

## Research Article

# Existence of Mild Solutions for a Class of Fractional Evolution Equations with Compact Analytic Semigroup

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This paper deals with the existence of mild solutions for a class of fractional evolution equations with compact analytic semigroup. We prove the existence of mild solutions, assuming that the nonlinear part satisfies some local growth conditions in fractional power spaces. An example is also given to illustrate the applicability of abstract results.

## 1. Introduction

The differential equations involving fractional derivatives in time have recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economics, and science. Numerous applications can be found in electrochemistry, control, porous media, electromagnetic, see for example, [1–5] and references therein. Hence the study of such equations has become an object of extensive study during recent years, see [6–23] and references therein.

In this paper, we consider the existence of the following fractional evolution equation:

$$\begin{aligned} D^q u(t) + Au(t) &= f(t, u(t), Gu(t)), \quad t \in J = [0, T], \\ u(0) &= x_0, \end{aligned} \tag{1.1}$$

where  $D^q$  is the Caputo fractional derivative of order  $q \in (0, 1)$ ,  $-A$  is the infinitesimal generator of a compact analytic semigroup  $S(\cdot)$  of uniformly bounded linear operators,  $f$  is the nonlinear term and will be specified later, and

$$Gu(t) = \int_0^t K(t,s)u(s)ds \quad (1.2)$$

is a Volterra integral operator with integral kernel  $K \in C(\Delta, R^+)$ ,  $\Delta = \{(t,s) : 0 \leq s \leq t \leq T\}$ ,  $R^+ = [0, +\infty)$ . Throughout this paper, we denote by  $K^* := \max_{(t,s) \in \Delta} K(t,s)$

In some existing articles, the fractional differential equations were treated under the hypothesis that nonlinear term satisfies Lipschitz conditions or linear growth conditions. It is obvious that these conditions are not easy to be verified sometimes. To make the things more applicable, in this work, we will prove the existence of mild solutions for (1.1) under some new conditions. More precisely, the nonlinear term only satisfies some local growth conditions (see conditions  $(H_1)$  and  $(H_2)$ ). These conditions are much weaker than Lipschitz conditions and linear growth conditions. The main techniques used here are fractional calculus, theory of analytic semigroup, and Schauder fixed point theorem.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on the fractional power of the generator of a compact analytic semigroup and the definition of mild solutions of (1.1). In Section 3, we study the existence of mild solutions for (1.1). In Section 4, an example is given to illustrate the applicability of abstract results obtained in Section 3.

## 2. Preliminaries

In this section, we introduce some basic facts about the fractional power of the generator of a compact analytic semigroup and the fractional calculus that are used throughout this paper.

Let  $X$  be a Banach space with norm  $\|\cdot\|$ . Throughout this paper, we assume that  $-A : D(A) \subset X \rightarrow X$  is the infinitesimal generator of a compact analytic semigroup  $S(t)$  ( $t \geq 0$ ) of uniformly bounded linear operator in  $X$ , that is, there exists  $M \geq 1$  such that  $\|S(t)\| \leq M$  for all  $t \geq 0$ . Without loss of generality, let  $0 \in \rho(-A)$ , where  $\rho(-A)$  is the resolvent set of  $-A$ . Then for any  $\alpha > 0$ , we can define  $A^{-\alpha}$  by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{(\alpha-1)} S(t) dt. \quad (2.1)$$

It follows that each  $A^{-\alpha}$  is an injective continuous endomorphism of  $X$ . Hence we can define  $A^\alpha$  by  $A^\alpha := (A^{-\alpha})^{-1}$ , which is a closed bijective linear operator in  $X$ . It can be shown that each  $A^\alpha$  has dense domain and that  $D(A^\beta) \subset D(A^\alpha)$  for  $0 \leq \alpha \leq \beta$ . Moreover,  $A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x$  for every  $\alpha, \beta \in \mathbb{R}$  and  $x \in D(A^\mu)$  with  $\mu := \max\{\alpha, \beta, \alpha + \beta\}$ , where  $A^0 = I$ ,  $I$  is the identity in  $X$ . (For proofs of these facts, we refer to the literature [24–26]).

