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Research Article

The Existence and Uniqueness of Periodic Solutions for Some Nonlinear nth-Order Differential Equations

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The nonlinear nth-order differential equations are considered. By using inequality techniques and coincidence degree theory, some criteria are obtained to guarantee the existence and uniqueness of T-periodic solutions for the equations. The obtained results are also valid and new for the problem discussed in the previous literature. Moreover, two illustrative examples are provided to illustrate the effectiveness of our results.

1. Introduction

In applied science, some practical problems are associated with the periodic solutions for nonlinear high-order differential equations, such as nonlinear oscillations [1, 2], electronic theory [3], biological model, and other models [4–6]. In particular, during the past thirty years, there has been a great amount of work on the existence and uniqueness of periodic solutions for the nth-order nonlinear differential equation

$$x^{(n)} + \sum_{j=1}^{n-1} a_j x^{(j)} + g(t, x) = e(t), \tag{1.1}$$

where $e : R \to R$ and $g : R \times R \to R$ are continuous functions, $e(t)$ is 2π-periodic with respect to t, g is 2π-periodic in the first variable, and a_i $(i = 1, 2, \ldots, n-1)$ are constants.

Many of these results can be found in [7–11] and the references cited therein. Among the known results, we find that the assumption

$(\overline{H})$ $g(t,x)$ is continuous, and there are positive constants m_0 and M_0 such that

$$m_0 \le \left|g_x(t,x)\right| \le M_0, \quad \forall(t,x) \in R \times R \tag{1.2}$$

is employed, and it plays an important role in the proofs of these known results (see, e.g., [9–11]). Recently, under some spectral conditions of linear differential operator, Li [12, 13] discussed the existence and uniqueness of T-periodic solutions of nonlinear differential equations.

However, to the best of our knowledge, there exist few results for the existence and uniqueness of periodic solutions of (1.1) without $(\overline{H})$ and the spectral conditions of linear differential operator. Thus, in this case, it is worth to study the problem of existence and uniqueness of periodic solutions of nth-order nonlinear differential equation (1.1).

The purpose of this paper is to investigate the existence and uniqueness of T-periodic solutions of (1.1). By using some inequality techniques and Mawhin's continuation theorem, we establish some sufficient conditions for the existence and uniqueness of T-periodic solutions of (1.1) when $(\overline{H})$ and the spectral conditions are avoided. Moreover, two illustrative examples are given in Section 4.

2. Preliminary Results

Let us introduce some notations. We will use Φ to denote the empty set. For $n \in \mathbf{N}$, we denote by C_T^n the Banach space

$$C_T^n = \{u \in C^n(R,R) : u(t) = u(t+T),\ \forall t \in R\}, \tag{2.1}$$

endowed with the norm

$$\|u\|_{(n)} = \sum_{k=0}^{n} \left|u^{(k)}\right|_\infty, \quad \left(u \in C_T^n\right), \tag{2.2}$$

where, for a function $v \in C_T^0$, we have that

$$|v|_\infty = \max_{[0,T]}|v|. \tag{2.3}$$

For $x \in C_T^0$, we will denote

$$|x|_p = \left(\int_0^T |x(t)|^p dt\right)^{1/p}, \quad (p > 0). \tag{2.4}$$

Now, let $\widetilde{f} : R^{n+1} \to R$ be a continuous function, T-periodic with respect to the first variable, and consider the nth-order differential equation

$$u^{(n)} = \widetilde{f}\left(t, u, u', u'', \ldots, u^{(n-1)}\right). \tag{2.5}$$

Lemma 2.1 (see [14]). *Assume that the following conditions hold.*

(i) *There exists $\rho > 0$ such that, for each $\lambda \in (0,1]$, one has that any possible T-periodic solution u of the problem*

$$u^{(n)} = \lambda \widetilde{f}\left(t, u, u', u'', \ldots, u^{(n-1)}\right) \tag{2.6}$$

satisfies the priori estimation $\|u\|_{(n-1)} < \rho$.

(ii) *The continuous function $F : \mathbf{R} \to \mathbf{R}$ defined by*

$$F(x) = \int_0^T \widetilde{f}(t, x, 0, 0, \ldots, 0)dt, \quad (x \in \mathbf{R}) \tag{2.7}$$

satisfies $F(-\rho)F(\rho) < 0$.

Then, (2.5) *has at least one T-periodic solution u such that $\|u\|_{(n-1)} < \rho$.*

From Lemma 2.2 in [15] and the proof of inequality (10) in [7, pp 3402], one obtains the following.

