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Research Article

The Stability of Nonlinear Differential Systems with Random Parameters

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The paper deals with nonlinear differential systems with random parameters in a general form. A new method for construction of the Lyapunov functions is proposed and is used to obtain sufficient conditions for L_2-stability of the trivial solution of the considered systems.

1. Introduction

1.1. The Aim of the Contribution

The method of Lyapunov functions is one of the most effective methods for investigation of self-regulating systems. It is important for determining the fact of stability or instability of given systems among other purposes. A successfully constructed Lyapunov function for given nonlinear self-regulating systems makes it possible to solve all the complex problems important in practical applications such as estimation of changes of a self-regulated variable, estimation of transient processes, estimation of integral criteria of the quality of self-regulation, or estimation of what is called guaranteed domain of stability.

In [1] it is explained why not every positive definite function can serve as a Lyapunov function for a system of differential equations. As experience shows, the most suitable Lyapunov functions have physical meaning. The Lyapunov function method is an effective method for the investigation of stability of linear or nonlinear differential systems that are explicitly independent of time (see, e.g., [1–9]). But there are no universal methods for constructing appropriate Lyapunov functions because, as well-known, in nonlinear differential systems, each case considered requires an individual method for constructing a Lyapunov function.

However, the method of Lyapunov functions is often difficult to apply to the investigation of some kinds of stability of nonstationary differential systems because the concept of Lyapunov stability can make the Lyapunov functions inconvenient to use. This problem was solved by a new definition of what is called L_2-stability of the trivial solution of the nonstationary differential (or difference) systems [10, 11], which is compatible with the method of Lyapunov functions.

In this paper, we deal with much more complicated investigation of the Lyapunov stability of differential systems with random parameters. We define a concept of L_2-stability of the trivial solution of the differential systems with semi-Markov coefficients and give an analogy between the L_2-stability and the stability obtained by Lyapunov functions. A new method of constructing Lyapunov functions is proposed for the study of stability of systems, and Lyapunov functions are derived for systems of differential equations with coefficients depending on a semi-Markov process. Sufficient conditions of stability are given, and it is proved that the condition of L_2-stability implies the existence of Lyapunov functions. In addition to this, the case of the coefficients of the considered systems depending on Markov process is analyzed.

1.2. Systems Considered

In this part, a new concept of semi-Markov function is proposed. It will be used later for the construction of Lyapunov functions.

Consider nonlinear n-dimensional differential system

$$\frac{dX(t)}{dt} = F(t, X(t), \xi(t)), \qquad F(t, 0, \xi) = 0, \tag{1.1}$$

on the probability space $(\Omega \equiv \{\omega\}, \mathfrak{T}, \mathbf{P}, \mathbf{F} \equiv \{\mathbf{F_t} : t \geq 0\})$. A vector-function $X = X(t)$, $t \geq 0$, is called a solution of (1.1) if $X(t)$ is a random vector-function from the set of random vector-functions defined on Ω, there exists mathematical expectation of $\{X^2(t)\}$, and (1.1) is satisfied for $t \geq 0$. The derivative is understood in the meaning of differentiability of a random process [12].

A space of solutions X can be interpreted as a phase space of states of a random environment. Measurable subsets of a random environment form a collection of its states. As a phase space of states serves a complete metric separable space (as a rule the Euclidean space or a finite space equipped with σ-algebra of all subsets of X). Under assumptions of our problem (and in similar problems as well), solutions are defined in the meaning of a strong solution of the Cauchy problem [13].

Together with (1.1), we consider the initial condition

$$X(0) = \varphi(\omega), \qquad \varphi : \Omega \longrightarrow \mathbb{R}^n. \tag{1.2}$$

In fact, any solution $X(t)$ of (1.1) depends on the random variable ω, that is, $X(t) \equiv X(t, \omega)$.

The random process $\xi(t)$, $t \geq 0$, is a semi-Markov process with the states

$$\theta_1, \theta_2, \ldots, \theta_n. \tag{1.3}$$

We assume $\xi(t_0) = 0$ where $t_0 = 0$, and moments of jumps t_j, $j = 0, 1, \ldots, n$, $t_0 < t_1 < \cdots < t_n$ of the process ξ are such that $\xi(t_j) = \lim_{t \to t_j + 0} \xi(t)$ and $\xi(t) = \theta_s$, $s \in \{1, 2, \ldots, n\}$ if $t_j \leq t < t_{j+1}$, $j = 0, 1, \ldots, n-1$.

