Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 918082, 17 pages
doi:10.1155/2012/918082

Research Article

Existence and Multiplicity of Positive Solutions of a Nonlinear Discrete Fourth-Order Boundary Value Problem

Ruyun Ma and Yanqiong Lu

Department of Mathematics, Northwest Normal University, Lanzhou 730070, China

Correspondence should be addressed to Ruyun Ma, ruyun_ma@126.com

Received 16 September 2011; Revised 14 December 2011; Accepted 25 December 2011

Academic Editor: Yuriy Rogovchenko

we show the existence and multiplicity of positive solutions of the nonlinear discrete fourth-order boundary value problem $\Delta^4 u(t-2) = \lambda h(t) f(u(t))$, $t \in \mathbb{T}_2$, $u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0$, where $\lambda > 0$, $h : \mathbb{T}_2 \to (0, \infty)$ is continuous, and $f : \mathbb{R} \to [0, \infty)$ is continuous, $T > 4$, $\mathbb{T}_2 = \{2, 3, \ldots, T\}$. The main tool is the Dancer's global bifurcation theorem.

1. Introduction

It's well known that the fourth order boundary value problem

$$\begin{gathered} u''''(t) = f(t, u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{1.1}$$

can describe the stationary states of the deflection of an elastic beam with both ends hinged, (it also models a rotating shaft). The existence and multiplicity of positive solutions of the boundary value problem (1.1) have been considered extensively in the literature, see [1–10]. The existence and multiplicity of positive solutions of the parameterized boundary value problem

$$\begin{gathered} u''''(t) = \lambda h(t) f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{gathered} \tag{1.2}$$

have also been studied by several authors, see Bai and Wang [11], Cid et al. [12], and the references therein.

However, relatively little is known about the corresponding discrete fourth-order problems. Let

$$T > 4, \quad \mathbb{T}_0 = \{0, 1, \ldots, T+2\}, \quad \mathbb{T}_1 = \{1, 2, \ldots, T+1\}, \quad \mathbb{T}_2 = \{2, 3, \ldots, T\}. \tag{1.3}$$

Zhang et al. [13], and He and Yu [14] used the fixed point index theory in cones to study the following discrete analogue

$$\Delta^4 u(t-2) = \lambda h(t) f(u(t)), \quad t \in \mathbb{T}_2, \tag{1.4}$$

$$u(0) = u(T+2) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \tag{1.5}$$

where $\Delta^4 u(t-2)$ denote the fourth forward difference operator and $\Delta u(t) = u(t+1) - u(t)$. It has been pointed out in [13, 14] that (1.4), (1.5) are equivalent to the equation of the form:

$$u(t) = \lambda \sum_{s=1}^{T+1} G(t,s) \sum_{j=2}^{T} G_1(s,j) h(j) f(u(j)) =: A_0 u(t), \quad t \in \mathbb{T}_0, \tag{1.6}$$

where

$$G(t,s) = \frac{1}{T+2} \begin{cases} s(T+2-t), & 1 \le s \le t \le T+2, \\ t(T+2-s), & 0 \le t \le s \le T+1, \end{cases}$$
$$G_1(s,j) = \frac{1}{T} \begin{cases} (T+1-s)(j-1), & 2 \le j \le s \le T+1, \\ (T+1-j)(s-1), & 1 \le s \le j \le T. \end{cases} \tag{1.7}$$

Notice that two distinct Green's functions used in (1.6) make the construction of cones and the verification of strong positivity of A_0 become more complex and difficult. Therefore, Ma and Xu [15] considered (1.4) with the boundary condition

$$u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \tag{1.8}$$

and introduced the definition of *generalized positive solutions*:

Definition 1.1. A function $y : \mathbb{T}_0 \to \mathbb{R}^+$ is called a *generalized positive solution* of (1.4), (1.8), if y satisfies (1.4), (1.8), and $y(t) \ge 0$ on $\mathbb{T}_1$ and $y(t) > 0$ on $\mathbb{T}_2$.

Remark 1.2. Notice that the fact $y : \mathbb{T}_0 \to \mathbb{R}^+$ is a generalized positive solution of (1.4), (1.8) does not means that $y(t) \ge 0$ on $\mathbb{T}_0$. In fact, y satisfies

(1) $y(t) \ge 0$ for $t \in \mathbb{T}_2$;

(2) $y(1) = y(T+1) = 0$;

(3) $y(0) = -y(2), \quad y(T+2) = -y(T)$.

Ma and Xu [15] also applied the fixed point theorem in cones to obtain some results on the existence of generalized positive solutions.

It is the purpose of this paper to show some new results on the existence and multiplicity of generalized positive solutions of (1.4), (1.8) by Dancer's global bifurcation theorem. To wit, we get the following.

Theorem 1.3. *Let* $h : \mathbb{T}_2 \to (0,\infty)$, $f \in C(\mathbb{R}, [0,\infty))$, *and*

$$\lim_{s\to 0^+} \frac{f(s)}{s} = f_0 \in (0,\infty), \quad \lim_{s\to\infty} \frac{f(s)}{s} = f_\infty = +\infty. \tag{1.9}$$

Assume that there exists $B \in [0,+\infty]$ *such that* f *is nondecreasing on* $[0,B)$. *Then*

(i) (1.4), (1.8) *have at least one generalized positive solution if* $0 < \lambda < \lambda_1/f_0$;

(ii) (1.4), (1.8) *have at least two generalized positive solutions if*

$$\frac{\lambda_1}{f_0} < \lambda < \sup_{s\in(0,B)} \frac{s}{\gamma^* f(s)}, \tag{1.10}$$

where $\gamma^* = \max_{t\in\mathbb{T}_1} \sum_{s=2}^{T} K(t,s)h(s)$, $K(t,s)$ *is defined as* (2.3) *and* λ_1 *is the first eigenvalue of*

$$\begin{gathered} \Delta^4 u(t-2) = \lambda h(t)u(t), \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0. \end{gathered} \tag{1.11}$$

The "dual" of Theorem 1.3 is as follows.