Lemma 2.2. *Let $x(t) \in C_T^1$. Suppose that there exists a constant $D \geq 0$ such that*

$$|x(\tau_0)| \leq D, \quad \tau_0 \in [0, T], \tag{2.8}$$

then

$$|x|_2 \leq \frac{T}{\pi}\left|x'\right|_2 + \sqrt{T}D, \qquad |x|_\infty \leq D + \frac{1}{2}\sqrt{T}\left(\int_0^T \left|x'(t)\right|^2 dt\right)^{1/2}. \tag{2.9}$$

Lemma 2.3. *For any $u \in C_T^2$, one has that*

$$\int_0^T \left|u'(t)\right|^2 dt \leq \left(\frac{T}{2\pi}\right)^2 \int_0^T \left|u''(t)\right|^2 dt. \tag{2.10}$$

Proof. Lemma 2.3 is a direct consequence of the Wirtinger inequality, and see [16, 17] for its proof. □

By the same approach used in the proof of Lemma 3 of [7], we have the following.

Lemma 2.4. *For any $u \in C_T^n$, one has that*

$$\left|u^{(j)}\right|_\infty \leq T^{(n-1)-j}\left(\frac{1}{2}\right)^{(n-1)-(j-1)}\int_0^T\left|u^{(n)}(t)\right|dt, \quad (j = 1, 2, \ldots, n-1). \tag{2.11}$$

Lemma 2.5. *Let k be an even number, $n = 2k$, and*

$$\Lambda = \left\{j : j \in \{1, 2, \ldots,\ k-1\},\ (-1)^j a_{2j} < 0\right\}. \tag{2.12}$$

Assume that one of the following conditions is satisfied:

(H_1) *for* $t, x_1, x_2 \in R, x_1 \neq x_2$,

$$1 + \sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j} > 0, \qquad (g(t, x_1) - g(t, x_2))(x_1 - x_2) > 0, \tag{2.13}$$

(H_2) *there exists a nonnegative constant B such that*

$$\begin{gathered}1 > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}, \qquad 1 + \sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j} > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}, \\ B(x_1 - x_2)^2 \geq -(g(t, x_1) - g(t, x_2))(x_1 - x_2) > 0,\end{gathered} \tag{2.14}$$

where $t, x_1, x_2 \in R, x_1 \neq x_2$, then (1.1) *has at most one T-periodic solution.*

Proof. Suppose that $u_1(t)$ and $u_2(t)$ are two T-periodic solutions of (1.1). Set $Z(t) = u_1(t) - u_2(t)$. Then, we obtain

$$(u_1(t) - u_2(t))^{(n)} + \sum_{j=1}^{n-1} a_j (u_1(t) - u_2(t))^{(j)} + (g(t, u_1(t)) - g(t, u_2(t))) = 0. \tag{2.15}$$

Integrating (2.15) from 0 to T, it results that

$$\int_0^T (g(t, u_1(t)) - g(t, u_2(t)))dt = 0. \tag{2.16}$$

Therefore, in view of integral mean value theorem, it follows that there exists a constant $\gamma \in [0,\ T]$ such that

$$g(\gamma, u_1(\gamma)) - g(\gamma, u_2(\gamma)) = 0. \tag{2.17}$$

Since $g(t,x)$ is a strictly monotone function in x, (2.17) implies that

$$Z(\gamma) = u_1(\gamma) - u_2(\gamma) = 0. \tag{2.18}$$

Then, from (2.9), we have

$$|Z|_2 \leq \frac{T}{\pi}|Z'|_2. \tag{2.19}$$

Multiplying (2.15) by $Z(t)$ and then integrating it from 0 to T, it follows that

$$\begin{aligned}\int_0^T \left|Z^{(k)}(t)\right|^2 dt + \sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|Z^{(j)}(t)\right|^2 dt &= -\int_0^T Z(t)\big(g(t,u_1(t)) - g(t,u_2(t))\big)dt \\ &= -\int_0^T (u_1(t)-u_2(t))\big(g(t,u_1(t)) - g(t,u_2(t))\big)dt.\end{aligned} \tag{2.20}$$

Now suppose that (H_1) (or (H_2)) holds, and we will consider two cases as follows.