We denote by  $X_\alpha$  the Banach space of  $D(A^\alpha)$  equipped with norm  $\|x\|_\alpha = \|A^\alpha x\|$  for  $x \in D(A^\alpha)$ , which is equivalent to the graph norm of  $A^\alpha$ . Then we have  $X_\beta \hookrightarrow X_\alpha$  for  $0 \leq \alpha \leq \beta \leq 1$  (with  $X_0 = X$ ), and the embedding is continuous. Moreover,  $A^\alpha$  has the following basic properties.

**Lemma 2.1** (see [24]).  *$A^\alpha$  has the following properties.*

- (i)  $S(t) : X \rightarrow X_\alpha$  for each  $t > 0$  and  $\alpha \geq 0$ .
- (ii)  $A^\alpha S(t)x = S(t)A^\alpha x$  for each  $x \in D(A^\alpha)$  and  $t \geq 0$ .

(iii) For every  $t > 0$ ,  $A^\alpha S(t)$  is bounded in  $X$  and there exists  $M_\alpha > 0$  such that

$$\|A^\alpha S(t)\| \leq M_\alpha t^{-\alpha}. \quad (2.2)$$

(iv)  $A^{-\alpha}$  is a bounded linear operator for  $0 \leq \alpha \leq 1$  in  $X$ .

In the following, we denote by  $C(J, X_\alpha)$  the Banach space of all continuous functions from  $J$  into  $X_\alpha$  with supnorm given by  $\|u\|_C = \sup_{t \in J} \|u(t)\|_\alpha$  for  $u \in C(J, X_\alpha)$ . From Lemma 2.1(iv), since  $A^{-\alpha}$  is a bounded linear operator for  $0 \leq \alpha \leq 1$ , there exists a constant  $C_\alpha$  such that  $\|A^{-\alpha}\| \leq C_\alpha$  for  $0 \leq \alpha \leq 1$ .

For any  $t \geq 0$ , denote by  $S_\alpha(t)$  the restriction of  $S(t)$  to  $X_\alpha$ . From Lemma 2.1(i) and (ii), for any  $x \in X_\alpha$ , we have

$$\begin{aligned} \|S(t)x\|_\alpha &= \|A^\alpha \cdot S(t)x\| = \|S(t) \cdot A^\alpha x\| \leq \|S(t)\| \cdot \|A^\alpha x\| = \|S(t)\| \cdot \|x\|_\alpha, \\ \|S(t)x - x_\alpha\| &= \|A^\alpha \cdot S(t)x - A^\alpha x\| = \|S(t) \cdot A^\alpha x - A^\alpha x\| \longrightarrow 0 \end{aligned} \quad (2.3)$$

as  $t \rightarrow 0$ . Therefore,  $S(t)$  ( $t \geq 0$ ) is a strongly continuous semigroup in  $X_\alpha$ , and  $\|S_\alpha(t)\|_\alpha \leq \|S(t)\|$  for all  $t \geq 0$ . To prove our main results, the following lemma is also needed.

**Lemma 2.2** (see [27]).  $S_\alpha(t)$  ( $t \geq 0$ ) is an immediately compact semigroup in  $X_\alpha$ , and hence it is immediately norm-continuous.

Let us recall the following known definitions in fractional calculus. For more details, see [16–20, 23].

*Definition 2.3.* The fractional integral of order  $\sigma > 0$  with the lower limits zero for a function  $f$  is defined by

$$I^\sigma f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} f(s) ds, \quad t > 0, \quad (2.4)$$

where  $\Gamma$  is the gamma function.

The Riemann-Liouville fractional derivative of order  $n-1 < \sigma < n$  with the lower limits zero for a function  $f$  can be written as

$${}^L D^\sigma f(t) = \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\sigma-1} f(s) ds, \quad t > 0, \quad n \in \mathbb{N}. \quad (2.5)$$

Also the Caputo fractional derivative of order  $n-1 < \sigma < n$  with the lower limits zero for a function  $f \in C^n[0, \infty)$  can be written as

$$D^\sigma f(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t-s)^{n-\sigma-1} f^{(n)}(s) ds, \quad t > 0, \quad n \in \mathbb{N}. \quad (2.6)$$

*Remark 2.4.* (1) The Caputo derivative of a constant is equal to zero.