Theorem 1.4. *Let* $h : \mathbb{T}_2 \to (0,\infty)$, $f \in C(\mathbb{R}, [0,\infty))$, *and*

$$\lim_{s\to 0^+} \frac{f(s)}{s} = f_0 \in (0,\infty), \quad \lim_{s\to\infty} \frac{f(s)}{s} = f_\infty = 0. \tag{1.12}$$

Assume that there exists $B \in [0,+\infty]$ *such that* f *is nondecreasing on* $[0,B)$. *Then*

(i) (1.4), (1.8) *have at least a generalized positive solution provided*

$$\lambda > \inf_{s\in(0,c_1 B)} \frac{s}{c_1 \gamma_* f(s)}, \tag{1.13}$$

where $\gamma_* = \min_{t\in\mathbb{T}_2} \sum_{s=2}^{T} K(t,s)h(s)$;

(ii) (1.4), (1.8) *have at least two generalized positive solutions provided*

$$\inf_{s\in(0,c_1B)} \frac{s}{c_1\gamma_* f(s)} < \lambda < \frac{\lambda_1}{f_0}. \tag{1.14}$$

The rest of the paper is organized as follows: in Section 2, we present the form of the Green's function of (1.4), (1.8) and its properties, and we enunciate the Dancer's global bifurcation theorem ([16, Corollary 15.2]). In Section 3, we use the Dancer's bifurcation theorem to prove Theorems 1.3 and 1.4 and in Section 4, we finish the paper presenting a couple of illustrative examples.

Remark 1.5. For other results on the existence and multiplicity of positive solutions and nodal solutions for fourth-order boundary value problems based on bifurcation techniques, see [17–21].

2. Preliminaries and Dancer's Global Bifurcation Theorem

Lemma 2.1. *Let $h : \mathbb{T}_2 \to \mathbb{R}$. Then the linear boundary value problem*

$$\begin{gathered}\Delta^4 u(t-2) = h(t), \quad t \in \mathbb{T}_2, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0\end{gathered} \tag{2.1}$$

has a solution

$$u(t) = \sum_{s=2}^{T} K(t,s)h(s), \quad t \in \mathbb{T}_1, \tag{2.2}$$

where

$$K(t,s) = \begin{cases} \dfrac{(s-1)(T+1-t)\left(2T(t-1)-(t-1)^2-(s-2)s\right)}{6T}, & 2 \le s \le t \le T+1, \\ \dfrac{(t-1)(T+1-t)\left(2T(s-1) \ -(s-1)^2-(t-2)t\right)}{6T}, & 1 \le t \le s \le T. \end{cases} \tag{2.3}$$

Proof. By a simple summing computation and $u(1) = \Delta^2 u(0) = 0$, we can obtain

$$u(t) = \Delta u(0)(t-1) + \frac{t(t-1)(t-2)}{6}\Delta^3 u(0) + \sum_{s=2}^{t-1} \frac{(t-s)(t-s-1)(t-s+1)}{6} h(s). \tag{2.4}$$

This together with $u(T+1) = \Delta^2 u(T) = 0$, it follows that

$$\begin{aligned} u(t) &= \sum_{s=2}^{T} \frac{(T+1-s)(t-1)\left[2T(s-1)-(s-1)^2-t(t-2)\right]}{6T} h(s) \\ &\quad + \sum_{s=2}^{t-1} \frac{(t-s)(t-s-1)(t-s+1)}{6T} h(s) \\ &= \sum_{s=t}^{T} \frac{(T+1-s)(t-1)\left[2T(s-1)-(s-1)^2-t(t-2)\right]}{6T} h(s) \\ &\quad + \sum_{s=2}^{t-1} \frac{(T+1-t)(s-1)\left[2T(t-1)-(t-1)^2-s(s-2)\right]}{6T} h(s). \end{aligned} \tag{2.5}$$

Therefore, (2.2) holds. □

Remark 2.2. It has been pointed out in [15] that (2.1) is equivalent to the summation equation of the form

$$u(t) = \sum_{s=2}^{T} G_1(t,s) \sum_{j=2}^{T} G_1(s,j) h(j), \quad t \in \mathbb{T}_1. \tag{2.6}$$

It is easy to verify that (2.2) and (2.6) are equivalent.

