Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 909520, 11 pages
doi:10.1155/2012/909520

Research Article

Necessary Conditions for Weak Sharp Minima in Cone-Constrained Optimization Problems

W. Y. Zhang,[1] S. Xu,[2] and S. J. Li[3]

[1] *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 610074, China*
[2] *Rear Services Office, Chongqing Police College, Chongqing 401331, China*
[3] *College of Mathematics and Statistics, Chongqing University, Chongqing 401331, China*

Correspondence should be addressed to W. Y. Zhang, wenyanzhang@126.com

Received 11 January 2012; Accepted 29 January 2012

Academic Editor: Bevan Thompson

We study weak sharp minima for optimization problems with cone constraints. Some necessary conditions for weak sharp minima of higher order are established by means of upper Studniarski or Dini directional derivatives. In particular, when the objective and constrained functions are strict derivative, a necessary condition is obtained by a normal cone.

1. Introduction

The notion of a weak sharp minimum in general mathematical program problems was first introduced by Ferris in [1]. It is an extension of a sharp (or strongly unique) minimum in [2]. Weak sharp minima play an important role in the sensitivity analysis [3, 4] and convergence analysis of a wide range of optimization algorithms [5–7]. Recently, the study of weak sharp solution set covers real-valued optimization problems [5, 8–10] and multiobjective optimization problems [11–13]. Moreover, it has been extended to convex-composite optimization with inequality constraints [14] and semi-infinite programs [15].

The weak sharp minima defined in [5] specified first-order growth of the objective function away from the set of optimal solutions. Recently, Studniarski [16] considered a special class of nonsmooth functions which are pointwise maxima of finite collections of strictly differentiable functions and presented a characterization of weak sharp local minima of order one. In addition, Studniarski [17] established the Kuhn-Tucker conditions for a nonlinear programming problem with constraints of both inequality and equality types, where the objective and inequality constrained functions are locally Lipschitzian and the equality constraints are differentiable.

Weak sharp minima of higher order are also of interest in sensitivity analysis in parametric optimization. In particular, the presence of weak sharp minima in parametric optimization leads to Hëlder continuity properties of the associated solution mappings in [18]. Bonnans and Ioffe [19] studied sufficient conditions and characterizations for weak sharp minima of order two in the case that the objective function is a pointwise maximum of twice continuously differentiable convex functions. In [20], Ward presented some necessary conditions for weak sharp minima of higher order for optimization problems with a set constraint. In [21], Studniarski and Ward obtained some sufficient conditions and characterizations for weak sharp local minimizer of higher order in terms of the limiting proximal normal cone and a generalization of the contingent cone.

However, to the best of our knowledge, there has no research concerning weak sharp minima for optimization with cone constraints although conic programming is a very hot research topic in optimization. In this paper, we first discuss necessary conditions for weak sharp minima of higher order in terms of the upper Studniarski and Dini directional derivatives and various tangent cones. In particular, by means of a normal cone, we provide a necessary condition for weak sharp minima of order one when the objective and constrained functions are strict derivative.

This paper is organized as follows. In Section 2, we recall the basic definitions. In Section 3, we establish several necessary conditions for a weak sharp minimizer of higher order.

2. Notions and Preliminaries

Consider the following optimization problem with cone constraints

$$\min f(x) \quad \text{s.t. } x \in Q, \quad g(x) \in -K, \tag{2.1}$$

where X is finite space and Y is a normed space. f is an extended real-valued function defined on X. S and K are nontrivial closed convex cones in Y, in which S defines an order. $g : X \to Y$ is a vector-valued mapping. Q is a closed convex subset of X. Let $G = \{x \in X : g(x) \in -K\}$. Denote by M the feasible set, that is, $M = Q \cap G$.

