^k c_k t^{\alpha-1} t^{k\nu}, \tag{2.21}$$

where $c_0 = 1$, $c_k = \prod_{i=0}^{k-1}(\Gamma(i\nu+\delta)/\Gamma(i\nu+\delta+\beta))$, $k \in N^+$. By virtue of Definition 2.5, we can arrive at inequality (2.8) and the proof of this theorem is completed. □

For the case $b(t) \equiv b > 0$ in Theorem 2.7, we can obtain the following corollary, which can be found in [17].

Corollary 2.8. *Let $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 > 0$, $\nu = \beta + \gamma - 1 > 0$, $a, b > 0$. And suppose that $u(t)$ is nonnegative and $t^{\gamma-1}u(t)$ is locally integrable on $0 \leq t < T$ ($T \leq \infty$) with*

$$u(t) \leq a t^{\alpha-1} + b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds, \quad 0 \leq t < T. \tag{2.22}$$

Then one has

$$u(t) \leq a t^{\alpha-1} F_{\nu,\delta,\delta+\beta}\left(\Gamma(\beta) b t^{\beta}\right), \quad 0 \leq t < T. \tag{2.23}$$

For $\alpha = \gamma = 1$ in Theorem 2.7, we can arrive at the following corollary, which can be found in [12].

Corollary 2.9. *Let β, $a > 0$, $b(t)$ be a nonnegative, nondecreasing continuous function on $0 \leq t < T$, $b(t) \leq M$, where $T \leq \infty$, M is a positive constant. And suppose that $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a + b(t) \int_0^t (t-s)^{\beta-1} u(s) ds, \quad 0 \leq t < T. \tag{2.24}$$

Then one has

$$u(t) \leq a E_{\beta}\left(\Gamma(\beta) b(t) t^{\beta}\right), \quad 0 \leq t < T. \tag{2.25}$$

3. Application

In this section, we will apply our established result to study the dependence of the solution on the initial condition of a fractional differential equation with the Riemann-Liouville derivative.

For the reader's convenience, we first recall several definitions of the Reimann-Liouville integral and derivative. From now on, we assume that T is a finite positive constant, that is, $T \neq \infty$.

Definition 3.1 (see [14, 16]). Let $0 < p < 1$. The Riemann-Liouville integral of order p is defined by

$$\left(I_{0^+}^{p} x\right)(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} x(s) ds, \quad 0 \le t \le T. \tag{3.1}$$

Definition 3.2 (see [14, 16]). Let $0 < p < 1$. The Riemann-Liouville derivative of order p is defined by

$$\left(D_{0^+}^{p} x\right)(t) = \frac{1}{\Gamma(1-p)} \frac{d}{dt} \int_0^t (t-s)^{-p} x(s) ds, \quad 0 \le t \le T. \tag{3.2}$$

Now we consider the following initial value problem of the form

$$\left(D_{0^+}^{p} x\right)(t) = f(t, x(t)), \quad \lim_{t \to 0^+} \left(I_{0^+}^{1-p} x\right)(t) = x_0, \quad 0 < p < 1, \ 0 < t \le T, \ x_0 \in R. \tag{3.3}$$

With regard to problem (3.3), the existence and uniqueness of the solution can be found in the book by Kilbas et al. [14]. For the completeness of this paper, we give the existence and uniqueness of the solution to (3.3) in the weighted space $C_{1-p}([0,T])$. The space $C_{1-p}([0,T])$ consists of all functions $x \in C((0,T])$ such that $t^{1-p}x(t) \in C([0,T])$, which turns out to be a Banach space when endowed with the norm $|x|_{1-p} = \max_{0 \le t \le T} |t^{1-p} x(t)|$.

Theorem 3.3 (see [14]). *Let $0 < p < 1$, and $f(t,x) : (0,T] \times R \to R$ be a function such that for any $x \in R$, $f(t,x) \in C_{1-p}([0,T])$. Further assume that for any $t \in (0,T]$, $x, y \in R$, the following inequality*

$$\left| f(t,x) - f(t,y) \right| \le L \left| x - y \right| \tag{3.4}$$

holds, where $L > 0$ is a constant. Then there exists a unique solution $x(t)$ to problem (3.3) *in the space $C_{1-p}([0,T])$.*

Theorem 3.4. *Let $0 < p < 1$, and $f(t,x) : (0,T] \times R \to R$ be a function such that for any $x \in R$, $f(t,x) \in C_{1-p}([0,T])$. Further assume that for any $t \in (0,T]$, $x, y \in R$, the following inequality*

$$\left| f(t,x) - f(t,y) \right| \le L \left| x - y \right| \tag{3.5}$$

holds, where $L > 0$ is a constant. Assume that x and y are the solutions of problem (3.3) *and*

$$\left(D_{0^+}^{p} y\right)(t) = f(t, y(t)), \quad \lim_{t \to 0^+} \left(I_{0^+}^{1-p} y\right)(t) = y_0, \quad 0 < t \le T, \ y_0 \in R, \tag{3.6}$$

respectively. Then, for $0 \le t \le T$, one has

$$t^{1-p}\left|x(t)-y(t)\right| \le \begin{cases} \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p-1,p,2p}(Lt), & \dfrac{1}{2} < p < 1,\ 0 \le t \le T, \\[2ex] \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p+q-1,p+q,2p+q}(L^{*}t), & 0 < p \le \dfrac{1}{2},\ 0 \le t < 1, \\[2ex] \dfrac{\left|x_0-y_0\right|}{\Gamma(p)} F_{2p+q'-1,p+q',2p+q'}(L't), & 0 < p \le \dfrac{1}{2},\ 1 \le t \le T, \end{cases} \tag{3.7}$$

where q, q', L^{*}, L' are positive constants such that

$$\begin{gathered} 1-2p < q < \log_t^{L/L^{*}}, \quad 0 < p \le \frac{1}{2},\ 0 < t < 1, \\ q' > \max\left\{1-2p, \log_t^{L/L'}\right\}, \quad 0 < p \le \frac{1}{2},\ 1 \le t \le T. \end{gathered} \tag{3.8}$$