Case i. If (H_1) holds, (2.10) and (2.20) yield that

$$\begin{aligned}0 &\leq \left(1 + \sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j}\right)\int_0^T \left|Z^{(k)}(t)\right|^2 dt \\ &\leq \int_0^T \left|Z^{(k)}(t)\right|^2 dt + \sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|Z^{(j)}(t)\right|^2 dt \\ &= -\int_0^T (u_1(t)-u_2(t))\big(g(t,u_1(t)) - g(t,u_2(t))\big)dt \\ &\leq 0, \quad \text{where } \Lambda \neq \emptyset, \\ 0 &\leq \int_0^T \left|Z^{(k)}(t)\right|^2 dt \\ &\leq \int_0^T \left|Z^{(k)}(t)\right|^2 dt + \sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|Z^{(j)}(t)\right|^2 dt \\ &= -\int_0^T (u_1(t)-u_2(t))\big(g(t,u_1(t)) - g(t,u_2(t))\big)dt \\ &\leq 0, \quad \text{where } \Lambda = \emptyset,\end{aligned} \tag{2.21}$$

which, together with (2.18), implies that

$$Z(t) \equiv Z'(t) \equiv \cdots \equiv Z^{(k)}(t) \equiv 0, \quad \forall t \in R. \tag{2.22}$$

Hence, (1.1) has at most one T-periodic solution.

Case ii. If (H_2) holds, using (2.9), (2.10), (2.19), and (2.20), we obtain that

$$\begin{aligned}
&\left(1+\sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j}\right)\int_0^T\left|Z^{(k)}(t)\right|^2dt\\
&\quad\leq\int_0^T\left|Z^{(k)}(t)\right|^2dt+\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T\left|Z^{(j)}(t)\right|^2dt\\
&\quad=-\int_0^T Z(t)\left(g(t,u_1(t))-g(t,u_2(t))\right)dt\\
&\quad\leq B\int_0^T|Z(t)|^2dt\\
&\quad\leq B\left(\frac{T}{\pi}\right)^2\int_0^T\left|Z'(t)\right|^2dt\\
&\quad\leq B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}\int_0^T\left|Z^{(k)}(t)\right|^2dt,\quad\text{where }\Lambda\neq\emptyset,\\
&\int_0^T\left|Z^{(k)}(t)\right|^2dt\leq\int_0^T\left|Z^{(k)}(t)\right|^2dt+\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T\left|Z^{(j)}(t)\right|^2dt\\
&\quad\leq B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}\int_0^T\left|Z^{(k)}(t)\right|^2dt,\quad\text{where }\Lambda=\emptyset.
\end{aligned}\tag{2.23}$$

From (2.18) and (H_2), (2.23) yield that

$$Z(t)\equiv Z'(t)\equiv\cdots\equiv Z^{(k)}(t)\equiv 0,\quad\forall t\in R.\tag{2.24}$$

Therefore, (1.1) has at most one T-periodic solution. The proof of Lemma 2.5 is now complete. □

Similar to the proof of Lemma 2.5, one can prove the following result.

Lemma 2.6. *Let k be an odd number, $n=2k$, and*

$$\overline{\Lambda}=\left\{j: j\in\{1,2,\ldots,k-1\},\ (-1)^{j+1}a_{2j}<0\right\}.\tag{2.25}$$

Assume that one of the following conditions is satisfied:

$(\overline{H_1})$ *for* $t,x_1,x_2\in R, x_1\neq x_2$,

$$1+\sum_{j\in\overline{\Lambda}}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^{j+1}a_{2j}>0,\qquad\left(g(t,x_1)-g(t,x_2)\right)(x_1-x_2)<0,\tag{2.26}$$

$(\overline{H_2})$ *there exists a nonnegative constant B such that*

$$1 > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}, \qquad 1+\sum_{j\in\overline{\Lambda}}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^{j+1}a_{2j} > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)},$$
$$B(x_1-x_2)^2 \geq \big(g(t,x_1)-g(t,x_2)\big)(x_1-x_2) > 0, \tag{2.27}$$

where $t, x_1, x_2 \in R, x_1 \neq x_2$, then (1.1) *has at most one T-periodic solution.*

Lemma 2.7. *Let k be an even number, $n = 2k+1$, and*

$$\Lambda^* = \left\{j : j \in \{1,2,\dots,k-1\},\ (-1)^j a_{2j} < 0\right\}. \tag{2.28}$$

Assume that one of the following conditions is satisfied:

(H_3) *for $t, x_1, x_2 \in R, x_1 \neq x_2$,*

$$a_{2k}+\sum_{j\in\Lambda^*}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j} > 0, \qquad \big(g(t,x_1)-g(t,x_2)\big)(x_1-x_2) > 0, \tag{2.29}$$