(2) If  $f$  is an abstract function with values in  $X$ , then integrals which appear in Definition 2.3 are taken in Bochner's sense.

**Lemma 2.5** (see [12]). *A measurable function  $h : J \rightarrow X$  is Bochner integrable if  $\|h\|$  is Lebesgue integrable.*

For  $x \in X$ , we define two families  $\{U(t)\}_{t \geq 0}$  and  $\{V(t)\}_{t \geq 0}$  of operators by

$$U(t)x = \int_0^\infty \eta_q(\theta) S(t^q \theta) x d\theta, \quad V(t)x = q \int_0^\infty \theta \eta_q(\theta) S(t^q \theta) x d\theta, \quad 0 < q < 1, \quad (2.7)$$

where

$$\eta_q(\theta) = \frac{1}{q} \theta^{-1-(1/q)} \rho_q(\theta^{-(1/q)}), \quad \rho_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \quad (2.8)$$

$$\theta \in (0, \infty),$$

$\eta_q$  is a probability density function defined on  $(0, \infty)$ , which has properties  $\eta_q(\theta) \geq 0$  for all  $\theta \in (0, \infty)$  and

$$\int_0^\infty \eta_q(\theta) d\theta = 1, \quad \int_0^\infty \theta \eta_q(\theta) d\theta = \frac{1}{\Gamma(q+1)}. \quad (2.9)$$

The following lemma follows from the results in [7, 11–13].

**Lemma 2.6.** *The operators  $U$  and  $V$  have the following properties.*

(i) *For fixed  $t \geq 0$  and any  $x \in X_\alpha$ , we have*

$$\|U(t)x\|_\alpha \leq M \|x\|_\alpha, \quad \|V(t)x\|_\alpha \leq \frac{qM}{\Gamma(1+q)} \|x\|_\alpha = \frac{M}{\Gamma(q)} \|x\|_\alpha. \quad (2.10)$$

(ii) *The operators  $U(t)$  and  $V(t)$  are strongly continuous for all  $t \geq 0$ .*

(iii)  *$U(t)$  and  $V(t)$  are norm-continuous in  $X$  for  $t > 0$ .*

(iv)  *$U(t)$  and  $V(t)$  are compact operators in  $X$  for  $t > 0$ .*

(v) *For every  $t > 0$ , the restriction of  $U(t)$  to  $X_\alpha$  and the restriction of  $V(t)$  to  $X_\alpha$  are norm-continuous.*

(vi) *For every  $t > 0$ , the restriction of  $U(t)$  to  $X_\alpha$  and the restriction of  $V(t)$  to  $X_\alpha$  are compact operators in  $X_\alpha$ .*

Based on an overall observation of the previous related literature, in this paper, we adopt the following definition of mild solution of (1.1).

*Definition 2.7.* By a mild solution of (1.1), we mean a function  $u \in C(J, X_\alpha)$  satisfying

$$u(t) = U(t)x_0 + \int_0^t (t-s)^{q-1}V(t-s)f(s, u(s), Gu(s))ds \quad (2.11)$$

for all  $t \in J$ .

### 3. Existence of Mild Solutions

In this section, we give the existence theorems of mild solutions of (1.1). The discussions are based on fractional calculus and Schauder fixed point theorem. Our main results are as follows.