Definition 2.1 (see [21]). Let $\|\cdot\|$ be the Euclidean norm on X. Suppose that f is a constant on the set $\overline{S} \subset X$, and let $\overline{x} \in \overline{S} \cap M$ and $m \geq 1$. For $x \in X$, let

$$\operatorname{dist}\left(x, \overline{S}\right)^m := \inf\left\{ \|y - x\|^m : y \in \overline{S} \right\}. \tag{2.2}$$

(a) We say that $\overline{x}$ is a weak sharp minimizer of order m with module $\alpha > 0$ for (2.1), if there exists $\alpha > 0$ such that

$$f(x) - f(\overline{x}) \geq \alpha \operatorname{dist}\left(x, \overline{S}\right)^m, \quad \forall x \in M. \tag{2.3}$$

Let $g : X \to Y$ be a vector-valued mapping. The Hadamard and Dini derivatives of g at $\bar{x}$ in a direction $v \in X$ are, respectively, defined by

$$d_H g(\bar{x}, v) = \lim_{t \to 0^+, u \to v} \frac{g(\bar{x} + tu) - g(\bar{x})}{t},$$
$$d_D g(\bar{x}, v) = \lim_{t \to 0^+} \frac{g(\bar{x} + tv) - g(\bar{x})}{t}. \tag{2.4}$$

Let $f : X \to R \cup \{+\infty\}$ be finite at $\bar{x}$ and $m \geq 1$ an integer number. The upper Studniarski and Dini derivatives of order m at $\bar{x}$ in a direction $v \in X$ are, respectively, defined by

$$\overline{d}_S^m f(\bar{x}, v) = \limsup_{t \to 0^+, u \to v} \frac{f(\bar{x} + tu) - f(\bar{x})}{t^m},$$
$$\overline{d}_D^m f(\bar{x}, v) = \limsup_{t \to 0^+} \frac{f(\bar{x} + tv) - f(\bar{x})}{t^m}. \tag{2.5}$$

If $A \subset X$, then, the indicator function of A is $i_A(x) = 0$ if $x \in A$ and $i_A(x) = +\infty$ if $x \notin A$. The support function for A is defined by $\varphi_A^*(x) := \sup\{\langle x^*, x \rangle : x^* \in A\}$.

Let A be a closed and convex subset of X, we define the projection of a point $x \in X$ onto the set A, denoted by $P(A, x)$ as follows:

$$P(A, x) = \left\{ y \in A : \|x - y\|_X = \min_{u \in A} \|x - u\|_X \right\}. \tag{2.6}$$

Let K be a cone in a norm space Z. Denote by K^* the dual cone of K

$$K^* = \{z^* : \langle z^*, z \rangle \geq 0, \ \forall z \in K\}, \tag{2.7}$$

where Z^* is the topological dual of Z. Note that K^* is a w^*-closed convex cone. Let us introduce the following set:

$$K_0 = \{z \in K : \langle z^*, z \rangle > 0, \ \forall z^* \in K^* \setminus \{0\}\}. \tag{2.8}$$

Definition 2.2. Let A be a subset of normed vector space Z and $x \in clA$, then

(a) (see [22]) the contingent cone to the set A is $T(A, x) = \{v \in R^n : \exists t_n \to 0^+, v_n \to v, \text{ with } x + t_n v_n \in A\}$,

(b) (see [23]) the Clarke tangent cone to the set A is $K(A, x) = \{v \in R^n : \forall A \ni x_n \to \bar{x}, \forall t_n \to 0^+, \exists v_n \to v, \text{ with } x_n + t_n v_n \in A\}$,

(c) (see [24]) $\widetilde{T}(A, x) = \{v \in X : \exists t_n \to 0^+ \text{ such that } x + t_n v \in A, \ \forall n \text{ large enough}\}$,

(d) (see [20]) $IK(A, x) = \{v \in X : \exists t_n \to 0^+ \text{ such that } \forall v_n \to v, x + t_n v_n \in A, \forall n \text{ large enough}\}$.