The transition from state θ_l to state θ_s is characterized by the intensity $q_{ls}(t)$, $l, s = 1, 2, \ldots, n$, and the semi-Markov process is defined by the intensity matrix

$$Q(t) = \left(q_{ls}(t)\right)_{l,s=1}^{n}, \tag{1.4}$$

whose elements satisfy the relationships

$$q_{ls}(t) \geq 0, \qquad \sum_{l=1}^{n} \int_0^{\infty} q_{ls}(t) dt = 1. \tag{1.5}$$

Let mutually different functions $w_s(t, x)$, $s = 1, 2, \ldots, n$, be defined for $t > 0$, $x \in \mathbb{R}^n$.

Definition 1.1. The function $w(t, x, \xi(t))$ is called a semi-Markov function if the equalities

$$w(t, x, \xi(t) = \theta_s) = w_s\left(t - t_j, x\right), \quad s = 1, 2, \ldots, n \tag{1.6}$$

hold for $t_j \leq t \leq t_{j+1}$.

It means that the semi-Markov function $w(t, x, \xi(t))$ is a functional of a random process $\xi(t)$. The value of $w(t, x, \xi(t))$ is determined by the values $t, x, \xi(t)$ at the time t and also by the value of the jump of the process $\xi(t)$ at time t_j, which precedes time t. In fact, the system (1.1) means n different differential systems in the form

$$\frac{dX(t)}{dt} = F_s(t, X(t)), \quad s = 1, 2, \ldots, n, \tag{1.7}$$

where

$$F_s(t, x) \equiv F(t, x, \theta_s). \tag{1.8}$$

We assume that there exists a unique solution of (1.7) for every point (t, x) such that $t \geq 0$, $\|x\| < \infty$ ($\|\cdot\|$ stands for Euclidean norm), continuable on $[0, \infty)$.

1.3. Auxiliaries

In the paper, in addition to what was mentioned above, the following notations and assumptions are introduced:

(1) the functions $F_s(t, x)$, $s = 1, 2, \ldots, n$, are Lipschitz functions with the Lipschitz constants ρ_s, that is, the inequalities

$$\left\|F_s(t, x) - F_s(t, y)\right\| \leq \rho_s \left\|x - y\right\|, \quad s = 1, 2, \ldots, n \tag{1.9}$$

hold.

(2) If $x = 0$, then

$$F_s(t,0) \equiv 0, \quad s = 1,2,\ldots,n, \ t \geq 0. \tag{1.10}$$

(3) The inequalities

$$\|N_s(t,x)\| \leq \rho_s e^{-\alpha_s t}, \quad s = 1,\ldots,n, \ t \geq 0 \tag{1.11}$$

are valid. Here ρ_s, $s = 1,\ldots,n$ are the Lipschitz constants, α_s, $s = 1,\ldots,n$ are positive constants, and $N_s(t, X(0))$, $s = 1,\ldots,n$ is the solution $X(t)$ of (1.7) in the Cauchy form, that is,

$$X(t) = N_s(t, X(0)), \quad s = 1,2,\ldots,n. \tag{1.12}$$

(4) We introduce the Lyapunov functional

$$V = \int_0^{\infty} E(w(t,x,\xi(t)))dt, \tag{1.13}$$

where $E(\cdot)$ denotes mathematical expectation, and we assume that the integral is convergent.

Definition 1.2. The trivial solution of the differential systems (1.1) is said to be L_2-stable if, for any solution $X(t)$ with bounded initial values of the mathematical expectation

$$E(X(0)X^*(0)), \tag{1.14}$$

the integral

$$J = \int_0^{\infty} E\left(\|X(t)\|^2\right)dt \tag{1.15}$$

converges.

Remark 1.3. It is easy to see that (1.15) converges if and only if the matrix integral

$$\int_0^{\infty} E(X(t)X^*(t))dt \tag{1.16}$$

is convergent.