**Theorem 3.1.** *Assume that the following condition on  $f$  is satisfied.*

(H<sub>1</sub>) *There exists a constant  $\beta \in [\alpha, 1]$  such that  $f : J \times X_\alpha \times X_\alpha \rightarrow X_\beta$  satisfies:*

- (i) *for each  $(x, y) \in X_\alpha \times X_\alpha$ , the function  $f(\cdot, x, y) : J \rightarrow X_\beta$  is measurable;*
- (ii) *for each  $t \in J$ , the function  $f(t, \cdot, \cdot) : X_\alpha \times X_\alpha \rightarrow X_\beta$  is continuous;*
- (iii) *for any  $r > 0$ , there exists a function  $g_r \in L^\infty(J, \mathbb{R}^+)$  such that*

$$\sup_{\|x\|_\alpha \leq r, \|y\|_\alpha \leq K^*Tr} \|f(t, x, y)\|_\beta \leq g_r(t), \quad t \in J, \quad (3.1)$$

*and there is a constant  $\gamma > 0$  such that*

$$\liminf_{r \rightarrow +\infty} \frac{1}{r} \int_0^t \frac{g_r(s)}{(t-s)^{1-q}} ds \leq \gamma < +\infty. \quad (3.2)$$

*If  $x_0 \in X_\alpha$  and  $MC_{\beta-\alpha}\gamma < \Gamma(q)$ , then (1.1) has at least one mild solution.*

*Proof.* Define an operator  $Q$  by

$$(Qu)(t) = U(t)x_0 + \int_0^t (t-s)^{q-1}V(t-s)f(s, u(s), Gu(s))ds, \quad t \in J. \quad (3.3)$$

It is not difficult to verify that  $Q : C(J, X_\alpha) \rightarrow C(J, X_\alpha)$ . We will use Schauder fixed point theorem to prove that  $Q$  has fixed points in  $C(J, X_\alpha)$ .

For any  $r > 0$ , let  $B_r := \{u \in C(J, X_\alpha) : \|u(t)\|_\alpha \leq r, t \in J\}$ . We first show that there is a positive number  $r$  such that  $Q(B_r) \subset B_r$ . If this were not the case, then for each  $r > 0$ ,

there would exist  $u_r \in B_r$  and  $t_r \in J$  such that  $\|(Qu_r)(t_r)\|_\alpha > r$ . Thus, from Lemma 2.6(i) and (H<sub>1</sub>)(iii), we see that

$$\begin{aligned} r < \|(Qu_r)(t_r)\|_\alpha &\leq \|U(t_r)x_0\|_\alpha + \int_0^{t_r} (t_r - s)^{q-1} \|V(t_r - s)f(s, u_r(s), Gu_r(s))\|_\alpha ds \\ &\leq M\|x_0\|_\alpha + \int_0^{t_r} (t_r - s)^{q-1} \|A^{\alpha-\beta}V(t_r - s) \cdot A^\beta f(s, u_r(s), Gu_r(s))\| ds \quad (3.4) \\ &\leq M\|x_0\|_\alpha + \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^{t_r} (t_r - s)^{q-1} g_r(s) ds. \end{aligned}$$

Dividing on both sides by  $r$  and taking the lower limit as  $r \rightarrow +\infty$ , we have

$$MC_{\beta-\alpha}\gamma \geq \Gamma(q), \quad (3.5)$$

which is a contradiction. Hence  $Q(B_r) \subset B_r$  for some  $r > 0$ .

To complete the proof, we separate the rest of proof into the following three steps.

*Step 1.*  $Q : B_r \rightarrow B_r$  is continuous.

Let  $\{u_n\} \subset B_r$  with  $u_n \rightarrow u \in B_r$  as  $n \rightarrow \infty$ . From the assumption (H<sub>1</sub>)(ii), for each  $s \in J$ , we have

$$f(s, u_n(s), Gu_n(s)) \rightarrow f(s, u(s), Gu(s)) \quad (3.6)$$

as  $n \rightarrow \infty$ . Since  $\|f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s))\|_\beta \leq 2g_r(s)$ , by the Lebesgue dominated convergence theorem, for each  $t \in J$ , we have

$$\begin{aligned} \|(Qu_n)(t) - (Qu)(t)\|_\alpha &\leq \int_0^t (t - s)^{q-1} \|V(t - s)[f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s))]\|_\alpha ds \\ &\leq \int_0^t (t - s)^{q-1} \|A^{\alpha-\beta}V(t - s) \\ &\quad \cdot A^\beta[f(s, u_n(s), Gu_n(s)) - f(s, u(s), Gu(s))]\| ds \\ &\leq \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t - s)^{q-1} \|f(s, u_n(s), Gu_n(s)) \\ &\quad - f(s, u(s), Gu(s))\|_\beta ds \rightarrow 0 \end{aligned} \quad (3.7)$$

as  $n \rightarrow \infty$ , which implies that  $Q : B_r \rightarrow B_r$  is continuous.