By a similar method in [9], it follows that $K(t,s)$ satisfies

$$\begin{aligned} &K(t,s) \le \Phi(s) \quad \text{for } s \in \mathbb{T}_1, \quad t \in \mathbb{T}_1, \\ &K(t,s) \ge c(t)\Phi(s) \quad \text{for } s \in \mathbb{T}_1, \quad t \in \mathbb{T}_1, \end{aligned} \tag{2.7}$$

where

$$\Phi(s) = \begin{cases} \dfrac{\sqrt{3}}{27T}(s-1)\left(T^2-(s-2)s\right)^{3/2}, & 1 \le s \le \dfrac{T}{2}+1, \\ \dfrac{\sqrt{3}}{27T}(T+1-s)\left(2T(s-1)-(s-2)s\right)^{3/2}, & \dfrac{T}{2}+1 < s \le T+1, \end{cases} \tag{2.8}$$

$$c(t) = \begin{cases} \dfrac{3\sqrt{3}\left[T^2-t(t-2)\right](t-1)}{2\left(T^2+1\right)^{3/2}}, & 1 \le t \le \dfrac{T}{2}+1, \\ \dfrac{3\sqrt{3}(T+1-t)[2T(t-1)-t(t-2)]}{2\left(T^2+1\right)^{3/2}}, & \dfrac{T}{2}+1 < t \le T+1. \end{cases} \tag{2.9}$$

Moreover, we have that

$$K(t,s) \geq c_1 \Phi(s), \quad \text{for } s \in \mathbb{T}_1, \ t \in \mathbb{T}_2. \tag{2.10}$$

here $c_1 = 3\sqrt{3}T^2/2(T^2+1)^{3/2}$.

Let X be a real Banach space with a cone K such that $X = K - K$. Let us consider the equation:

$$x = \mu(Lx + Nx), \quad \mu \in \mathbb{R}, \quad x \in X \tag{2.11}$$

under the assumptions:

(A1) The operators $L, N : X \to X$ are compact. Furthermore, L is linear, $\|Nx\|_X / \|x\|_X \to 0$ as $\|x\|_X \to 0$, and $(L+N)(K) \subseteq K$.

(A2) The spectral radius $r(L)$ of L is positive. Denote $\mu_0 = r(L)^{-1}$.

(A3) L is strongly positive.

Dancer's global bifurcation theorem is the following.

Theorem 2.3 (see [16, Corollary 15.2]). *Let*

$$S_+ := \{(\mu, x) \in \mathbb{R} \times X \mid (\mu, x) \ \text{is a solution of } (2.11) \text{ with } x > 0 \text{ and } \mu > 0\}. \tag{2.12}$$

If (A1) and (A2) are satisfied, then $(\mu_0, 0)$ is a bifurcation point of (2.11) *and $\overline{S_+}$ has an unbounded solution component C_+ which passes through $(\mu_0, 0)$. Additionally, if (A3) is satisfied, then $(\mu, x) \in C_+$ and $\mu \neq \mu_0$ always implies $x > 0$ and $\mu > 0$.*

3. Proof of the Main Results

Before proving Theorem 1.3, we state some preliminary results and notations. Let

$$\rho := 4 \sin^2 \frac{\pi}{2T}, \qquad e(t) := \sin \frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_1, \tag{3.1}$$

$$X := \left\{ u \mid u : \mathbb{T}_0 \longrightarrow \mathbb{R}, \ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0 \right\}. \tag{3.2}$$

Then X is a Banach space under the normal:

$$\|u\|_X := \inf \left\{ \frac{\gamma}{\rho} \mid -\gamma e(t) \leq -\Delta^2 u(t-1) \leq \gamma e(t), \quad t \in \mathbb{T}_1 \right\}. \tag{3.3}$$

See [22] for the detail.

Let

$$K := \left\{ u \in X \mid \Delta^2 u(t-1) \leq 0, \ u(t) \geq 0, \ t \in \mathbb{T}_1 \right\}. \tag{3.4}$$

Then K is normal and has a nonempty interior and $X = K - K$.

Let $Y = \{u \mid u : \mathbb{T}_2 \to \mathbb{R}\}$. Then Y is a Banach space under the norm:

$$\|u\|_\infty = \max_{t\in\mathbb{T}_2}|u(t)|. \tag{3.5}$$

Define $\mathcal{L} : X \to Y$ by setting

$$\mathcal{L}u := \Delta^4 u(t-2), \quad u \in X. \tag{3.6}$$

It is easy to check that $\mathcal{L}^{-1} : Y \to X$ is compact.

Lemma 3.1. *Let $h \in Y$ with $h \geq 0$ and $h(t_0) > 0$ for some $t_0 \in \mathbb{T}_2$, and*

$$\mathcal{L}u - h = 0. \tag{3.7}$$

Then $u \in \operatorname{int} K$.

Proof. It is enough to show that there exist two constants $r_1,\ r_2 \in (0,\infty)$ such that

$$r_1 e(t) \leq -\Delta^2 u(t-1) \leq r_2 e(t), \quad t \in \mathbb{T}_1. \tag{3.8}$$

In fact, we have from (3.7) that

$$-\Delta^2 u(t-1) = \sum_{s=2}^{T} G_1(t,s)h(s), \quad t \in \mathbb{T}_1. \tag{3.9}$$

This together with the relation $((t-1)(T+1-t)/T)G_1(s,s) \leq G_1(t,s) \leq (t-1)(T+1-t)/T$ implies that

$$\left[\sum_{s=2}^{T} G_1(s,s)h(s)\right]\frac{(t-1)(T+1-t)}{T} \leq \sum_{s=2}^{T} G_1(t,s)h(s) \leq \|h\|_\infty \frac{(t-1)(T+1-t)}{T}. \tag{3.10}$$

Combining (3.9) with (3.10) and the fact that

$$c_1 \sin\frac{\pi(t-1)}{T} \leq \frac{(t-1)(T+1-t)}{T} \leq c_2 \sin\frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_1 \tag{3.11}$$

for some constants $c_1, c_2 \in (0,\infty)$, we conclude that (3.8) is true. □

Let $\zeta \in C(\mathbb{R},\mathbb{R})$ be such that

$$f(u) = f_0 u + \zeta(u), \tag{3.12}$$

clearly

$$\lim_{|u|\to 0}\frac{\zeta(u)}{u}=0. \tag{3.13}$$

Let us consider

$$\mathcal{L}u=\lambda\big(h(\cdot)f_0u+h(\cdot)\zeta(u)\big) \tag{3.14}$$

as a bifurcation problem from the trivial solution $u\equiv 0$.