Lemma 1.4. *Let the function $w(t,x,\xi)$ be bounded, that is, there exists a constant β such that the inequalities*

$$0 \leq w(x) \leq w(t,x,\xi) \leq \beta\|x\|^2, \quad \text{for } \xi = \theta_s, \ s = 1,2,\ldots,n, \tag{1.17}$$

or the inequalities

$$0 \le w(x) \le w_s(t,x) \le \beta \|x\|^2, \quad s = 1,\dots,n \tag{1.18}$$

hold where $w(x)$ is a positive definite and differentiable function satisfying the inequality

$$E(w(X(t))) \le E(w(t,X(t),\xi(t))) \le \beta E\left(\|X(t)\|^2\right). \tag{1.19}$$

Let, moreover, the Lyapunov functional (1.13) *exist for the system* (1.7) *with an L_2-stable trivial solution.*

Then the Lyapunov functional (1.13) *can be expressed in the form*

$$V = \int_{E_n} \sum_{s=1}^{n} v_s(x) f_s(0,x)dx, \quad dx \equiv dx_1 \cdots dx_n \tag{1.20}$$

if the particular Lyapunov functions

$$v_s(x) = \int_0^\infty E(w(t,X(t),\xi(t)) \mid X(0) = x, \xi(0) = \theta_s)dt, \quad s = 1,\dots,n \tag{1.21}$$

are known.

Proof. The functions $v_s(x)$, $s = 1,\dots,n$, will be defined using auxiliary functions

$$u_s(t,x) = E(w(t,X(t),\xi(t)) \mid X(0) = x, \xi(0) = \theta_s), \quad s = 1,\dots,n. \tag{1.22}$$

The mathematical expectation in (1.22) can be calculated by the transition intensities $q_{ls}(t)$, $l,s = 1,\dots,n$, $t \ge 0$

$$\Psi_s(t) = \int_t^\infty q_s(\tau)d\tau, \quad q_s(t) \equiv \sum_{l=1}^{n} q_{ls}(t), \quad s = 1,\dots,n, \tag{1.23}$$

whence the system

$$u_s(t,x) = \Psi_s(t) w_s(t, X_s(t,x)) + \sum_{l=1}^{n} \int_0^t q_{ls}(\tau) u_l(t-\tau, N_s(\tau,x))d\tau, \quad s = 1,\dots,n \tag{1.24}$$

is obtained. Integrating the system of equations (1.24) with respect to t, we get the system of functional equations

$$\begin{aligned} v_s(x) = {} & \int_0^\infty \Psi_s(t) w_s(t, N_s(t,x))dt \\ & + \sum_{l=1}^{n} \int_0^\infty \left[\int_0^t q_{ls}(\tau) v_l(t-\tau, N_s(\tau,x))d\tau \right] dt, \quad s = 1,\dots,n, \end{aligned} \tag{1.25}$$

for

$$v_s(x) = \int_0^{\infty} u_s(t, x)dt, \quad s = 1, \ldots, n. \tag{1.26}$$

The system (1.25) thus obtained can be solved by successive approximations

$$\begin{aligned} &v_s^0(x) \equiv 0, \quad s = 1, \ldots, n, v_s^{(\alpha+1)}(x) \\ &\quad = \int_0^{\infty} \Psi_s(t) w_s(t, N_s(t, x))dt \\ &\qquad + \sum_{l=1}^{n} \int_0^{\infty} q_{ls}(t) v_l^{(\alpha)}(N_s(t, x))dt, \quad s = 1, \ldots, n, \alpha = 0, 1, 2, \ldots. \end{aligned} \tag{1.27}$$

□

2. Main Results

2.1. The Case of a Semi-Markovian Random Process $\xi(t)$

Theorem 2.1. *Let the functions* $F_s(t, x)$*,* $s = 1, 2, \ldots, n$*, in the system* (1.7) *satisfy conditions* (1.9)*,* (1.11)*, let the semi-Markov process* $\xi(t)$ *be determined by the transition intensities* $q_{ls}(t), l, s = 1, \ldots, n$*,* $t \geq 0$ *satisfying* (1.5)*, and let the functions* $w(t, x, \xi(t))$ *satisfy* (1.17)*. Then the following statements are true.*