Proof. The proof is rather technical. We first prove the case $1/2 < p < 1$ and $0 \le t \le T$. Since $x(t)$ and $y(t)$ are the solutions of (3.3) and (3.6), we have

$$x(t) = \frac{x_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, x(s))\,ds, \tag{3.9}$$

$$y(t) = \frac{y_0 t^{p-1}}{\Gamma(p)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s, y(s))\,ds. \tag{3.10}$$

Subtracting (3.10) from (3.9) and using the Lipschitz condition (3.5), we obtain

$$\left|x(t)-y(t)\right| \le \frac{\left|x_0-y_0\right| t^{p-1}}{\Gamma(p)} + \frac{L}{\Gamma(p)} \int_0^t (t-s)^{p-1} \left|x(s)-y(s)\right| ds. \tag{3.11}$$

Taking into account that $x(t), y(t) \in C_{1-p}([0,T])$, we multiply at both sides of inequality (3.11) by t^{1-p} to get

$$t^{1-p}\left|x(t)-y(t)\right| \le \frac{\left|x_0-y_0\right|}{\Gamma(p)} + \frac{Lt^{1-p}}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^{p-1} s^{1-p} \left|x(s)-y(s)\right| ds. \tag{3.12}$$

Denote $u(t) = t^{1-p}|x(t)-y(t)|$. Then, (3.12) can be written as

$$u(t) \le \frac{\left|x_0-y_0\right|}{\Gamma(p)} + \frac{Lt^{1-p}}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^{p-1} u(s)\,ds. \tag{3.13}$$

Putting $a = |x_0 - y_0|/\Gamma(p)$, $b(t) = Lt^{1-p}/\Gamma(p)$, $\alpha = 1$, $\beta = p$, $\gamma = p$, we see that $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 = p > 0$, $\nu = \beta + \gamma - 1 = 2p - 1 > 0$, $a > 0$, and $b(t)$ is a nondecreasing continuous function due to $p, L > 0$. So, applying Theorem 2.7 to (3.13), we obtain

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} F_{2p-1,p,2p}(Lt), \quad \frac{1}{2} < p < 1, \; 0 \leq t \leq T. \tag{3.14}$$

Next, we prove the case $0 < p \leq 1/2$ and $0 \leq t < 1$. Notice that the Lipschitz condition (3.5) holds for each t in $t \in (0, T]$. Since $t > 0$ and $L > 0$, we can always choose two positive constants q, L^* such that

$$1 - 2p < q < \log_t^{L/L^*}, \quad 0 < t < 1. \tag{3.15}$$

Condition (3.15) means that $0 \leq 1 - 2p < q$ and $L < L^* t^q$. That is to say, if the Lipschitz condition (3.5) holds for each t in $t \in (0, T]$, then we can always choose two positive constants q, L^* such that the following condition

$$0 \leq 1 - 2p < q, \quad \left|f(t, u) - f(t, v)\right| \leq L^* t^q |u - v| \tag{3.16}$$

holds for each t in $t \in (0, T]$.

Subtracting (3.10) from (3.9) and using condition (3.16), we obtain

$$\left|x(t) - y(t)\right| \leq \frac{\left|x_0 - y_0\right| t^{p-1}}{\Gamma(p)} + \frac{L^*}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^q \left|x(s) - y(s)\right| ds. \tag{3.17}$$

Multiplying t^{1-p} on both sides of (3.17), we get

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} + \frac{L^* t^{1-p}}{\Gamma(p)} \int_0^t (t-s)^{p-1} s^{p+q-1} u(s) ds, \tag{3.18}$$

where $u(t)$ is defined as before. Now, putting $a = |x_0 - y_0|/\Gamma(p)$, $b(t) = L^* t^{1-p}/\Gamma(p)$, $\alpha = 1$, $\beta = p$, $\gamma = p + q$, we see that $\alpha, \beta, \gamma > 0$, $\delta = \alpha + \gamma - 1 = p + q > 0$, $\nu = \beta + \gamma - 1 = 2p + q - 1 = q - (1 - 2p) > 0$, $a > 0$, $b(t)$ is a nondecreasing continuous function due to $p, L^* > 0$. So, applying Theorem 2.7 to (3.18), we have

$$u(t) \leq \frac{\left|x_0 - y_0\right|}{\Gamma(p)} F_{2p+q-1,p+q,2p+q}(L^* t), \quad 0 < p \leq \frac{1}{2}, \; 0 \leq t < 1. \tag{3.19}$$

Finally, we prove the case $0 < p \leq 1/2$ and $1 \leq t \leq T$. Since $t > 0$ and $L > 0$, we can always choose two positive constants q', L' such that

$$q' > \max\left\{1 - 2p, \log_t^{L/L'}\right\}, \quad 1 \leq t \leq T. \tag{3.20}$$

Condition (3.20) means that $0 \le 1-2p < q'$ and $L < L' t^q$. Using the same arguments as above, we can obtain that

$$u(t) \le \frac{|x_0 - y_0|}{\Gamma(p)} F_{2p+q'-1,p+q',2p+q'}(L't), \quad 0 < p \le \frac{1}{2},\ 1 \le t \le T, \tag{3.21}$$

where $u(t)$ is defined as before. So the conclusion of this theorem is true. □

From the proof of Theorem 3.4, we can see that the integral inequality in Theorem 2.7 is very useful. This demonstrates that our investigation is meaningful.

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