Step 2.  $(QB_r)(t) := \{(Qu)(t) : u \in B_r\}$  is relatively compact in  $X_\alpha$  for all  $t \in J$ .

It follows from (2.9) and (3.3) that  $(QB_r)(0) = \{(Qu)(0) : u \in B_r\} = \{x_0\}$  is compact in  $X_\alpha$ . Hence it is only necessary to consider the case of  $t > 0$ . For each  $t \in (0, T]$ ,  $\epsilon \in (0, t)$ , and any  $\delta > 0$ , we define a set  $(Q_{\epsilon, \delta}B_r)(t)$  by

$$(Q_{\epsilon, \delta}B_r)(t) := \{(Q_{\epsilon, \delta}u)(t) : u \in B_r\}, \quad (3.8)$$

where

$$\begin{aligned} (Q_{\epsilon, \delta}u)(t) &= \int_{\delta}^{\infty} \eta_q(\theta) S(t^q \theta) d\theta x_0 \\ &\quad + q \int_0^{t-\epsilon} (t-s)^{q-1} \int_{\delta}^{\infty} \theta \eta_q(\theta) S((t-s)^q \theta) d\theta \cdot f(s, u(s), Gu(s)) ds \\ &= S(\epsilon^q \delta) \left[ \int_{\delta}^{\infty} \eta_q(\theta) S(t^q \theta - \epsilon^q \delta) d\theta x_0 \right. \\ &\quad \left. + q \int_0^{t-\epsilon} (t-s)^{q-1} \int_{\delta}^{\infty} \theta \eta_q(\theta) S((t-s)^q \theta - \epsilon^q \delta) d\theta \cdot f(s, u(s), Gu(s)) ds \right]. \end{aligned} \quad (3.9)$$

Then the set  $(Q_{\epsilon, \delta}B_r)(t)$  is relatively compact in  $X_\alpha$  since by Lemma 2.2, the operator  $S_\alpha(\epsilon^q \delta)$  is compact in  $X_\alpha$ . For any  $u \in B_r$  and  $t \in (0, T]$ , from the following inequality:

$$\begin{aligned} \|(Qu)(t) - (Q_{\epsilon, \delta}u)(t)\|_\alpha &\leq \left\| \int_0^\delta \eta_q(\theta) S(t^q \theta) d\theta x_0 \right\|_\alpha \\ &\quad + \left\| q \int_0^t (t-s)^{q-1} \int_0^\delta \theta \eta_q(\theta) S((t-s)^q \theta) d\theta \right. \\ &\quad \left. \cdot f(s, u(s), Gu(s)) ds \right\|_\alpha \\ &\quad + \left\| q \int_0^t (t-s)^{q-1} \int_\delta^\infty \theta \eta_q(\theta) S((t-s)^q \theta) d\theta \cdot f(s, u(s), Gu(s)) ds \right. \\ &\quad \left. - q \int_0^{t-\epsilon} (t-s)^{q-1} \int_\delta^\infty \theta \eta_q(\theta) S((t-s)^q \theta) d\theta \right. \\ &\quad \left. \cdot f(s, u(s), Gu(s)) ds \right\|_\alpha \\ &\leq M \|x_0\|_\alpha \int_0^\delta \eta_q(\theta) d\theta + qMC_{\beta-\alpha} \|g_r\|_{L^\infty} \int_0^t (t-s)^{q-1} ds \int_0^\delta \theta \eta_q(\theta) d\theta \\ &\quad + qMC_{\beta-\alpha} \|g_r\|_{L^\infty} \int_{t-\epsilon}^t (t-s)^{q-1} ds \int_0^\infty \theta \eta_q(\theta) d\theta \end{aligned}$$

$$\begin{aligned}
&\leq M\|x_0\|_\alpha \int_0^\delta \eta_q(\theta) d\theta + MC_{\beta-\alpha} T^q \|g_r\|_{L^\infty} \\
&\quad \times \int_0^\delta \theta \eta_q(\theta) d\theta + \frac{MC_{\beta-\alpha} \|g_r\|_{L^\infty}}{\Gamma(q+1)} \epsilon^q.
\end{aligned} \tag{3.10}$$