It is easy to see that $IK(A, x) \subset \widetilde{T}(A, x) \subset T(A, x)$.

A set A is said to be regular at $x \in A$ if $T(A,x) = K(A,x)$. Obviously, every convex set is regular. Moreover, if A is a convex set, we call both the contingent cone and Clarke tangent cone to the set A as tangent cone to the set A.

For a nonempty set $A \subset X$, we define the polar of A to be the set $A^\circ = \{x^* \in X^* : \langle x^*, x\rangle \le 1 \ \forall x \in A\}$. The classic normal cone to A at x is defined dually by the relation $N(A,x) = T(A,x)^\circ$.

Definition 2.3 (see [17]). Let E and S be subset of R^n, and let $\overline{x} \in clE$. The normal cone to E at $\overline{x}$ relative to S is defined by

$$N_S(E,\overline{x}) := \Big\{ y \in R^n : \exists y_n \longrightarrow y,\ x_n \longrightarrow \overline{x},\ t_n \in (0,+\infty),\ w_n \in R^n \\ \text{with } x_n \in S,\ w_n \in P(E,x_n) \text{ and } y_n = \frac{(x_n - w_n)}{t_n}, (\forall n) \Big\}. \tag{2.9}$$

Definition 2.4. Let f map X to another Banach space Y. We say that f admits a strict derivative at x, an element $L(X,Y)$ denoted $\nabla f(x)$, provided that for each the following holds

$$\lim_{x' \to x, y \to v, t \to 0^+} \frac{f(x' + ty) - f(x')}{t} = \langle \nabla f(x), v\rangle. \tag{2.10}$$

3. Necessary Conditions

In the section, we provide necessary optimality conditions for the problem (2.1), which are formulated in terms of the upper Studniarski and Dini derivatives of the objective function, respectively. Simultaneously, we also apply the indicator function of a set to state the necessary conditions.

Theorem 3.1. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in \overline{S}$ be a weak sharp minimizer of order m with module $\alpha > 0$ for the problem* (2.1)*. Suppose that g is Hadamard derivative at $\overline{x}$ in all directions $v \in X$. Then,*

$$\overline{d}_S^m f(\overline{x}, v) \ge \alpha \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in T(Q,\overline{x}) \cap \{u : d_H g(\overline{x}, u) \in -\operatorname{int} K\}. \tag{3.1}$$

In particular, if $\overline{S}$ is regular at $\overline{x}$, then

$$\overline{d}_S^m f(\overline{x}, v) \ge \alpha \operatorname{dist}\left(v, K\left(\overline{S}, \overline{x}\right)\right), \quad \forall v \in T(Q,\overline{x}) \cap \{u : d_H g(\overline{x}, u) \in -\operatorname{int} K\}. \tag{3.2}$$

Proof. Let $v \in T(Q,\overline{x}) \cap \{u : d_H g(\overline{x}, u) \in -\operatorname{int} K\}$. By the definition of contingent cone, there exist $t_n \to 0^+$ and $v_n \to v$ such that $\overline{x} + t_n v_n \in Q$. In addition, $v \neq 0$, by assumption $d_H g(\overline{x}, v) \in -\operatorname{int} K$.

Since g is Hadamard derivative at $\overline{x}$ in the direction $v \in X$, we have that, for $t_n \to 0^+$,

$$\lim_{n\to\infty} \frac{g(\overline{x} + t_n v_n) - g(\overline{x})}{t_n} = d_H g(\overline{x}, v). \tag{3.3}$$

Moreover, $d_H g(\overline{x}, v) \in -\operatorname{int} K$, then there exists a natural number N_1 such that for $n \geq N_1$,

$$\frac{g(\overline{x} + t_n v_n) - g(\overline{x})}{t_n} \in -K, \tag{3.4}$$

which implies that

$$g(\overline{x} + t_n v_n) \in g(\overline{x}) - K \subset -K. \tag{3.5}$$

Hence,

$$\overline{x} + t_n v_n \in M = Q \cap G. \tag{3.6}$$

According to the definition of weak sharp minimizer of order m, we get

$$f(\overline{x} + t_n v_n) - f(\overline{x}) \geq \alpha \operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m. \tag{3.7}$$

