Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 924645, 9 pages
doi:10.1155/2012/924645

Research Article

The Univalence Conditions of Some Integral Operators

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Received 26 January 2012; Revised 4 March 2012; Accepted 8 March 2012

Academic Editor: Ondřej Došlý

We introduce new integral operators of analytic functions f and g defined in the open unit disk $\mathbb{U}$. For these operators, we discuss some univalence conditions.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\} \tag{1.2}$$

and satisfy the following usual normalization condition:

$$f(0) = f'(0) - 1 = 0. \tag{1.3}$$

Also, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f, which are univalent in $\mathbb{U}$ (see, for details [1]; see also [2, 3]).

In [4, 5], Pescar gave the following univalence conditions for the functions $f \in \mathcal{A}$.

Theorem 1.1 (see [4]). *Let α be a complex number,* $\operatorname{Re}\alpha > 0$, *and c a complex number,* $|c| \leq 1$, $c \neq -1$, *and $f(z) = z + \cdots$ a regular function in $\mathbb{U}$. If*

$$\left| c|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \tag{1.4}$$

for all $z \in \mathbb{U}$, then the function

$$F_\alpha(z) = \left(\alpha \int_0^z t^{\alpha-1} f'(t) dt\right)^{1/\alpha} = z + \cdots \tag{1.5}$$

is regular and univalent in $\mathbb{U}$.

Theorem 1.2 (see [5]). *Let α be a complex number,* $\operatorname{Re}\alpha > 0$, *and c a complex number,* $|c| \leq 1$, $c \neq -1$, *and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - |c| \tag{1.6}$$

for all $z \in \mathbb{U}$, then for any complex number β, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the function*

$$F_\beta(z) = \left(\beta \int_0^z t^{\beta-1} f'(t) dt\right)^{1/\beta} \tag{1.7}$$

is in the class $\mathcal{S}$.

On the other hand, for the functions $f \in \mathcal{A}$, Ozaki and Nunokawa [6] proved another univalence condition asserted by Theorem 1.3.

Theorem 1.3 (see [6]). *Let $f \in \mathcal{A}$ satisfy the condition*

$$\left| \frac{z^2 f'(z)}{[f(z)]^2} - 1 \right| < 1 \quad (z \in \mathbb{U}). \tag{1.8}$$

Then f is univalent in $\mathbb{U}$.

In the paper [7], Pescar determined some univalence conditions for the following integral operators.

Theorem 1.4 (see [7]). *Let the function g satisfy* (1.8), *M a positive real number fixed, and c a complex number. If $\alpha \in [(2M+1)/(2M+2), (2M+1)/2M]$,*

$$|c| \leq 1 - \left| \frac{\alpha - 1}{\alpha} \right| (2M+1), \quad c \neq -1, \qquad |g(z)| \leq M \tag{1.9}$$

for all $z \in \mathbb{U}$, *then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z [g(t)]^{\alpha-1} dt\right)^{1/\alpha} \tag{1.10}$$

is in the class $\mathcal{S}$.

Theorem 1.5 (see [7]). *Let* $g \in \mathcal{A}$, α *a real number,* $\alpha \geq 1$, *and* c *a complex number,* $|c| \leq 1/\alpha$, $c \neq -1$. *If*

$$\left|\frac{g''(z)}{g'(z)}\right| \leq 1 \quad (z \in \mathbb{U}), \tag{1.11}$$

then the function

$$H_\alpha(z) = \left(\alpha \int_0^z \left[t g'(t)\right]^{\alpha-1} dt\right)^{1/\alpha} \tag{1.12}$$

is in the class $\mathcal{S}$.

Theorem 1.6 (see [7]). *Let* $g \in \mathcal{A}$ *satisfies* (1.8), α *a complex number,* $M > 1$ *fixed,* $\operatorname{Re}\alpha > 0$, *and* c *a complex number,* $|c| < 1$. *If* $|g(z)| \leq M$ *for all* $z \in \mathbb{U}$, *then for any complex number* β

$$\operatorname{Re}\beta \geq \operatorname{Re}\alpha \geq \frac{2M+1}{|\alpha|(1-|c|)} \tag{1.13}$$

the function

$$H_\beta(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{g(t)}{t}\right)^{1/\alpha} dt\right)^{1/\beta} \tag{1.14}$$

is in the class $\mathcal{S}$.