One can obtain that the set  $(QB_r)(t)$  is relatively compact in  $X_\alpha$  for all  $t \in (0, T)$ . And since it is compact at  $t = 0$ , we have the relatively compactness of  $(QB_r)(t)$  in  $X_\alpha$  for all  $t \in J$ .

*Step 3.*  $QB_r := \{Qu \in C(J, X_\alpha) : u \in B_r\}$  is equicontinuous.

For  $\tau \in [0, T)$ , by (3.3), we have

$$\begin{aligned}
\|(Qu)(\tau) - (Qu)(0)\|_\alpha &\leq \|U(\tau)x_0 - x_0\|_\alpha \\
&\quad + \left\| \int_0^\tau (\tau - s)^{q-1} V(\tau - s) f(s, u(s), Gu(s)) ds \right\|_\alpha \\
&\leq \|U(\tau) - I\| \cdot \|x_0\|_\alpha + \frac{MC_{\beta-\alpha} \|g_r\|_{L^\infty}}{\Gamma(q+1)} \tau^q.
\end{aligned} \tag{3.11}$$

Hence it is only necessary to consider the case of  $t > 0$ . For  $0 < t_1 < t_2 \leq T$ , by Lemma 2.1 and Lemma 2.6(i), we have

$$\begin{aligned}
\|(Qu)(t_2) - (Qu)(t_1)\|_\alpha &\leq \|U(t_2)x_0 - U(t_1)x_0\|_\alpha \\
&\quad + \left\| \int_0^{t_2} (t_2 - s)^{q-1} V(t_2 - s) f(s, u(s), Gu(s)) ds \right. \\
&\quad \left. - \int_0^{t_1} (t_1 - s)^{q-1} V(t_1 - s) f(s, u(s), Gu(s)) ds \right\|_\alpha \\
&\leq \|U(t_2) - U(t_1)\| \cdot \|x_0\|_\alpha \\
&\quad + \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} V(t_2 - s) f(s, u(s), Gu(s)) ds \right\|_\alpha \\
&\quad + \left\| \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] V(t_2 - s) f(s, u(s), Gu(s)) ds \right\|_\alpha \\
&\quad + \left\| \int_0^{t_1} (t_1 - s)^{q-1} f(s, u(s), Gu(s)) [V(t_2 - s) - V(t_1 - s)] ds \right\|_\alpha \\
&\leq \|U(t_2) - U(t_1)\| \cdot \|x_0\|_\alpha + \frac{MC_{\beta-\alpha} \|g_r\|_{L^\infty}}{\Gamma(q+1)} (t_2 - t_1)^q \\
&\quad + \frac{MC_{\beta-\alpha} \|g_r\|_{L^\infty}}{\Gamma(q+1)} [t_2^q - t_1^q - (t_2 - t_1)^q]
\end{aligned}$$

$$\begin{aligned}
 & + \left\| \int_0^{t_1} (t_1 - s)^{q-1} [V(t_2 - s) - V(t_1 - s)] \cdot f(s, u(s), Gu(s)) ds \right\|_{\alpha} \\
 & \triangleq I_1 + I_2 + I_3 + I_4.
 \end{aligned} \tag{3.12}$$