Consequently, it follows from (3.7) that

$$\frac{f(\overline{x} + t_n v_n) - f(\overline{x})}{t_n^m} \geq \alpha \frac{\operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m}{t_n^m}. \tag{3.8}$$

Taking lim sups of both sides in (3.8) as $n \to \infty$, we have

$$\begin{aligned} \overline{d}_S^m f(\overline{x}, v) &\geq \limsup_{n \to \infty} \frac{f(\overline{x} + t_n v_n) - f(\overline{x})}{t_n^m} \\ &\geq \alpha \limsup_{n \to \infty} \frac{\operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m}{t_n^m} \\ &\geq \alpha \liminf_{n \to \infty} \frac{\operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m}{t_n^m}. \end{aligned} \tag{3.9}$$

To establish (3.1), we suffice to show that

$$\liminf_{n \to \infty} \frac{\operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m}{t_n^m} \geq \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m. \tag{3.10}$$

Set

$$L := \liminf_{n \to \infty} \frac{\operatorname{dist}\left(\overline{x} + t_n v_n, \overline{S}\right)^m}{t_n^m}. \tag{3.11}$$

If $L = +\infty$, then (3.10) holds true. Hence, we may assume that $L < +\infty$. Then, for any $\epsilon > 0$, there exists a number N, such that, for all $k > N$,

$$t_{n_k} \in B(0,\epsilon), \qquad v_{n_k} \in B(v,\epsilon), \qquad w_{n_k} \in \overline{S},$$
$$\frac{\|\overline{x} + t_{n_k} v_{n_k} - w_{n_k}\|^m}{t_{n_k}^m} \leq L + \epsilon. \tag{3.12}$$

Set

$$z_{n_k} := \frac{(\overline{x} + t_{n_k} v_{n_k} - w_{n_k})}{t_{n_k}}. \tag{3.13}$$

Since the sequence $\{z_{n_k}\}$ is bounded, we may assume, taking a subsequence if necessary, that it converges to some $z \in X$ with $\|z\|^m \leq L$. For each $k > N$, we have

$$\overline{x} + t_{n_k}(v_{n_k} - z_{n_k}) = w_{n_k} \in \overline{S},$$
$$v - z \in T\left(\overline{S}, \overline{x}\right). \tag{3.14}$$

Hence

$$\operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m \leq \|v - (v - z)\|^m \leq L, \tag{3.15}$$

and (3.10) holds.

Observe that the regularity at $\overline{x} \in \overline{S}$ implies that $T(\overline{S}, \overline{x}) = K(\overline{S}, \overline{x})$. Hence, the inequality (3.2) holds. □

Theorem 3.2. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in \overline{S}$ be a weak sharp minimizer of order m with module $\alpha > 0$ for the problem* (2.1). *Suppose that g is Hadamard derivative at $\overline{x}$ in all directions $v \in X$. Then,*

$$\overline{d}_S^m(f + i_Q)(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in \{u : d_H g(\overline{x}, u) \in -\operatorname{int} K\}. \tag{3.16}$$

In particular, if $\overline{S}$ is regular at $\overline{x}$, then

$$\overline{d}_S^m(f + i_Q)(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, K\left(\overline{S}, \overline{x}\right)\right), \quad \forall v \in \{u : d_H g(\overline{x}, u) \in -\operatorname{int} K\}. \tag{3.17}$$

Proof. Suppose that $\overline{x}$ is a weak sharp minimizer of order m with module $\alpha > 0$ for the problem (2.1), then

$$f(x) - f(\overline{x}) \geq \alpha \operatorname{dist}\left(x, \overline{S}\right)^m, \quad \forall x \in M. \tag{3.18}$$