(H_4) *there exists a nonnegative constant B such that*

$$a_{2k} > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)}, \qquad 1+\sum_{j\in\Lambda^*}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j} > B\left(\frac{T}{\pi}\right)^2\left(\frac{T}{2\pi}\right)^{2(k-1)},$$
$$B(x_1-x_2)^2 \geq -\big(g(t,x_1)-g(t,x_2)\big)(x_1-x_2) > 0, \tag{2.30}$$

where $t, x_1, x_2 \in R, x_1 \neq x_2$, then (1.1) *has at most one T-periodic solution.*

Proof. Multiplying (2.15) by $Z(t)$ and then integrating it from 0 to T, yields that

$$\begin{aligned} a_{2k}\int_0^T\left|Z^{(k)}(t)\right|^2 dt+\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T\left|Z^{(j)}(t)\right|^2 dt &= -\int_0^T Z(t)\big(g(t,u_1(t))-g(t,u_2(t))\big)dt \\ &= -\int_0^T (u_1(t)-u_2(t)) \\ &\quad \times\big(g(t,u_1(t))-g(t,u_2(t))\big)dt. \end{aligned} \tag{2.31}$$

Now the proof proceeds in the same way as in Lemma 2.5. □

Similar to the proof of Lemma 2.7, we can prove the following results.

Lemma 2.8. *Let k be an odd number, $n = 2k + 1$, and*

$$\overline{\Lambda^*} = \left\{ j : j \in \{1, 2, \ldots, k-1\},\ (-1)^{j+1} a_{2j} < 0 \right\}. \tag{2.32}$$

Assume that one of the following conditions is satisfied:

$(\overline{H_3})$ *for $t, x_1, x_2 \in R, x_1 \neq x_2$,*

$$a_{2k} + \sum_{j \in \overline{\Lambda^*}} \left(\frac{T}{2\pi}\right)^{2(k-j)} (-1)^{j+1} a_{2j} > 0, \qquad \big(g(t, x_1) - g(t, x_2)\big)(x_1 - x_2) < 0, \tag{2.33}$$

$(\overline{H_4})$ *there exists a nonnegative constant B such that*

$$a_{2k} > B\left(\frac{T}{\pi}\right)^2 \left(\frac{T}{2\pi}\right)^{2(k-1)}, \qquad a_{2k} + \sum_{j \in \overline{\Lambda^*}} \left(\frac{T}{2\pi}\right)^{2(k-j)} (-1)^{j+1} a_{2j} > B\left(\frac{T}{\pi}\right)^2 \left(\frac{T}{2\pi}\right)^{2(k-1)},$$
$$B(x_1 - x_2)^2 \geq \big(g(t, x_1) - g(t, x_2)\big)(x_1 - x_2) > 0, \tag{2.34}$$

where $t, x_1, x_2 \in R, x_1 \neq x_2$, then (1.1) *has at most one T-periodic solution.*

3. Main Results

Theorem 3.1. *Let k be an even number and $n = 2k$. Assume that one of the following conditions is satisfied:*

$(H_1)^*$ *let (H_1) hold, and there exists a nonnegative constant d_0 such that*

$$\big(g(t, u) - e(t)\big)u > 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.1}$$

$(H_2)^*$ *there exist nonnegative constants d_0 and B such that (H_2) holds,*

$$\big(g(t, u) - e(t)\big)u < 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.2}$$

then (1.1) *has a unique T-periodic solution.*

Proof. From Lemma 2.5, together with $(H_1)^*$ (or $(H_2)^*$), it is easy to see that (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that (1.1) has at least one T-periodic solution. To do this, we shall use Lemma 2.1 with the nonlinearity $\widetilde{f} : \mathbf{R}^{n+1} \to \mathbf{R}$ given by

$$\widetilde{f}\left(t, u, u', u'', \ldots, u^{(n-1)}\right) = e(t) - \sum_{j=1}^{n-1} a_j u^{(j)} - g(t, u). \tag{3.3}$$

For $\lambda \in (0, 1]$, we consider the nth-order differential equation

$$u^{(n)}(t) + \lambda \sum_{j=1}^{n-1} a_j u^{(j)} + \lambda g(t, u(t)) = \lambda e(t). \tag{3.4}$$