From Lemma 2.6(v), we see that  $I_1 \rightarrow 0$  as  $t_2 \rightarrow t_1$  independently of  $u \in B_r$ . From the expressions of  $I_2$  and  $I_3$ , it is clear that  $I_2 \rightarrow 0$  and  $I_3 \rightarrow 0$  as  $t_2 \rightarrow t_1$  independently of  $u \in B_r$ . For any  $\epsilon \in (0, t_1)$ , we have

$$\begin{aligned}
 I_4 & \leq \left\| \int_0^{t_1-\epsilon} (t_1 - s)^{q-1} [V(t_2 - s) - V(t_1 - s)] \cdot f(s, u(s), Gu(s)) ds \right\|_{\alpha} \\
 & + \left\| \int_{t_1-\epsilon}^{t_1} (t_1 - s)^{q-1} [V(t_2 - s) - V(t_1 - s)] \cdot f(s, u(s), Gu(s)) ds \right\|_{\alpha} \\
 & \leq \frac{1}{q} C_{\beta-\alpha} \|g_r\|_{L^\infty} (T^q + \epsilon^q) \sup_{0 \leq s \leq t_1-\epsilon} \|V(t_2 - s) - V(t_1 - s)\| + \frac{2MC_{\beta-\alpha} \|g_r\|_{L^\infty}}{\Gamma(q+1)} \epsilon^q.
 \end{aligned} \tag{3.13}$$

It follows from Lemma 2.6(v) that  $I_4 \rightarrow 0$  as  $t_2 \rightarrow t_1$  and  $\epsilon \rightarrow 0$  independently of  $u \in B_r$ . Therefore, we prove that  $QB_r$  is equicontinuous.

Thus, the Arzela-Ascoli theorem guarantees that  $Q$  is a compact operator. By the Schauder fixed point theorem, the operator  $Q$  has at least one fixed point  $u^*$  in  $B_r$ , which is a mild solution of (1.1). This completes the proof.  $\square$

*Remark 3.2.* In assumption (H<sub>1</sub>)(iii), if the function  $g_r(t)$  is independent of  $t$ , then we can easily obtain a constant  $\gamma > 0$  satisfying (3.2). For example, if there is a constant  $a_f > 0$  such that

$$\|f(t, x, y)\|_{\beta} \leq a_f(1 + \|x\|_{\alpha} + \|y\|_{\alpha}) \tag{3.14}$$

for all  $x, y \in X_{\alpha}$  and  $t \in J$ , then for any  $r > 0$ ,  $x, y \in X_{\alpha}$  with  $\|x\|_{\alpha} \leq r$ ,  $\|y\|_{\alpha} \leq K^*Tr$ , we have  $\|f(t, x, y)\|_{\beta} \leq a_f + a_f(1 + K^*T)r \triangleq g_r(t)$ , where  $g_r(t)$  is independent of  $t$ . Thus,  $\gamma := (1/q)a_fT^q(1 + K^*T) > 0$  is the constant in (3.2).

More generally, if  $f$  satisfies the following condition:

(H2) there is a constant  $\beta \in [\alpha, 1]$  such that  $f : J \times X_{\alpha} \times X_{\alpha} \rightarrow X_{\beta}$  satisfies:

- (i) for each  $(x, y) \in X_{\alpha} \times X_{\alpha}$ , the function  $f(\cdot, x, y) : J \rightarrow X_{\beta}$  is measurable,
- (ii) for any  $r > 0$ , there exists a function  $\ell \in L^\infty(J, \mathbb{R}^+)$  such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\|_{\beta} \leq \ell(t)(\|x_1 - x_2\|_{\alpha} + \|y_1 - y_2\|_{\alpha}) \tag{3.15}$$

for any  $x_i, y_i \in X_{\alpha}$  with  $\|x_i\|_{\alpha} \leq r$ ,  $\|y_i\|_{\alpha} \leq K^*Tr$  ( $i = 1, 2$ ) and  $t \in J$ , then we have the following existence and uniqueness theorem.

**Theorem 3.3.** Assume that the condition  $(H_2)$  is satisfied. If  $x_0 \in X_\alpha$  and  $MC_{\beta-\alpha}T^q(1 + K^*T)\|\ell\|_{L^\infty} < \Gamma(q+1)$ , then (1.1) has a unique mild solution.