Since g is the Hadamard derivative at $\overline{x}$ in the direction $v \in X$, there exist $t_n \to 0^+$ and $v_n \to v$ such that

$$\lim_{n\to\infty} \frac{g(\overline{x}+t_n v_n)-g(\overline{x})}{t_n} = d_H g(\overline{x}, v). \tag{3.19}$$

Moreover, $dg(\overline{x}, v) \in -\operatorname{int} K$, then there exists a natural number N_2 such that, for $n \geq N_2$,

$$\frac{g(\overline{x}+t_n v_n)-g(\overline{x})}{t_n} \in -K, \tag{3.20}$$

which implies that

$$g(\overline{x}+t_n v_n) \in g(\overline{x}) - K \subset -K. \tag{3.21}$$

Therefore,

$$\overline{x}+t_n v_n \in G. \tag{3.22}$$

If $\overline{x}+t_n v_n \in Q$, then, by (3.18),

$$f(\overline{x}+t_n v_n) + i_Q(\overline{x}+t_n v_n) - f(\overline{x}) - i_Q(\overline{x}) \geq \alpha \operatorname{dist}\left(\overline{x}+t_n v_n, \overline{S}\right)^m. \tag{3.23}$$

On the other hand, if $\overline{x}+t_n v_n \notin Q$, then $i_Q(\overline{x}+t_n v_n) = +\infty$ and (3.23) still holds true.

Hence, from (3.23), it follows that

$$\frac{f(\overline{x}+t_n v_n) - f(\overline{x}) + i_Q(\overline{x}+t_n v_n) - i_Q(\overline{x})}{t_n^m} \geq \alpha \frac{\operatorname{dist}\left(\overline{x}+t_n v_n, \overline{S}\right)^m}{t_n^m}. \tag{3.24}$$

Taking the lim sups on both sides in (3.24) as $n \to +\infty$, we get

$$\overline{d}^m (f+i_Q)(\overline{x}, v) \geq \alpha \liminf_{n\to\infty} \frac{\operatorname{dist}\left(\overline{x}+t_n v_n, \overline{S}\right)^m}{t_n^m}. \tag{3.25}$$

The rest of the proof is similar to Theorem 3.1 and hence omitted. □

In what follows, we state other necessary conditions for the weak sharp minimizer of order m for the problem (2.1) in terms of the cone K_0 and $\widetilde{T}(Q, \overline{x})$. Note that the necessary conditions do not require the cone K to have nonempty interior.

Theorem 3.3. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in \overline{S}$ be a weak sharp minimizer of order m with module $\alpha > 0$ for the problem* (2.1)*. Suppose that g is Dini derivative at $\overline{x}$ in all directions $v \in X$. Then,*

$$\overline{d}_D^m f(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in \widetilde{T}(Q, \overline{x}) \cap \{u : d_D g(\overline{x}, u) \in -K_0\}. \tag{3.26}$$

In particular, if $\overline{S}$ is regular at $\overline{x}$, then

$$\overline{d}_D^m f(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, K\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in \widetilde{T}(Q, \overline{x}) \cap \{u : d_D g(\overline{x}, u) \in -K_0\}. \tag{3.27}$$

Proof. Let $v \in \widetilde{T}(Q, \overline{x}) \cap \{u : d_D g(\overline{x}, u) \in -K_0\}$, then there exists a sequence $t_n \to 0^+$ such that $\overline{x} + t_n v \in Q$ for sufficiently large n. In addition,

$$\lim_{n \to \infty} \frac{g(\overline{x} + t_n v) - g(\overline{x})}{t_n} = d_D g(\overline{x}, v), \tag{3.28}$$

which leads to the following relation

$$g(\overline{x} + t_n v) = g(\overline{x}) + t_n \cdot d_D g(\overline{x}, v) + o(t_n), \tag{3.29}$$