By (1.4), (1.8), it follows that if $u(t)\in X$ is one solution of (1.4), (1.8), then $u(t)$ satisfies $u(0)=-u(2)$, $\quad u(T+2)=-u(T)$. So, $(u(0),0,u(2),\dots,u(T),0,u(T+2))$ is a solution of (1.4), (1.8), if and only if, $(0,u(2),\dots,u(T),0)$ solves the operator equation

$$u(t)=\lambda\sum_{s=2}^{T}G(t,s)h(s)f(u(s)),\quad t\in\mathbb{T}_1. \tag{3.15}$$

Now, let $J:Y\to X$ be the linear operator:

$$J(u(2),u(3),\dots,u(T))=(-u(2),0,u(2),u(3),\dots,u(T),0,-u(T)),\quad u\in Y. \tag{3.16}$$

Let $L,N:X\to X$ be the operators:

$$Lu:=(J\circ\mathcal{L})^{-1}\big(h(\cdot)f_0u\big), \tag{3.17}$$

$$Nu:=(J\circ\mathcal{L})^{-1}(h(\cdot)\zeta(u)), \tag{3.18}$$

respectively. Then Lemma 3.1 yields that $L:X\to X$ is strongly positive. Moreover, [16, Theorem 7.c] implies $r(L)>0$.

Now, it follows from Theorem 2.3 that there exists a continuum

$$\mathcal{C}_+\subseteq\overline{\big\{\big(\mu,x\big)\in\mathbb{R}\times X\mid\big(\mu,x\big)\text{ is a solution of (1.4), (1.8) with }x>0\text{ and }\mu>0\big\}}, \tag{3.19}$$

which joins $(r(L)^{-1},0)$ with infinity in $(0,\infty)\times K$ and

$$\big(\mu,x\big)\in\mathcal{C}_+,\quad \mu\neq r(L)^{-1}\implies x>0,\quad \mu>0. \tag{3.20}$$

It is easy to check that

$$r(L)^{-1}=\frac{\lambda_1}{f_0}. \tag{3.21}$$

Lemma 3.2. *Let $h_1, h_2 \in Y$ with $h_1 \geq h_2 > 0$. Then the eigenvalue problems*

$$\mathcal{L}u(t) = \lambda h_i(t)u(t), \quad t \in \mathbb{T}_2, \quad i = 1,\ 2 \tag{3.22}$$

have the principal eigenvalue $\overline{\lambda}_i$, $i = 1,\ 2$ such that $\overline{\lambda}_1 \leq \overline{\lambda}_2$. Moreover, the corresponding eigenfunctions φ_i are positive in $\mathbb{T}_2$.

Proof. Let $L_i : X \to X$ be the operator

$$L_i u := \lambda (J \circ \mathcal{L})^{-1}(h_i(\cdot)u), \quad i = 1, 2. \tag{3.23}$$

Then Lemma 3.1 yields that $L_i : X \to X$ is strongly positive. By Krein-Rutman theorem [16, Theorem 7.c] the spectral radius $r(L_i) > 0$ and there exist $\varphi_i \in X$ with $\varphi_i > 0$ on $\mathbb{T}_2$ such that

$$L_i \varphi_i(t) = r(L_i)\varphi_i(t), \quad i = 1,\ 2. \tag{3.24}$$

That is, the eigenvalue problems (3.22) have the principal eigenvalues $\overline{\lambda}_i = 1/r(L_i)$, and $\varphi_i(t)$ is the corresponding eigenfunctions of $\overline{\lambda}_i$, $i = 1,\ 2$.

Next, we prove $\overline{\lambda}_1 \leq \overline{\lambda}_2$. Since $\sum_{t=2}^{T} \Delta^4 \varphi_1(t-2)\ \varphi_2(t) = \sum_{t=2}^{T} \varphi_1(t)\Delta^4\varphi_2(t-2)$, it follows that

$$\begin{aligned}
\sum_{t=2}^{T} h_1(t)\varphi_1(t)\varphi_2(t) &\geq \sum_{t=2}^{T} \frac{\overline{\lambda}_2}{\overline{\lambda}_2} h_2(t)\varphi_2(t)\varphi_1(t) = \sum_{t=2}^{T} \frac{1}{\overline{\lambda}_2} \Delta^4 \varphi_2(t-2)\varphi_1(t) \\
&= \sum_{t=2}^{T} \frac{1}{\overline{\lambda}_2} \varphi_2(t)\Delta^4\varphi_1(t-2) = \sum_{t=2}^{T} \frac{\varphi_2(t)}{\overline{\lambda}_2} \overline{\lambda}_1 h_1(t)\varphi_1(t) \\
&= \frac{\overline{\lambda}_1}{\overline{\lambda}_2} \sum_{t=2}^{T} h_1(t)\varphi_1(t)\varphi_2(t).
\end{aligned} \tag{3.25}$$

Therefore, $\overline{\lambda}_1 \leq \overline{\lambda}_2$. □

Suppose that $\mathbb{T}_a = \{a+1, a+2, \ldots, b-1\}$ is a strict subset of $\mathbb{T}_2$ and h_a denote the restriction of h on $\mathbb{T}_a$. Consider the linear eigenvalue problems:

$$\begin{gathered}
\Delta^4 u(t-2) = \lambda h_a(t) f_0 u(t), \quad t \in \mathbb{T}_a, \\
u(a) = u(b) = \Delta^2 u(a-1) = \Delta^2 u(b-1) = 0.
\end{gathered} \tag{3.26}$$

Then we get the following result.