In this paper, we introduce the following integral operators as follows:

$$F_1(f,g)(z) = \left(\alpha \int_0^z \left(f(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha} \quad (f,g \in \mathcal{A};\ \alpha \in \mathbb{C}), \tag{1.15}$$

$$G_1(f,g)(z) = \left(\alpha \int_0^z \left(t f'(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha} \quad (f,g \in \mathcal{A};\ \alpha \in \mathbb{C}), \tag{1.16}$$

$$H_1(f,g)(z) = \left(\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{1/\alpha} dt\right)^{1/\beta} \quad (f,g \in \mathcal{A};\ \alpha,\beta \in \mathbb{C}-\{0\}). \tag{1.17}$$

Remark 1.7. For $e^{g(z)} = 1$ and $f(z) = g(z)$, the integral operators (1.15), (1.16), and (1.17) would reduce to the integral operators (1.10), (1.12), and (1.14).

In this paper, we generalize the integral operators given by Pescar [7], and we study the univalence conditions for the integral operators defined by (1.15), (1.16), and (1.17).

For this purpose, we need the following result.

Lemma 1.8 (General Schwarz Lemma [8]). *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$\left|f(z)\right| \leq \frac{M}{R^m}|z|^m \quad (z \in \mathbb{U}_R). \tag{1.18}$$

The equality can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m, \tag{1.19}$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let $f, g \in \mathcal{A}$, where g satisfies the condition* (1.8), *M_1 and M_2 are real positive numbers, and α a complex number,* $\operatorname{Re}\alpha > 0$. *If*

$$\left|\frac{f'(z)}{f(z)}\right| \leq M_1 \quad (z \in \mathbb{U}), \qquad \left|g(z)\right| \leq M_2 \quad (z \in \mathbb{U}), \tag{2.1}$$

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right|\left(M_1 + 2M_2^2 + 1\right), \quad c \in \mathbb{C},\ c \neq -1, \tag{2.2}$$

then the integral operator $F_1(f, g)(z)$ defined by (1.15) *is in the class $\mathcal{S}$.*

Proof. From (1.15), we have

$$F_1(f, g)(z) = \left(\alpha \int_0^z t^{\alpha-1}\left(\frac{f(t)}{t}e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha}. \tag{2.3}$$

Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t}e^{g(t)}\right)^{\alpha-1} dt. \tag{2.4}$$

The function h is regular in $\mathbb{U}$. From (2.4), we get

$$h'(z) = \left(\frac{f(z)}{z}e^{g(z)}\right)^{\alpha-1},$$
$$h''(z) = (\alpha-1)\left(\frac{f(z)}{z}e^{g(z)}\right)^{\alpha-2}\left(\frac{zf'(z)-f(z)}{z^2}e^{g(z)} + \frac{f(z)}{z}g'(z)e^{g(z)}\right). \tag{2.5}$$

Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = (\alpha-1)\left[\left(\frac{zf'(z)}{f(z)}-1\right) + zg'(z)\right], \tag{2.6}$$

which readily shows that

$$\begin{aligned}\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)}\right| &= \left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{\alpha-1}{\alpha}\left(\left(\frac{zf'(z)}{f(z)}-1\right)+zg'(z)\right)\right| \\ &\leq |c| + \left|\frac{\alpha-1}{\alpha}\right|\left(\left(\left|\frac{zf'(z)}{f(z)}\right|+1\right) + \left|\frac{z^2g'(z)}{[g(z)]^2}\right|\left|\frac{[g(z)]^2}{z}\right|\right) \quad (z\in\mathbb{U}).\end{aligned} \tag{2.7}$$

From the hypothesis of Theorem 2.1, we have

$$\left|\frac{f'(z)}{f(z)}\right| \leq M_1 \quad (z\in\mathbb{U}), \qquad |g(z)| \leq M_2 \quad (z\in\mathbb{U}), \tag{2.8}$$

then by *General Schwarz Lemma* for the function g, we obtain

$$|g(z)| \leq M_2|z| \quad (z\in\mathbb{U}). \tag{2.9}$$

Using the inequality (2.7), we have

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)}\right| \leq |c| + \left|\frac{\alpha-1}{\alpha}\right|\left(M_1 + 1 + \left(\left|\frac{z^2g'(z)}{[g(z)]^2}-1\right|+1\right)M_2^2\right). \tag{2.10}$$