where $o(t_n)/t_n \to 0$ as $n \to \infty$. Since $d_D g(\overline{x}, v) \in -K_0$, for any $z^* \in K^* \setminus \{0\}$, $\langle z^*, d_D g(\overline{x}, v)\rangle < 0$, which yields that

$$\langle z^*, g(\overline{x} + t_n v)\rangle = \langle z^*, g(\overline{x})\rangle + t_n \left[\langle z^*, d_D g(\overline{x}, v)\rangle + \frac{o(t_n)}{t_n}\right] \leq 0 \tag{3.30}$$

for large enough n. Observing that K is a closed convex cone, it follows that K is weakly closed and

$$g(\overline{x} + t_n v) \in -K^{**} = -K. \tag{3.31}$$

Consequently, $\overline{x} + t_n v \in M$ for sufficiently large n. The rest of the proof is analogue to Theorem 3.1 and thus omitted. □

By using the method of Theorems 3.2 and 3.3, we easily establish the following results.

Theorem 3.4. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in \overline{S}$ be a weak sharp minimizer of order m with module $\alpha > 0$ for the problem* (2.1)*. Suppose that g is Dini derivative at $\overline{x}$ in all directions $v \in X$. Then,*

$$\overline{d}_D^m (f + i_Q)(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in \{u : d_D g(\overline{x}, u) \in -K_0\}. \tag{3.32}$$

In particular, if $\overline{S}$ is regular at $\overline{x}$, then

$$\overline{d}_D^m(f + i_Q)(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, K\left(\overline{S}, \overline{x}\right)\right)^m, \quad \forall v \in \{u : d_D g(\overline{x}, u) \in -K_0\}. \tag{3.33}$$

Since $IK(A, x) \subset \widetilde{T}(A, x)$, the following result is direct consequence of Theorem 3.3.

Corollary 3.5. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in bd\overline{S}$ be a weak sharp minimizer of order m with module $\alpha > 0$ for the problem* (2.1)*. Suppose that g is Dini derivative at $\overline{x}$ in all directions $v \in X$. Then,*

$$\overline{d}_D^m f(\overline{x}, v) \geq \alpha \operatorname{dist}\left(v, T\left(\overline{S}, \overline{x}\right)\right), \quad \forall v \in IK(Q, \overline{x}) \cap \{u : d_D g(\overline{x}, u) \in -K_0\}. \tag{3.34}$$

Suppose that X and Y are finite spaces. In what follow, we apply the normal cone to present a necessary optimality condition for problem (2.1), where the objective and constrained functions are strict derivative.

Theorem 3.6. *Suppose that $\overline{S}$ is a closed set. Let $\overline{x} \in bd\overline{S}$ be a weak sharp minimizer of order one with module $\alpha > 0$ for the problem* (2.1)*. Suppose that f and g are strict derivatives at $\overline{x}$. Then, for any $v \in N_Q(\overline{S}, \overline{x}) \cap \{u : \langle \nabla g(\overline{x}), u\rangle \in -\operatorname{int} K\}$ with $\|v\| = 1$,*

$$\left\langle \nabla f(\overline{x}), v\right\rangle > 0. \tag{3.35}$$

Proof. Assume that $\langle \nabla f(\overline{x}), v\rangle \leq 0$ for some $v \in N_Q(\overline{S}, \overline{x}) \cap \{u : \langle \nabla g(\overline{x}), u\rangle \in -\operatorname{int} K\}$. Then, by the definition of normal cone, there exist $v_n \longrightarrow v$, $x_n \longrightarrow \overline{x}$, $t_n \in (0, +\infty)$, $w_n \in X$ with $x_n \in Q$, $w_n \in P(\overline{S}, x_n)$ and

$$v_n = \frac{x_n - w_n}{t_n}. \tag{3.36}$$

Since $\|v\| = 1$ and $v_n \longrightarrow v$, we have $v_n \neq 0$ for n sufficiently large, and consequently, $\|x_n - w_n\| = d(x_n, \overline{S})$.