(1) *The relationships*

$$v_s^{(\alpha)}(x) \leq C_s^{(\alpha)} \|x\|^2, \quad s = 1, \ldots, n, \ \alpha = 0, 1, \ldots, \tag{2.1}$$

$$\int_0^{\infty} \Psi_s(t) e^{-2\alpha_s t} dt < \infty, \quad \int_0^{\infty} q_s(t) e^{-2\alpha_s t} dt < \infty, \quad s = 1, \ldots, n, \tag{2.2}$$

imply that, for the system (1.7)*, the particular Lyapunov functions can be established in the form*

$$v_s(x) = \Psi_s(t) w_s(t, N_s(t, x)) + \sum_{l=1}^{n} \int_0^{t} q_{ls}(\tau) v_l(N_s(\tau, x))d\tau, \quad s = 1, \ldots, n, \tag{2.3}$$

(2) *If the spectral radius of the matrix* $\Gamma = (\gamma_{ls})_{l,s=1}^{n}$ *is less than one, then the particular Lyapunov functions* $v_s(x)$*,* $s = 1, \ldots, n$*, can be found by the method of successive approximations* (1.27)*.*

(3) *Under assumption* (1.11)*, the method of successive approximations* (1.27) *converges and the inequalities*

$$v_s^{(\alpha+1)} \geq v_s^{(\alpha)}(x), \quad v_s^{(\alpha)}(x) \leq v_s(x), \quad s = 1, \ldots, n \tag{2.4}$$

hold. Then the sequence of functions $v_s^{(\alpha)}(x)$*,* $s = 1, \ldots, n$*,* $\alpha = 0, 1, 2, \ldots$*, is monotone increasing and bounded from above by the functions* $v_s(x)$*,* $s = 1, \ldots, n$*.*

Proof. Applying estimation (2.1) and assumption (2.2) to the successive approximations (1.27), we get

$$C_s^{(\alpha+1)} \le \beta\rho_s^2 \int_0^\infty \Psi_s(t) e^{-2\alpha_s t} dt + \rho_s^2 \sum_{l=1}^n \int_0^\infty q_{ls}(t) e^{-2\alpha_s t} dt C_l^{(\alpha)}, \quad s = 1, \ldots, n, \ \alpha = 0, 1, 2, \ldots. \tag{2.5}$$

It is sufficient to assume the existence of a bounded solution of the system of inequalities (2.1) whence the existence follows of a positive solution of the system of linear algebraic equations (2.3). Moreover, assumption (2.2) guarantees the convergence of the improper integrals in the system (1.27) and so, for the existence of a positive solution of the system (2.3), it is sufficient that the spectral radius $\rho(\Gamma)$ of the matrix

$$\Gamma = (\gamma_{ls})_{l,s=1}^n \tag{2.6}$$

is less than one. For this, it is sufficient that

$$\sum_{l=1}^n \gamma_{ls} \equiv \rho_s^2 \int_0^\infty q_s(t) t^{-2\alpha_s t} dt < 1, \quad s = 1, \ldots, n. \tag{2.7}$$

The convergence of the sequence $v_s^{(\alpha)}(x)$, $s = 1, \ldots, n$, $\alpha = 0, 1, 2, \ldots$, can be determined by the system

$$v_s^{(\alpha+1)}(x) - v_s^{(\alpha)}(x) = \sum_{l=1}^n \int_0^\infty q_{ls}(t) \left[v_l^{(\alpha)}(N_s(t,x)) - v_l^{(\alpha-1)}(N_s(t,x)) \right] dt, \quad s = 1, \ldots, n, \ \alpha = 1, 2, 3, \ldots. \tag{2.8}$$

If there exist the inequalities

$$\left| v_s^{(\alpha)}(x) - v_s^{(\alpha-1)}(x) \right| \le \sum_{l=1}^n \int_0^\infty q_{ls}(t) \rho_s^2 e^{-2\alpha_s t} dt \, d_s^{(\alpha)} \|x\|^2, \quad s = 1, \ldots, n, \tag{2.9}$$

where

$$d_s^{(\alpha+1)} = \sum_{l=1}^n \gamma_{sl} d_l^{(\alpha)}, \quad s = 1, \ldots, n, \tag{2.10}$$

hold, then estimation (2.4) is true for all $\alpha = 2, 3, \ldots$, $d_s^{(1)} = C_s$, $s = 1, \ldots, n$.