Let us show that (i) in Lemma 2.1 is satisfied, which means that there exists $\rho > 0$ such that any possible T-periodic solution u of (3.4) is such that

$$\|u\|_{(n-1)} < \rho. \tag{3.5}$$

Let $\lambda \in (0, 1]$ and let u be a possible T-periodic solution of (3.4). In what follows, C_j denotes a fixed constant independent of λ and u. Integrating (3.4) from 0 to T, it results that

$$\int_0^T \left[g(t, u(t)) - e(t)\right] dt = 0, \tag{3.6}$$

which together with $(H_1)^*$ (or $(H_2)^*$) implies that

$$\exists \xi \in [0, T] : |u(\xi)| < d_0. \tag{3.7}$$

Hence, from (2.9), we have that

$$|u|_\infty \leq d_0 + \frac{1}{2}\sqrt{T}\left(\int_0^T \left|u'(t)\right|^2 dt\right)^{1/2}. \tag{3.8}$$

In view of (2.10), (3.8) implies that

$$|u|_\infty \leq d_0 + \frac{1}{2}\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}. \tag{3.9}$$

It follows that

$$\begin{gathered}
\left|\int_0^T e(t)u(t)dt\right| \leq T d_0 |e|_\infty + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} |e|_\infty \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}, \\
\left|\int_0^T g(t,0)u(t)dt\right| \leq T d_0 \left|g(t,0)\right|_\infty + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} \left|g(t,0)\right|_\infty \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}.
\end{gathered} \tag{3.10}$$

On the other hand, multiplying (3.4) by u and integrating from 0 to T, it follows that

$$\int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda \sum_{j=1}^{k-1} (-1)^j a_{2j} \int_0^T \left|u^{(j)}(t)\right|^2 dt = -\lambda \int_0^T u(t) g(t, u(t)) dt + \lambda \int_0^T e(t) u(t) dt. \quad (3.11)$$

Now suppose that $(H_1)^*$(or $(H_2)^*$) holds, and we will consider two cases as follows.

Case 1. If $(H_1)^*$ holds, using (2.10), (3.10), and (3.11), we have

$$\begin{aligned}
&\left(1+\sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j}\right)\int_0^T \left|u^{(k)}(t)\right|^2 d\mathrm{t} \\
&\quad\le \int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda \sum_{j=1}^{k-1} (-1)^j a_{2j} \int_0^T \left|u^{(j)}(t)\right|^2 dt \\
&\quad\le -\lambda \int_0^T (u(t)-0)\left(g(t,u(t))-g(t,0)\right) dt - \lambda \int_0^T u(t) g(t,0) dt \\
&\qquad + T d_0 |e|_\infty + \frac{1}{2} T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} |e|_\infty \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2} \\
&\quad\le T d_0 \left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \\
&\qquad + \frac{1}{2} T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} \left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}, \quad \text{where } \Lambda \neq \emptyset, \\
&\int_0^T \left|u^{(k)}(t)\right|^2 dt \\
&\quad\le \int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda \sum_{j=1}^{k-1} (-1)^j a_{2j} \int_0^T \left|u^{(j)}(t)\right|^2 dt \\
&\quad\le -\lambda \int_0^T (u(t)-0)\left(g(t,u(t))-g(t,0)\right) dt - \lambda \int_0^T u(t) g(t,0) dt \\
&\qquad + T d_0 |e|_\infty + \frac{1}{2} T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} |e|_\infty \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2} \\
&\quad\le T d_0 \left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \\
&\qquad + \frac{1}{2} T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1} \left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}, \quad \text{where } \Lambda = \emptyset,
\end{aligned} \quad (3.12)$$

which imply that there exists a positive constant C_1 satisfying

$$\int_0^T \left|u^{(k)}(t)\right|^2 dt \le C_1, \qquad \int_0^T \left|u^{(k)}(t)\right| dt \le \sqrt{TC_1}. \tag{3.13}$$