Observe that the condition (3.36) implies that for $n \longrightarrow \infty$,

$$t_n = \frac{\|x_n - w_n\|}{\|v_n\|} \leq \frac{\|x_n - \overline{x}\|}{\|v_n\|} \longrightarrow 0. \tag{3.37}$$

Moreover, for $n \longrightarrow \infty$, $\|w_n - \overline{x}\| \leq \|w_n - x_n\| + \|x_n - \overline{x}\| \leq 2\|x_n - \overline{x}\| \longrightarrow 0$. We can assume that $\lim_{n\to\infty}(f(w_n + t_n v_n) - f(w_n))/t_n \leq 0$ since $\langle \nabla f(\overline{x}), v\rangle \leq 0$. Hence, for any $\epsilon > 0$, there exist N_1 such that, for all $n > N_1$,

$$\frac{f(w_n + t_n v_n) - f(w_n)}{t_n} < \epsilon. \tag{3.38}$$

Since g is strict derivative at $\overline{x}$,

$$\lim_{n\to\infty}\frac{g(w_n+t_nv_n)-g(w_n)}{t_n}=\langle\nabla g(\overline{x}),v\rangle. \tag{3.39}$$

It follows that from $\langle\nabla g(\overline{x}),v\rangle\in-\operatorname{int}K$, there exists a natural number N_2 such that, for $n\geq N_2$,

$$\frac{g(w_n+t_nv_n)-g(w_n)}{t_n}\in-K, \tag{3.40}$$

which implies that

$$g(w_n+t_nv_n)\in g(w_n)-K\subset-K. \tag{3.41}$$

Hence,

$$w_n+t_nv_n\in M=Q\cap G. \tag{3.42}$$

On the other hand, by assumption, for $n\geq\max\{N_1,N_2\}$, $x_n=w_n+t_nv_n$,

$$f(w_n+t_nv_n)-f(\overline{x})=f(w_n+t_nv_n)-f(w_n)\geq\alpha d\left(x_n,\overline{S}\right)=\alpha\|x_n-w_n\|. \tag{3.43}$$

Together with relation (3.38), for $n\geq\max\{N_1,N_2\}$, we have

$$\epsilon>\frac{f(w_n+t_nv_n)-f(w_n)}{t_n}\geq\alpha\|v_n\|. \tag{3.44}$$

Taking the limit when $n\to\infty$ and for the arbitrary $\epsilon>0$, we deduce that $\lim_{n\to\infty}\|v_n\|=0$, which is a contradiction to the fact that $v\neq0$. □

Acknowledgment

This research was partially supported by the National Natural Science Foundation of China (Grant nos. 10871216, A011201).

References

[1] M. C. Ferris, *Weak sharp minima and penalty functions in mathematical programming*, Ph.D. thesis, Universiy of Cambridge, Cambridge, UK, 1988.

[2] B. T. Polyak, "Sharp minima," in *Proceedings of the IIASA Workshop on Generalized Lagrangians and Their Applications*, Institue of control sciences lecture notes, IIASA, Laxenburg, Austria, USSR, Moscow, Russia, 1979.

[3] R. Henrion and J. Outrata, "A subdifferential condition for calmness of multifunctions," *Journal of Mathematical Analysis and Applications*, vol. 258, no. 1, pp. 110–130, 2001.

[4] A. S. Lewis and J. S. Pang, "Error bounds for convex inequality systems," in *Proceedings of the 5th Symposium on Generalized Convexity*, J. P. Crouzeix, Ed., Luminy, Marseille, France, 1996.

[5] J. V. Burke and M. C. Ferris, "Weak sharp minima in mathematical programming," *SIAM Journal on Control and Optimization*, vol. 31, no. 5, pp. 1340–1359, 1993.