From (2.10) and since g satisfies the condition (1.8), we have

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)}\right| \leq |c| + \left|\frac{\alpha-1}{\alpha}\right|\left(M_1 + 2M_2^2 + 1\right), \tag{2.11}$$

from which, by (2.2), we get

$$\left|c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)}\right| \leq 1 \quad (z\in\mathbb{U}). \tag{2.12}$$

Applying Theorem 1.1, we conclude that the integral operator $F_1(f,g)(z)$ defined by (1.15) is in the class $\mathcal{S}$. □

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.1, we immediately arrive at the following application of Theorem 2.1.

Corollary 2.2. *Let $f, g \in \mathcal{A}$, where g satisfies the condition* (1.8) *and α a complex number,* $\operatorname{Re}\alpha > 0$. *If*

$$\left|\frac{f'(z)}{f(z)}\right| \leq 1 \quad (z \in \mathbb{U}), \qquad |g(z)| \leq 1 \quad (z \in \mathbb{U}),$$
$$|c| \leq 1 - \left|\frac{4(\alpha-1)}{\alpha}\right|, \quad c \in \mathbb{C},\ c \neq -1, \tag{2.13}$$

then the integral operator $F_1(f,g)(z)$ defined by (1.15) *is in the class $\mathcal{S}$.*

Theorem 2.3. *Let $f, g \in \mathcal{A}$, where g satisfies the inequality $|g(z)| \leq M$, $M \geq 1$. Also, let α be a real number, $\alpha \geq 1$, and c a complex number with*

$$|c| \leq 1 - \left|\frac{\alpha-1}{\alpha}\right|(M+1), \quad c \neq -1. \tag{2.14}$$

If

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1 \quad (z \in \mathbb{U}), \qquad \left|\frac{g'(z)}{g(z)}\right| \leq 1 \quad (z \in \mathbb{U}), \tag{2.15}$$

then the integral operator $G_1(f,g)(z)$ defined by (1.16) *is in the class $\mathcal{S}$.*

Proof. We observe that

$$G_1(f,g)(z) = \left(\alpha \int_0^z t^{\alpha-1}\left(f'(t)e^{g(t)}\right)^{\alpha-1} dt\right)^{1/\alpha}. \tag{2.16}$$

Let us consider the function

$$h(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha-1} dt. \tag{2.17}$$

The function h is regular in $\mathbb{U}$. From (2.17), we have

$$h'(z) = \left(f'(z)e^{g(z)}\right)^{\alpha-1},$$
$$\frac{zh''(z)}{h'(z)} = (\alpha-1)\left(\frac{zf''(z)}{f'(z)} + zg'(z)\right), \tag{2.18}$$

which readily shows that

$$\left| c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)} \right| \le |c| + \left|\frac{\alpha-1}{\alpha}\right| \left(\left|\frac{zf''(z)}{f'(z)}\right| + \left|\frac{zg'(z)}{g(z)}\right| \left|g(z)\right| \right) \quad (z \in \mathbb{U}). \tag{2.19}$$

From (2.19) and the conditions of Theorem 2.3, we get

$$\left| c|z|^{2\alpha} + \left(1-|z|^{2\alpha}\right)\frac{zh''(z)}{\alpha h'(z)} \right| \le |c| + \left|\frac{\alpha-1}{\alpha}\right| (1+M) \le 1 \quad (z \in \mathbb{U}). \tag{2.20}$$

Applying Theorem 1.1, we conclude that the integral operator $G_1(f,g)(z)$ defined by (1.16) is in the class $\mathcal{S}$. □

Setting $M = 1$ in Theorem 2.3, we obtain the following consequence of Theorem 2.3.