Under assumptions (2.8), it follows

$$\lim_{\alpha \to +\infty} d_s^{(\alpha)} = 0, \quad s = 1, \ldots, n, \tag{2.11}$$

which implies a uniform convergence of the sequence $v_s^{(\alpha)}(x)$, $s = 1, \ldots, n$, $\alpha = 0, 1, 2, \ldots$. □

Corollary 2.2. *If the trivial solution of the differential systems* (1.1) *is L_2-stable, then there exist particular Lyapunov functions $v_s(x)$, $s = 1, \dots, n$ that satisfy* (2.3).

Corollary 2.3. *Let the function $w(t, x, \xi(t))$ satisfy the inequality:*

$$\beta_1 \|x\|^2 \leq w(t, x, \xi(t)) \leq \beta \|x\|^2, \quad \beta_1 > 0. \tag{2.12}$$

If there exist the Lyapunov functions $v_s(x)$, $s = 1, \dots, n$ for the system (1.21), *then the trivial solution of the differential systems* (1.1) *is L_2-stable.*

Corollary 2.4. *Let the semi-Markov process $\xi(t)$ in the system* (1.1) *have jumps at the times t_j, $j = 0, 1, 2, \dots$, $t_0 = 0$, in the transition from state θ_s to state θ_l, and let the jumps satisfy the equation*

$$X(t_j) = \Phi_{ls}(X(t_j - 0)), \quad \Phi_{ls}(0) = 0, \; j = 1, 2, \dots, \tag{2.13}$$

where $\Phi_{ls}(x)$ are any continuous Lipschitz vector functions. Then the system (2.3) *has the form*

$$\begin{aligned} v_s(x) = &\int_0^\infty \Psi_s(t) w_s(t, N_s(t, x)) dt \\ &+ \sum_{l=1}^n \int_0^\infty q_{ls}(t) v_s(\Phi_{ls}(N_s(t, x))) dt, \quad s = 1, \dots, n, \end{aligned} \tag{2.14}$$

and its solution can be found by the method of successive approximations.

2.2. The Case of a Markovian Random Process $\xi(t)$

Next result relates to the case of the semi-Markov process $\xi(t)$ being transformed into a Markov process described by the system of ordinary differential equations:

$$\frac{dP}{dt} = AP(t), \qquad A = (a_{ls})_{l,s=1}^n, \tag{2.15}$$

under the influence of which the considered system

$$\frac{dX(t)}{dt} = F(X(t), \xi(t)), \quad F(0, \xi(t)) \equiv 0, \tag{2.16}$$

takes the form

$$\frac{dX(t)}{dt} = F_s(X(t)), \quad s = 1, \dots, n. \tag{2.17}$$

We also assume that, if $t_j \leq t < t_{j+1}$, $\xi(t) = \theta_s$, then

$$w(t, x, \xi(t)) = w_s(x), \quad w_s(0) = 0, \quad s = 1, \dots, n. \tag{2.18}$$

Then the system of equations (2.14) has the form

$$v_s(x) = \int_0^\infty e^{a_{ss}t} w_s(N_s(t,x))dt + \sum_{\substack{l=1\\ l\neq s}}^{n} \int_0^\infty a_{ls} e^{a_{ss}t} v_l(N_s(t,x))dt, \quad s = 1,\dots,n. \tag{2.19}$$

Theorem 2.5. *Let the nonlinear differential system* (1.1), *depending on the Markov process* $\xi(t)$, *be described by* (2.15). *Then the particular Lyapunov functions* $v_s(x)$ *satisfy the linear differential system:*

$$\frac{Dv_s(x)}{Dx} F_s(x) + w_s(x) + \sum_{l=1}^{n} a_{ls} v_l(x) = 0, \quad s = 1,\dots,n. \tag{2.20}$$

Proof. Let us write the solution of the system (2.17) in the Cauchy form:

$$X(t) = N_s(t-\tau, X(\tau)), \quad s = 1,\dots,n. \tag{2.21}$$

Differentiating (2.21) with respect to τ, we get

$$-\frac{\partial N_s(t-\tau, X(\tau))}{\partial t} + \frac{DN_s(t-\tau, X(\tau))}{DX(\tau)} F_s(X(\tau)) = 0, \tag{2.22}$$

which, for $\tau = 0$, $X(0) = x$, takes the form:

$$\frac{DN_s(t,x)}{Dx} F_s(x) \equiv \frac{\partial N_s(t,x)}{\partial x}, \quad s = 1,\dots,n. \tag{2.23}$$