[6] J. Burke and S.-P. Han, "A Gauss-Newton approach to solving generalized inequalities," *Mathematics of Operations Research*, vol. 11, no. 4, pp. 632–643, 1986.

[7] M. C. Ferris, "Iterative linear programming solution of convex programs," *Journal of Optimization Theory and Applications*, vol. 65, no. 1, pp. 53–65, 1990.

[8] J. V. Burke and S. Deng, "Weak sharp minima revisited. I. Basic theory," *Control and Cybernetics*, vol. 31, no. 3, pp. 439–469, 2002.

[9] J. V. Burke and S. Deng, "Weak sharp minima revisited. II. Application to linear regularity and error bounds," *Mathematical Programming*, vol. 104, no. 2-3, pp. 235–261, 2005.

[10] J. V. Burke and S. Deng, "Weak sharp minima revisited. III. Error bounds for differentiable convex inclusions," *Mathematical Programming*, vol. 116, no. 1-2, Ser. B, pp. 37–56, 2009.

[11] S. Deng and X. Q. Yang, "Weak sharp minima in multicriteria linear programming," *SIAM Journal on Optimization*, vol. 15, no. 2, pp. 456–460, 2004.

[12] X. Y. Zheng and X. Q. Yang, "Weak sharp minima for piecewise linear multiobjective optimization in normed spaces," *Nonlinear Analysis*, vol. 68, no. 12, pp. 3771–3779, 2008.

[13] X. Y. Zheng, X. M. Yang, and K. L. Teo, "Sharp minima for multiobjective optimization in Banach spaces," *Set-Valued Analysis*, vol. 14, no. 4, pp. 327–345, 2006.

[14] X. Y. Zheng and K. F. Ng, "Strong KKT conditions and weak sharp solutions in convex-composite optimization," *Mathematical Programming. Series A*, vol. 126, no. 2, pp. 259–275, 2011.

[15] X. Y. Zheng and X. Q. Yang, "Weak sharp minima for semi-infinite optimization problems with applications," *SIAM Journal on Optimization*, vol. 18, no. 2, pp. 573–588, 2007.

[16] M. Studniarski, "On weak sharp minima for a special class of nonsmooth functions," *Discussiones Mathematicae. Differential Inclusions, Control and Optimization*, vol. 20, no. 2, pp. 195–207, 2000.

[17] M. Studniarski, "Characterizations of weak sharp minima of order one in nonlinear programming," in *Systems Modelling and Optimization*, vol. 396 of *Chapman & Hall/CRC Res. Notes Math.*, pp. 207–215, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[18] D. Klatte, "On quantitative stability for non-isolated minima," *Control and Cybernetics*, vol. 23, no. 1-2, pp. 183–200, 1994.

[19] J. F. Bonnans and A. Ioffe, "Second-order sufficiency and quadratic growth for nonisolated minima," *Mathematics of Operations Research*, vol. 20, no. 4, pp. 801–817, 1995.

[20] D. E. Ward, "Characterizations of strict local minima and necessary conditions for weak sharp minima," *Journal of Optimization Theory and Applications*, vol. 80, no. 3, pp. 551–571, 1994.

[21] M. Studniarski and D. E. Ward, "Weak sharp minima: characterizations and sufficient conditions," *SIAM Journal on Control and Optimization*, vol. 38, no. 1, pp. 219–236, 1999.

[22] J. P. Aubin and H. Frankowska, *Set-Valued Analysis*, vol. 2 of *Systems & Control: Foundations & Applications*, Birkhäuser Boston Inc., Boston, Mass, USA, 1990.

[23] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1983.

[24] Do Van Luu, "Higher-order necessary and sufficient conditions for strict local Pareto minima in terms of Studniarski's derivatives," *Optimization*, vol. 57, no. 4, pp. 593–605, 2008.