Lemma 3.3. *Let $\widetilde{\lambda}_1$ is the principal eigenvalue of* (3.17), *then* (3.26) *has only one principal eigenvalue λ_a such that $0 < \widetilde{\lambda}_1 < \lambda_a$.*

Proof. It is not difficult to prove that (3.26) has only one principal eigenvalue $\lambda_a > 0$ by Lemma 3.1, and the corresponding eigenfunction $\psi_a > 0$ on $\mathbb{T}_a$. So we only to verify that $0 < \widetilde{\lambda}_1 < \lambda_a$.

Let ψ_1 be the corresponding eigenfunction of $\widetilde{\lambda}_1$, we have that

$$\begin{aligned}\sum_{t=a+1}^{b-1} \Delta^4\psi_a(t-2)\psi_1(t) &= \sum_{t=a+1}^{b-1} \Delta^4\psi_1(t-2)\psi_a(t) - \psi_1(b)\Delta^2\psi_a(b-2) - \psi_a(b-1)\Delta^2\psi_1(b-1) \\ &\quad - \Delta^2\psi_a(a)\psi_1(a) - \psi_a(a+1)\Delta^2\psi_1(a-1) \\ &> \sum_{t=a+1}^{b-1} \Delta^4\psi_1(t-2)\psi_a(t).\end{aligned} \tag{3.27}$$

So

$$\begin{aligned}\sum_{t=a+1}^{b-1} h(t)\psi_a(t)\psi_1(t) &= \sum_{t=a+1}^{b-1} \frac{1}{\lambda_a}\Delta^4\psi_a(t-2)\psi_1(t) \\ &> \sum_{t=a+1}^{b-1} \frac{1}{\lambda_a}\Delta^4\psi_1(t-2)\psi_a(t) \\ &= \sum_{t=a+1}^{b-1} \frac{\psi_a(t)}{\lambda_a}\widetilde{\lambda}_1 h(t)\psi_1(t) \\ &= \frac{\widetilde{\lambda}_1}{\lambda_a}\sum_{t=a+1}^{b-1} h(t)\ \psi_a(t)\psi_1(t).\end{aligned} \tag{3.28}$$

Thus $0 < \widetilde{\lambda}_1 < \lambda_a$. □

Proof of Theorem 1.3. We divide the proof into three steps.

Let $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ be such that

$$|\mu_n| + \|y_n\|_X \to \infty, \quad n \to \infty. \tag{3.29}$$

Then

$$\begin{gathered}\Delta^4 y_n(t-2) = \mu_n h(t) f(y_n(t)), \quad t \in \mathbb{T}_2, \\ y_n(1) = y_n(T+1) = \Delta^2 y_n(0) = \Delta^2 y_n(T) = 0.\end{gathered} \tag{3.30}$$

Step 1. We show that there exists a constant M such that $\mu_n \in (0, M]$ for all n.

Suppose on the contrary that

$$\lim_{n\to\infty} \mu_n = \infty. \tag{3.31}$$

Let $v_n = y_n/\|y_n\|_X$. Then it follows from (3.30) that

$$\Delta^4 v_n(t-2) = \mu_n h(t)\frac{f(y_n(t))}{y_n(t)}v_n(t), \quad t \in \mathbb{T}_2,$$
$$v_n(1) = v_n(T+1) = \Delta^2 v_n(0) = \Delta^2 v_n(T) = 0. \tag{3.32}$$

Since

$$\inf\left\{\frac{f(s)}{s} \mid s > 0\right\} := M_0 > 0, \tag{3.33}$$

there exists a constant $M_0 > 0$, such that

$$\frac{f(y_n(t))}{y_n(t)} > M_0 > 0. \tag{3.34}$$

Let λ^* be the principal eigenvalue of the linear eigenvalue problems:

$$\Delta^4 v(t-2) = \lambda h(t)\ M_0 v(t), \quad t \in \mathbb{T}_2,$$
$$v(1) = v(T+1) = \Delta^2 v(0) = \Delta^2 v(T) = 0. \tag{3.35}$$

Combining (3.31) and (3.34) with the relation (3.32), using Lemma 3.2, we get

$$0 < \mu_n \le \lambda^*. \tag{3.36}$$

This contradicts (3.31). So $\mu_n \in (0, M]$ for all n.

Step 2. We show that $\mathcal{C}_+$ joins $(\lambda_1/f_0, 0)$ with $(0, \infty)$.