Case 2. If $(H_2)^*$ holds, using (2.9), (2.10), (3.10), and (3.11), we obtain

$$\begin{aligned}
&\left(1+\sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^j a_{2j}\right)\int_0^T \left|u^{(k)}(t)\right|^2 dt \\
&\quad\le \int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|u^{(j)}(t)\right|^2 dt \\
&\quad\le -\lambda\int_0^T (u(t)-0)\left(g(t,u(t))-g(t,0)\right)dt - \lambda\int_0^T u(t)g(t,0)dt \\
&\qquad + Td_0|e|_\infty + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}|e|_\infty\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2} \\
&\quad\le B|u|_2^2 + Td_0\left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \\
&\qquad + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}\left(\left|g(t,0)\right|_\infty + |e|_\infty\right)\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2} \\
&\quad\le B\left(\frac{T}{\pi}\left|u'\right|_2 + \sqrt{T}d_0\right)^2 + Td_0\left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \\
&\qquad + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}\left(\left|g(t,0)\right|_\infty + |e|_\infty\right)\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2} \\
&\quad\le B\left(\frac{T}{\pi}\left(\frac{T}{2\pi}\right)^{k-1}\left|u^{(k)}\right|_2 + \sqrt{T}d_0\right)^2 + Td_0\left(\left|g(t,0)\right|_\infty + |e|_\infty\right) \\
&\qquad + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}\left(\left|g(t,0)\right|_\infty + |e|_\infty\right)\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}, \qquad \text{where } \Lambda \ne \emptyset, \\
&\int_0^T \left|u^{(k)}(t)\right|^2 dt \\
&\quad\le \int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|u^{(j)}(t)\right|^2 dt \\
&\quad\le -\lambda\int_0^T (u(t)-0)\left(g(t,u(t))-g(t,0)\right)dt - \lambda\int_0^T u(t)g(t,0)dt \\
&\qquad + Td_0|e|_\infty + \frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}|e|_\infty\left(\int_0^T \left|u^{(k)}(t)\right|^2 dt\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq B\left(\frac{T}{\pi}\left(\frac{T}{2\pi}\right)^{k-1}\left|u^{(k)}\right|_2+\sqrt{T}d_0\right)^2+Td_0\left(\left|g(t,0)\right|_\infty+|e|_\infty\right)\\
&\quad+\frac{1}{2}T\sqrt{T}\left(\frac{T}{2\pi}\right)^{k-1}\left(\left|g(t,0)\right|_\infty+|e|_\infty\right)\left(\int_0^T\left|u^{(k)}(t)\right|^2dt\right)^{1/2},\quad \text{where } \Lambda=\emptyset,
\end{aligned}\tag{3.14}$$

which together with (H_2) yield that (3.13) holds.

Using (3.9) and (3.13), it follows that there exists C_2 such that

$$|u|_\infty \leq C_2. \tag{3.15}$$

Now, we shall estimate $x^{(j)}$ $(j = 1,2,\ldots,2k-1)$, multiplying (3.4) by $x^{(2k)}$ and integrating from 0 to T, we have

$$\begin{aligned}
\int_0^T\left|u^{(2k)}(t)\right|^2dt+\lambda\sum_{j=1}^{k-1}(-1)^ja_{2j}\int_0^T\left|u^{(k+j)}(t)\right|^2dt &= -\lambda\int_0^T u^{(2k)}(t)g(t,u(t))dt\\
&\quad+\lambda\int_0^T e(t)u^{(2k)}(t)dt
\end{aligned}\tag{3.16}$$

Using (2.10), (3.15), and (3.16), we have

$$\begin{aligned}
&\left(1+\sum_{j\in\Lambda}\left(\frac{T}{2\pi}\right)^{2(k-j)}(-1)^ja_{2j}\right)\int_0^T\left|u^{(2k)}(t)\right|^2dt\\
&\quad\leq\int_0^T\left|u^{(2k)}(t)\right|^2dt+\lambda\sum_{j=1}^{k-1}(-1)^ja_{2j}\int_0^T\left|u^{(k+j)}(t)\right|^2dt\\
&\quad=-\lambda\int_0^T u^{(2k)}(t)g(t,u(t))dt+\lambda\int_0^T e(t)u^{(2k)}(t)dt\\
&\quad\leq\left(\sqrt{T}\sup_{|u|\leq C_2,t\in R}\left|g(t,u)\right|+\sqrt{T}|e|_\infty\right)\left(\int_0^T\left|u^{(2k)}(t)\right|^2dt\right)^{1/2},\quad\text{where } \Lambda\neq\emptyset,\\
&\int_0^T\left|u^{(2k)}(t)\right|^2dt\\
&\quad\leq\int_0^T\left|u^{(2k)}(t)\right|^2dt+\lambda\sum_{j=1}^{k-1}(-1)^ja_{2j}\int_0^T\left|u^{(k+j)}(t)\right|^2dt\\
&\quad\leq\left(\sqrt{T}\sup_{|u|\leq C_2,t\in R}\left|g(t,u)\right|+\sqrt{T}|e|_\infty\right)\left(\int_0^T\left|u^{(2k)}(t)\right|^2dt\right)^{1/2},\quad\text{where } \Lambda=\emptyset,
\end{aligned}\tag{3.17}$$

which imply that there exists a positive constant C_3 satisfying

$$\int_0^T \left|u^{(2k)}(t)\right|^2 dt \leq C_3, \qquad \int_0^T \left|u^{(2k)}(t)\right| dt \leq \sqrt{TC_3}. \tag{3.18}$$

This implies the existence of a constant $\rho > d_0$ such that (3.5) holds.