Then

$$\begin{aligned}
\frac{DN_s(t,x)}{Dx} F_s + a_{ss} v_s(x) &= -\int_0^\infty a_{ss} e^{a_{ss}t} (w_s(N_s(t,x))) + \sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls} e^{a_{ss}t} v_l(N_s(t,x))dt \\
&\quad + \int_0^\infty e^{a_{ss}t} \frac{\partial}{\partial t} (w_s(N_s(t,x))) + \sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls} v_l(N_s(t,x))dt \\
&= \int_0^\infty \frac{\partial}{\partial t} \left(e^{a_{ss}t} (w_s(N_s(t,x)))\right) \\
&\quad + \sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls} v_l(N_s(t,x))dt \\
&= -\sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls} v_l(x),
\end{aligned} \tag{2.24}$$

which implies (2.20) if

$$\lim_{t\to+\infty} e^{\alpha_{ss}t}(w_s(N_s(t,x))) + \sum_{\substack{l=1\\ l\neq s}}^{n} a_{ls}v_l(N_s(t,x)) = 0, \quad s = 1,\dots,n \tag{2.25}$$

and the functions $w_s(x)$, $v_s(x)$, $s = 1,\dots,n$ are differentiable. □

Corollary 2.6. *If the solutions of the system* (2.16) *have the same jumps as the solution of the system* (2.14) *and converge to the jumps of the Markov process* $\xi(t)$ *such that* $\Phi_{ls}(x) \equiv E$, $l = 1,\dots,n$, *then the system* (2.19) *takes the form*

$$\begin{aligned} v_s(x) = &\int_0^\infty e^{a_{ss}t} w_s(N_s(t,x))dt \\ &+ \sum_{\substack{l=1\\ l\neq s}}^{n} \int_0^\infty a_{ls}e^{a_{ss}t} v_l(\Phi_{ls}(N_s(t,x)))dt, \quad s = 1,\dots,n, \end{aligned} \tag{2.26}$$

and the system (2.20) *takes the form*

$$\frac{Dv_s(x)}{Dx} = F_s(x) + \sum_{l=1}^{n} a_{ls}v_l(\Phi_{ls}(x)) = -w_s(x), \quad s = 1,\dots,n. \tag{2.27}$$

Example 2.7. Let us investigate the stability of solutions of two-dimensional system

$$\frac{dX(t)}{dt} = (\nu - \lambda - \alpha)X(t) + G(X(t),\xi(t)), \quad \alpha + \lambda > \nu,\ \alpha > 0, \tag{2.28}$$

where $\xi(t)$ is a random Markov process having two states θ_1, θ_2 with probabilities $p_k = P\{\xi(t) = \theta_k\}$, $k = 1,2$, that satisfy the equations

$$\begin{aligned} \frac{dp_1(t)}{dt} &= -\lambda p_1(t) + \nu p_2(t), \\ \frac{dp_2(t)}{dt} &= \lambda p_1(t) - \nu p_2(t), \end{aligned} \tag{2.29}$$

where $\lambda > 0$. The random matrix function G is known:

$$\begin{aligned} G_1(x) = G(x,\theta_1) &= \begin{pmatrix} -\gamma_1 x_2 & -x_1^3 \\ \gamma_1 x_1 & -x_2^3 \end{pmatrix}, \\ G_2(x) = G(x,\theta_2) &= \begin{pmatrix} \gamma_2 x_2 & -x_1^3 \\ -\gamma_2 x_1 & -x_2^3 \end{pmatrix}. \end{aligned} \tag{2.30}$$

Taking the positive definite functions

$$w_1(x) = w_2(x) = x_1^2 + x_2^2 + \frac{1}{\alpha}\left(x_1^4 + x_2^4\right), \quad x = (x_1, x_2), \tag{2.31}$$

we can verify that the positive definite particular Lyapunov functions

$$v_1(x) = v_2(x) = \frac{1}{2\alpha}\left(x_1^2 + x_2^2\right) \tag{2.32}$$

are the solutions to (2.20). Consequently, since the integral (1.13)

$$v = \int_0^\infty \left\langle x_1^2 + x_2^2 + \frac{1}{\alpha}\left(x_1^4 + x_2^4\right)\right\rangle dt, \tag{2.33}$$

is convergent, the zero solution of the considered system is L_2-stable.

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