Assume that there exist $\delta > 0$ and $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ such that

$$0 < \delta \le \mu_n \le M; \quad \|y_n\|_X \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{3.37}$$

First, we show that

$$\|y_n\|_X \longrightarrow \infty \implies \|y_n\|_\infty \longrightarrow \infty. \tag{3.38}$$

Suppose on the contrary that

$$\|y_n\|_\infty \le M_1 \tag{3.39}$$

for some $M_1 > 0$ (independent on n). Then it follows from (3.30) and $0 < \delta \le |\mu_n| \le M$ that

$$\left\|\Delta^4 y_n\right\|_\infty \le M\|h\|_\infty \sup\{f(s) \mid 0 < s \le M_1\}, \tag{3.40}$$

and subsequently, $\{\|y_n\|_X\}$ is bounded. This is a contradiction. So, (3.38) holds.

Next, we show that

$$\|y_n\|_\infty \longrightarrow \infty \;\Rightarrow\; \min\{y_n(t) \mid t \in \mathbb{T}_2\} \longrightarrow \infty. \tag{3.41}$$

In fact,

$$y_n(t) = \mu_n \sum_{s=2}^{T} K(t,s)h(s)f(y_n(s)), \quad t \in \mathbb{T}_1. \tag{3.42}$$

This together with (2.7) imply that (3.41) is valid.

Finally, we have from the facts that $\min\{y_n(t) \mid t \in \mathbb{T}_2\} \rightarrow \infty$ and $0 < \delta \le |\mu_n| \le M$ that

$$\mu_n \frac{f(y_n(t))}{y_n(t)} \longrightarrow \infty, \qquad n \longrightarrow \infty \text{ for any } t \in \mathbb{T}_a. \tag{3.43}$$

Consider the following linear eigenvalue problems:

$$\begin{gathered} \Delta^4 v(t-2) = \lambda h_a(t)\, v(t), \qquad t \in \mathbb{T}_a, \\ v(a) = v(b) = \Delta^2 v(a-1) = \Delta^2 v(b-1) = 0. \end{gathered} \tag{3.44}$$

By Lemma 3.3 and (3.32), (3.44) has a positive principal eigenvalue λ_a, and

$$\mu_n \frac{f(y_n(t))}{y_n(t)} \le \lambda_a, \tag{3.45}$$

which contradicts (3.43). Thus $\lim_{n\to\infty}\mu_n = 0$.

Step 3. Fixed λ such that

$$0 < \lambda < \sup_{s\in(0,B)} \frac{s}{\gamma^* f(s)}. \tag{3.46}$$

Then there exists $b \in (0, B]$ such that

$$0 < \lambda < \frac{b}{\gamma^* f(b)}. \tag{3.47}$$

We show that there is no $(\mu, u) \in \mathcal{C}_+$ such that

$$\|u\|_\infty = b, \quad 0 < \mu < \frac{b}{\gamma^* f(b)}. \tag{3.48}$$

In fact, if there exists $(\eta, y) \in \mathcal{C}_+$ satisfying (3.48), then

$$y(t) = \eta \sum_{s=2}^{T} K(t,s)h(s)f(y(s)) \\ \leq \eta\gamma^* f(b) \\ = \eta\gamma^* \frac{f(b)}{b} \cdot b \tag{3.49}$$

for $t \in \mathbb{T}_1$, and subsequently, $\eta \geq b/\gamma^* \ f(b)$. Therefore, no $(\mu, u) \in \mathcal{C}_+$ satisfies (3.48).

Now, combining the conclusions in Steps 2 and 3, using the fact that no $(\mu, u) \in \mathcal{C}_+$ satisfies (3.48), it concludes that for every $\lambda \in (\lambda_1/f_0, b/\gamma^* f(b))$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. For arbitrary $\lambda \in (0, \sup_{s\in(0,B)}(s/\gamma^* f(s)))$, we may find $b = b(\lambda)$ satisfying (3.47). So, for every $\lambda \in (\lambda_1/f_0, \sup_{s\in(0,B)}(s/\gamma^* f(s)))$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. □

Proof of Theorem 1.4. We divide the proof into three steps.

Step 1. We show that there exists a positive constant $\beta > 0$ such that

$$\inf \{\mu \mid (\mu, u) \in \mathcal{C}_+\} =: \beta > 0. \tag{3.50}$$

Suppose on the contrary that there exists $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ such that

$$\mu_n \to 0^+, \quad \text{as} \quad n \to \infty. \tag{3.51}$$

Then we have from (3.32), (3.51), $f_0 \in (0, \infty)$ and $f_\infty = 0$ that

$$\|v_n\|_X \to 0, \quad \text{as} \quad n \to \infty. \tag{3.52}$$

However, this contradicts with the fact that $\|v_n\|_X = 1$ for all $n \in \mathbb{N}$. Therefore, (3.50) holds.

Step 2. We show that for any closed interval $I \subset [\beta, \infty)$, there exists $M_I > 0$ such that

$$\sup\{\|u\| \mid (\mu, u) \in \mathcal{C}_+\} \leq M_I. \tag{3.53}$$

Suppose on the contrary that there exists $\{(\mu_n, y_n)\} \subset \mathcal{C}_+$ with

$$\{\mu_n\} \subset I, \quad \|y_n\|_X \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty. \tag{3.54}$$

Then by (3.38),

$$\|y_n\|_\infty \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{3.55}$$

and subsequently

$$\min_{t\in\mathbb{T}_2} y_n(t) \geq c_1 \|y_n\|_\infty \longrightarrow \infty. \tag{3.56}$$

This together with (3.32) and $f_0 \in (0,\infty)$ and $f_\infty = 0$ that

$$\|(v_n|_{\mathbb{T}_2})\|_\infty \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.57}$$

However, this contradicts with the fact that

$$\min_{t\in\mathbb{T}_2} v_n(t) \geq c_1, \quad n \in \mathbb{N}. \tag{3.58}$$

Therefore, (3.53) holds.