Now, to show that (ii) in Lemma 2.1 is satisfied, it suffices to remark that

$$F(x) = \int_0^T \left[-g(t,x) + e(t)\right] dt, \quad (x \in \mathbf{R}). \tag{3.19}$$

Hence, from $(H_1)^*$ (or $(H_2)^*$) and $\rho > d_0$, it results that $F(-\rho)F(\rho) < 0$. Then, using Lemma 2.1, it follows that (3.4) has at least one T-periodic solution u satisfying (3.5). This completes the proof.

□

Similar to the proof of Theorem 3.1, from Lemma 2.6, one can prove the following results.

Theorem 3.2. *Let k be an odd number and $n = 2k + 1$. Assume that one of the following conditions is satisfied:*

$(\overline{H_1})^*$ *let $(\overline{H_1})$ hold, and there exists a nonnegative constant d_0 such that*

$$\left(g(t,u) - e(t)\right)u < 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.20}$$

$(\overline{H_2})^*$ *there exist nonnegative constants d_0 and B such that $(\overline{H_2})$ holds,*

$$\left(g(t,u) - e(t)\right)u > 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.21}$$

then (1.1) *has a unique T-periodic solution.*

Theorem 3.3. *Let k be an even number and $n = 2k + 1$. Assume that one of the following conditions is satisfied:*

$(H_3)^*$ *let (H_3) hold, and there exists a nonnegative constant d_0 such that*

$$\left(g(t,u) - e(t)\right)u > 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.22}$$

$(H_4)^*$ *there exist nonnegative constants d_0 and B such that (H_4) holds,*

$$\left(g(t,u) - e(t)\right)u < 0, \quad \forall t \in R,\ |u| \geq d_0, \tag{3.23}$$

then (1.1) *has a unique T-periodic solution.*

Proof. From Lemma 2.7, together with $(H_3)^*$ (or $(H_4)^*$), it is easy to see that (1.1) has at most one T-periodic solution. Thus, to prove Theorem 3.3, it suffices to show that (1.1) has at least one T-periodic solution.

Multiplying (3.4) by $u^{(i)}(i = 0, 2k)$ and integrating from 0 to T, it follows that

$$\begin{aligned}
&\lambda a_{2k}\int_0^T \left|u^{(k)}(t)\right|^2 dt + \lambda\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|u^{(j)}(t)\right|^2 dt \\
&\quad = -\lambda\int_0^T u(t)g(t,u(t))dt + \lambda\int_0^T e(t)u(t)dt, \\
&\lambda a_{2k}\int_0^T \left|u^{(2k)}(t)\right|^2 dt + \lambda\sum_{j=1}^{k-1}(-1)^j a_{2j}\int_0^T \left|u^{(k+j)}(t)\right|^2 dt \\
&\quad = -\lambda\int_0^T u^{(2k)}(t)g(t,u(t))dt + \lambda\int_0^T e(t)u^{(2k)}(t)dt.
\end{aligned} \tag{3.24}$$

Then, from (3.24), by using similar arguments in proof of (3.15) and (3.18), we can obtain that there exists a constant C_5 such that

$$|u|_\infty \le C_5, \qquad \int_0^T \left|u^{(2k)}(t)\right| dt \le C_5, \tag{3.25}$$

$$\int_0^T \left|u^{(j)}(t)\right|^2 dt \le \left(\frac{T}{2\pi}\right)^{2k-j}\int_0^T \left|u^{(2k)}(t)\right|^2 dt \le C_5, \qquad j = 1, 2, \ldots, 2k-1. \tag{3.26}$$

In view of (3.4), (3.25) and (3.26) yield that

$$\begin{aligned}
\int_0^T \left|u^{(2k+1)}(t)\right| dt &= \int_0^T \left|-\lambda\sum_{j=1}^{2k} a_j u^{(j)} - \lambda g(t,u(t)) + \lambda e(t)\right| dt \\
&\le \sum_{j=1}^{2k}\left|a_j\right|\sqrt{T}\left(\int_0^T \left|u^{(j)}\right|^2 dt\right)^{1/2} + T\left(\sup_{|u|\le C_5, t\in R}\left|g(t,u)\right| + |e|_\infty\right) \\
&\le \sum_{j=1}^{2k}\left|a_j\right|\sqrt{TC_5} + T\left(\sup_{|u|\le C_5, t\in R}\left|g(t,u)\right| + |e|_\infty\right),
\end{aligned} \tag{3.27}$$

which together with (2.11) and (3.25) implies the existence of a constant $\rho > d_0$ such that (3.5) holds.