Corollary 2.4. *Let $f, g \in \mathcal{A}$, where g satisfies the condition $|g'(z)/g(z)| \le 1$, α a real number, $\alpha \ge 1$, and c a complex number with $|c| \le 1 - |2(\alpha-1)/\alpha|$, $c \ne -1$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le 1 \quad (z \in \mathbb{U}), \qquad \left|g(z)\right| \le 1 \quad (z \in \mathbb{U}), \tag{2.21}$$

then the integral operator $G_1(f,g)(z)$ defined by (1.16) *is in the class $\mathcal{S}$.*

Theorem 2.5. *Let $f, g \in \mathcal{A}$, where g satisfies the condition* (1.8)*, α a complex number,* $\operatorname{Re}\alpha > 0$, *$M_1$ and M_2 are real positive numbers, and c a complex number, $|c| < 1$. If*

$$\left|\frac{f'(z)}{f(z)}\right| \le M_1 \quad (z \in \mathbb{U}), \qquad \left|g(z)\right| \le M_2 \quad (z \in \mathbb{U}), \tag{2.22}$$

then for any complex number β,

$$\operatorname{Re}\beta \ge \operatorname{Re}\alpha \ge \frac{M_1 + 2M_2^2 + 1}{|\alpha|(1-|c|)}, \tag{2.23}$$

the integral operator $H_1(f,g)(z)$ defined by (1.17) *is in the class $\mathcal{S}$.*

Proof. Let us consider the function

$$h(z) = \int_0^z \left(\frac{f(t)}{t} e^{g(t)}\right)^{1/\alpha} dt. \tag{2.24}$$

The function h is regular in $\mathbb{U}$. From (2.24), we have

$$h'(z) = \left(\frac{f(z)}{z}e^{g(z)}\right)^{1/\alpha},$$
$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\left(\frac{zf'(z)}{f(z)} - 1\right) + zg'(z)\right), \tag{2.25}$$

which readily shows that

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\left(\left|\frac{zf'(z)}{f(z)}\right| + 1 + \left|\frac{z^2g'(z)}{[g(z)]^2}\right|\left|\frac{[g(z)]^2}{z}\right|\right)$$
$$\le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\left(\left|\frac{zf'(z)}{f(z)}\right| + 1 + \left(\left|\frac{z^2g'(z)}{[g(z)]^2} - 1\right| + 1\right)\left|\frac{[g(z)]^2}{z}\right|\right). \tag{2.26}$$

By the *General Schwarz Lemma* for the function g, we obtain

$$|g(z)| \le M_2|z| \quad (z \in \mathbb{U}), \tag{2.27}$$

and using the inequality (2.26), we have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\left(M_1 + 1 + \left(\left|\frac{z^2g'(z)}{[g(z)]^2} - 1\right| + 1\right)M_2^2\right). \tag{2.28}$$

From (2.28) and since g satisfies the condition (1.8), we get

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\frac{(M_1 + 2M_2^2 + 1)}{|\alpha|}$$
$$\le \frac{M_1 + 2M_2^2 + 1}{|\alpha|\operatorname{Re}\alpha}. \tag{2.29}$$

From (2.23), we have

$$\frac{M_1 + 2M_2^2 + 1}{|\alpha|\operatorname{Re}\alpha} \le 1 - |c|, \tag{2.30}$$

and using (2.29), we obtain

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le 1 - |c| \quad (z \in \mathbb{U}). \tag{2.31}$$

Applying Theorem 1.2, we conclude that the integral operator $H_1(f, g)(z)$ defined by (1.17) is in the class $\mathcal{S}$. □

Setting $M_1 = 1$ and $M_2 = 1$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let* $f, g \in \mathcal{A}$, *where* g *satisfies the condition* (1.8), α *a complex number,* $\operatorname{Re}\alpha > 0$, *and* c *a complex number,* $|c| < 1$. *If*

$$\left|\frac{f'(z)}{f(z)}\right| \leq 1 \quad (z \in \mathbb{U}), \qquad \left|g(z)\right| \leq 1 \quad (z \in \mathbb{U}), \tag{2.32}$$

then for any complex number β,

$$\operatorname{Re}\beta \geq \operatorname{Re}\alpha \geq \frac{4}{|\alpha|(1-|c|)}, \tag{2.33}$$

the integral operator $H_1(f, g)(z)$ *defined by* (1.17) *is in the class* $\mathcal{S}$.

Acknowledgments

This paper was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007–2013 cofinanced by the European Social Fund-Investing in People.

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