Step 3. Fixed λ such that

$$\lambda > \inf_{s\in(0,c_1B)} \frac{s}{c_1\gamma_* f(s)}. \tag{3.59}$$

Then there exists $l \in (0, c_1B)$ such that

$$\lambda > \frac{l}{\gamma_* c_1 f(l)}. \tag{3.60}$$

We show that there is no $(\eta, y) \in \mathcal{C}_+$ such that

$$\|y\|_\infty = \frac{l}{c_1} \quad \eta > \frac{l}{\gamma_* c_1 f(l)}. \tag{3.61}$$

Suppose on the contrary that there exists $(\eta, y) \in C_+$ satisfying (3.61). Then for $t \in \mathbb{T}_2$,

$$\begin{aligned} y(t) &= \eta \sum_{s=2}^{T} K(t,s)h(s)f(y(s)) \\ &\geq \eta \sum_{s=2}^{T} K(t,s)h(s)f(c_1\|y\|_\infty) \\ &= \eta \sum_{s=2}^{T} K(t,s)h(s)f(l) \\ &\geq \eta\gamma_* f(l) \ = \eta\gamma_* \frac{f(l)}{l} \cdot l, \end{aligned} \tag{3.62}$$

and subsequently, $\eta \leq l/c_1\gamma_* f(l)$. Therefore, there is no $(\eta, y) \in \mathcal{C}_+$ such that (3.61) holds.

Now, combining the conclusions in Steps 2 and 3, using the fact that no $(\mu, u) \in \mathcal{C}_+$ satisfying (3.61), it concludes that for every $\lambda \in (l/c_1\gamma_* f(l), \lambda_1/f_0)$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. For arbitrary $\lambda \in (\inf_{s\in(0,c_1B)}(s/c_1\gamma_* f(s)),\ \infty)$, we may find $l = l(\lambda)$ satisfying (3.60). So, for every $\lambda \in (\inf_{s\in(0,c_1B)}(s/\gamma_* f(s)),\ \lambda_1/f_0)$, (1.4), (1.8) has at least two generalized positive solutions in $\mathcal{C}_+$. □

4. Some Examples

In this section, we will apply our results to two examples.

For convenience, set $T = 12$, then $\mathbb{T}_1 = \{1, 2, \ldots, 13\}$, $\mathbb{T}_2 = \{2, 3, \ldots, 12\}$.

Example 4.1. Let us consider the boundary value problem

$$\begin{gathered} \Delta^4 u(t-2) = \lambda f(u(t)), \quad t \in \mathbb{T}_2, \\ u(1) = u(13) = \Delta^2 u(0) = \Delta^2 u(12) = 0, \end{gathered} \tag{4.1}$$

where

$$f(u) = \begin{cases} \arctan u, & u \in (0, 1000], \\ (u-1000)^2 + \arctan 1000, & u \in (1000, \infty). \end{cases} \tag{4.2}$$

Clearly, $f(u)$ is nondecreasing, $f_0 = 1$, $f_\infty = \infty$. Take $B = 1000$. By a simple computation, it follows that $\inf_{s\in(0,1000)}(f(s)/s) = \arctan 1000/1000 \approx 0.00157$, $\lambda_1 = 16\sin^4(\pi/24) \approx 0.0048$ and $\gamma^* = \max_{t\in\mathbb{T}_1}\sum_{s=2}^{12} K(t,s) = 1629/6$, then

$$0.0048 \approx 16\sin^4\frac{\pi}{24} = \frac{\lambda_1}{f_0} < \sup_{s\in(0,1000)}\frac{s}{f(s)\gamma^*} = \frac{6}{1629\inf_{s\in(0,1000)}(f(s)/s)} \approx 2.34631. \tag{4.3}$$

So, Theorem 1.3(i) implies that (4.1) has at least one generalized positive solution for

$$0 < \lambda < \frac{\lambda_1}{f_0} \approx 0.0048; \tag{4.4}$$

Theorem 1.3(ii) implies that (4.1) has at least two generalized positive solutions for

$$\frac{\lambda_1}{f_0} < \lambda < \frac{1}{\gamma^*\left(16\sin^4(\pi/24) - 1\right)} \approx 2.34631. \tag{4.5}$$

Example 4.2. Let us consider the boundary value problem:

$$\begin{gathered} \Delta^4 u(t-2) = \lambda \widetilde{f}(u(t)), \quad t \in \mathbb{T}_2, \\ u(1) = u(13) = \Delta^2 u(0) = \Delta^2 u(12) = 0, \end{gathered} \tag{4.6}$$

where

$$\tilde{f}(u) = \begin{cases} \dfrac{e^u - 1}{u}, & u \in (0, 30], \\ \sqrt{u - 30} + \dfrac{e^{30} - 1}{30}, & u \in (30, \infty). \end{cases} \tag{4.7}$$