Now the proof proceeds in the same way as in Theorem 3.1. □

Similar to the proof of Theorem 3.3, from Lemma 2.8, we obtain the following.

Theorem 3.4. *Let k be an odd number and $n = 2k+1$. Assume that one of the following conditions is satisfied:*

$(\overline{H_3})^*$ *let $(\overline{H_3})$ hold, and there exists a nonnegative constant d_0 such that*

$$\big(g(t,u) - e(t)\big)u < 0, \qquad \forall t \in R,\ |u| \geq d_0, \tag{3.28}$$

$(\overline{H_4})^*$ *there exist nonnegative constants d_0 and B such that $(\overline{H_4})$ holds,*

$$\big(g(t,u) - e(t)\big)u > 0, \qquad \forall t \in R,\ |u| \geq d_0, \tag{3.29}$$

then (1.1) *has a unique T-periodic solution.*

Remark 3.5. If $\int_0^T e(t)dt = 0$ and $g(t,u)$ satisfies the following condition:

(H_5) there exist $d > 0$ and $\epsilon \in \{-1,1\}$ such that, for any continuous T-periodic function u, we have

$$\begin{aligned} &\epsilon\int_0^T g(t,u(t))dt < 0, \qquad \text{if } \min_R u \geq d, \\ &\epsilon\int_0^T g(t,u(t))dt > 0, \qquad \text{if } \min_R u \leq -d. \end{aligned} \tag{3.30}$$

Moreover, one of conditions (H_1)–(H_4) holds. Then, by using the methods similarly to those used in Theorem 3.1, one may also establish the results similar to those in Theorems 3.1–3.4.

4. Examples and Remarks

Example 4.1. Let $a, b, c : R \to R$ be three continuous, strictly positive, and T-periodic functions, and let $e : R \to R$ be continuous, T-periodic, then the fourth-order differential equation

$$u''''(t) + 400u'''(t) + \left(\frac{1.6\pi}{T}\right)^2 u''(t) + 20u'(t) + a(t)u(t) + b(t)u^3(t) + c(t)u^5(t) = e(t) \tag{4.1}$$

has a unique T-periodic solution. For the proof, it suffices to remark that the function $g(t,u) \equiv a(t)u + b(t)u^3 + c(t)u^5$ satisfies

$$\big(g(t,u) - e(t)\big)u > 0, \qquad \forall t \in R,\ |u| \geq d_0, \tag{4.2}$$

where $d_0 \in \mathbf{R}$ is sufficiently large. Hence, $p = -(1.6\pi/T)^2$ and g satisfy $(H_1)^*$, and the result follows from Theorem 3.1.

Example 4.2. Let $\beta : R \to R$ be continuous, strictly positive, and T-periodic, let $\delta, \tau \in R$ be constants, and let $e : R \to R$ be continuous, T-periodic, then the five-order differential equations

$$u'''''(t) + 100000\sqrt{3}u''''(t) + \delta u'''(t) - 20u''(t) + \tau u'(t) + \beta(t)u^3(t) = e(t) \tag{4.3}$$

have a unique T-periodic solution. For the proof, it suffices to remark that the function $g(t,u) \equiv \beta(t)u^3(t)$ with $a_4 = 100000\sqrt{3}$, $a_3 = \delta, a_2 = -20$, and $a_1 = \tau$ satisfies $(H_3)^*$. Hence, the result follows from Theorem 3.3.

Remark 4.3. Since $g(t,u)$ in Examples 4.1 and 4.2 does not satisfy $(\overline{H})$, the main results in [9–11] and the references therein cannot be applicable to (4.1)–(4.3) to obtain the existence and uniqueness of 2-periodic solutions. Moreover, all the results in this present paper avoid the spectral conditions in [12, 13]. This implies that the results of this paper are new, and they complement previously known results.

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