Obviously, $\tilde{f}(u)$ is nondecreasing in $[0, \infty)$, so $\tilde{f}_0 = \lim_{u \to 0}(\tilde{f}(u)/u) = 1$, $\tilde{f}_\infty = \lim_{u \to 0}(\tilde{f}(u)/u) = 0$. By a simple computation, it follows that $\lambda_1 = 16 \sin^4(\pi/24)$ and $\gamma_* = \min_{t \in \mathbb{T}_2} \sum_{s=2}^{12} K(t, s) = 143/2$, $c_1 = 216\sqrt{3}/145\sqrt{145} \approx 0.214$. Take $B = 50$. Since $\sup_{s \in (0, 10.715\,)}(\tilde{f}(s)/s) = (e^{10.715} - 1)/10.715 \approx 4202.0726$, it follows that

$$0.0000155 \approx \inf_{s \in (0, c_1 B)} \frac{s}{c_1 \gamma_* \tilde{f}(s)} = \frac{1}{\sup_{s \in (0, 10.715)}\left(\tilde{f}(s)/s\right) c_1 \gamma_*} < \frac{\lambda_1}{\tilde{f}_0} = 16 \sin^4 \frac{\pi}{24} \approx 0.0048. \tag{4.8}$$

Therefore, (i) of Theorem 1.4 implies that (4.6) has at least one generalized positive solution for

$$\lambda > \inf_{s \in (0, c_1 B)} \frac{s}{c_1 \gamma_* \tilde{f}(s)} \approx 0.0000155; \tag{4.9}$$

(ii) of Theorem 1.4 implies that (4.6) has at least two generalized positive solutions for

$$0.0000155 < \lambda < 16 \sin^4 \frac{\pi}{24}. \tag{4.10}$$

Acknowledgments

This paper was written when the first author visited Tabuk University, Tabuk, during May 16-June 13, 2011 and he is very thankful to the administration of Tabuk University for providing him the hospitalities during the stay. The second author gratefully acknowledges the partial financial support from the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah. The authors thank the referees for their valuable comments.

References

[1] R. P. Agarwal, “On fourth order boundary value problems arising in beam analysis,” *Differential and Integral Equations*, vol. 2, no. 1, pp. 91–110, 1989.

[2] A. Cabada, “The method of lower and upper solutions for second, third, fourth, and higher order boundary value problems,” *Journal of Mathematical Analysis and Applications*, vol. 185, no. 2, pp. 302–320, 1994.

[3] C. De Coster and L. Sanchez, “Upper and lower solutions, Ambrosetti-Prodi problem and positive solutions for fourth order O.D.E,” *Rivista di Matematica Pura ed Applicata*, no. 14, pp. 57–82, 1994.

[4] Z. Liu and F. Li, "Multiple positive solutions of nonlinear two-point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 3, pp. 610–625, 1996.
[5] Z.-C. Hao and L. Debnath, "On eigenvalue intervals and eigenfunctions of fourth-order singular boundary value problems," *Applied Mathematics Letters*, vol. 18, no. 5, pp. 543–553, 2005.
[6] Y. Guo and Y. Gao, "The method of upper and lower solutions for a Lidstone boundary value problem," *Czechoslovak Mathematical Journal*, vol. 55(130), no. 3, pp. 639–652, 2005.
[7] J. Chu and D. O'Regan, "Positive solutions for regular and singular fourth-order boundary value problems," *Communications in Applied Analysis*, vol. 10, no. 2-3, pp. 185–199, 2006.
[8] G. Han and Z. Xu, "Multiple solutions of some nonlinear fourth-order beam equations," *Nonlinear Analysis*, vol. 68, no. 12, pp. 3646–3656, 2008.
[9] J. R. L. Webb, G. Infante, and D. Franco, "Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions," *Proceedings of the Royal Society of Edinburgh A*, vol. 138, no. 2, pp. 427–446, 2008.
[10] R. Ma and H. Wang, "On the existence of positive solutions of fourth-order ordinary differential equations," *Applicable Analysis*, vol. 59, no. 1–4, pp. 225–231, 1995.
[11] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth-order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357–368, 2002.
[12] J. A. Cid, D. Franco, and F. Minhós, "Positive fixed points and fourth-order equations," *Bulletin of the London Mathematical Society*, vol. 41, no. 1, pp. 72–78, 2009.
[13] B. Zhang, L. Kong, Y. Sun, and X. Deng, "Existence of positive solutions for BVPs of fourth-order difference equations," *Applied Mathematics and Computation*, vol. 131, no. 2-3, pp. 583–591, 2002.
[14] Z. He and J. Yu, "On the existence of positive solutions of fourth-order difference equations," *Applied Mathematics and Computation*, vol. 161, no. 1, pp. 139–148, 2005.
[15] R. Ma and Y. Xu, "Existence of positive solution for nonlinear fourth-order difference equations," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3770–3777, 2010.
[16] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, I: Fixed-Point Theorems* , Springer, New York, NY, USA, 1986, Translated from the German by Peter R. Wadsac.
[17] B. P. Rynne, "Infinitely many solutions of superlinear fourth order boundary value problems," *Topological Methods in Nonlinear Analysis*, vol. 19, no. 2, pp. 303–312, 2002.
[18] R. Ma, "Existence of positive solutions of a fourth-order boundary value problem," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1219–1231, 2005.
[19] R. Ma and J. Xu, "Bifurcation from interval and positive solutions of a nonlinear fourth-order boundary value problem," *Nonlinear Analysis*, vol. 72, no. 1, pp. 113–122, 2010.
[20] R. Ma, "Nodal solutions of boundary value problems of fourth-order ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 424–434, 2006.
[21] R. Ma, "Nodal solutions for a fourth-order two-point boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 1, pp. 254–265, 2006.
[22] Y. Xu, C. Gao, and R. Ma, "Solvability of a nonlinear fourth-order discrete problem at resonance," *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 662–670, 2010.