*Proof.* For any  $r > 0$ , if  $x, y \in X_\alpha$  with  $\|x\|_\alpha \leq r, \|y\|_\alpha \leq K^*Tr$ , then from  $(H_2)(ii)$ , we have

$$\|f(t, x, y)\|_\beta \leq \ell(t)(1 + K^*T)r + b(t) \triangleq g_r(t), \quad (3.16)$$

where  $b(t) = \|f(t, 0, 0)\|_\beta$ . Therefore, the condition  $(H_1)(iii)$  is satisfied with  $\gamma = ((1 + K^*T)T^q\|\ell\|_{L^\infty})/q$ . By Theorem 3.1, (1.1) has at least one mild solution  $u^* \in B_r$ .

Let  $u_1, u_2 \in B_r$  be the solutions of (1.1). We show that  $u_1 \equiv u_2$ . Since  $u_1(t) = (Qu_1)(t)$  and  $u_2(t) = (Qu_2)(t)$  for all  $t \in J$ , we have

$$\begin{aligned} \|u_1(t) - u_2(t)\|_\alpha &= \|(Qu_1)(t) - (Qu_2)(t)\|_\alpha \\ &\leq \int_0^t (t-s)^{q-1} \|V(t-s)[f(s, u_1(s), Gu_1(s)) - f(s, u_2(s), Gu_2(s))]\|_\alpha ds \\ &= \int_0^t (t-s)^{q-1} \|A^{\alpha-\beta}V(t-s) \cdot A^\beta[f(s, u_1(s), Gu_1(s)) - f(s, u_2(s), Gu_2(s))]\| ds \\ &\leq \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \cdot \|f(s, u_1(s), Gu_1(s)) - f(s, u_2(s), Gu_2(s))\|_\beta ds \\ &\leq \frac{MC_{\beta-\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \cdot \ell(s)(\|u_1(s) - u_2(s)\|_\alpha + \|Gu_1(s) - Gu_2(s)\|_\alpha) ds \\ &\leq \frac{MC_{\beta-\alpha}\|\ell\|_{L^\infty}}{\Gamma(q)}(1 + K^*T) \int_0^t (t-s)^{q-1} \cdot \|u_1(s) - u_2(s)\|_\alpha ds. \end{aligned} \quad (3.17)$$

By using the Gronwall-Bellman inequality (see [14, Theorem 1]), we can deduce that  $\|u_1(t) - u_2(t)\|_\alpha = 0$  for all  $t \in J$ , which implies that  $u_1 \equiv u_2$ . Hence (1.1) has a unique mild solution  $u^* \in B_r$ . This completes the proof.  $\square$

*Remark 3.4.* In Theorem 3.3, we only assume that  $f$  satisfies a local Lioschitz condition (see condition  $(H_2)$ ), and an existence and uniqueness result is obtained. If  $f(t, u, v) \equiv f(t, u) : J \times X_\alpha \rightarrow X$ , then the assumption  $(H_2)$  deletes the linear growth condition (3) of assumption  $(Hf)$  in [12]. Therefore, the Theorem 3.3 extends and improves the main result in [12].

#### 4. An Example

Assume that  $X = L^2[0, \pi]$  equipped with its natural norm and inner product defined, respectively, for all  $u, v \in L^2[0, \pi]$ , by

$$\|u\|_X = \left( \int_0^\pi |u(x)|^2 dx \right)^{1/2}, \quad \langle u, v \rangle = \int_0^\pi u(x)\overline{v(x)} dx. \quad (4.1)$$

Consider the following fractional partial differential equation:

$$\begin{aligned} \frac{\partial^{1/2}}{\partial t^{1/2}} u(x, t) - \frac{\partial^2}{\partial x^2} u(x, t) &= g\left(x, t, u(x, t), \int_0^t K(t, s) u(x, s) ds\right), \quad t \in [0, T], x \in [0, \pi] \\ u(0, t) = u(\pi, t) &= 0, \quad t \in [0, T], \\ u(x, 0) &= u_0(x), \quad x \in [0, \pi], \end{aligned} \tag{4.2}$$

where  $T > 0$  is a constant.

Let the operator  $A : D(A) \subset X \rightarrow X$  be defined by

$$D(A) := \{v \in X : v'' \in X, v(0) = v(\pi) = 0\}, \quad Au = -\frac{\partial^2 u}{\partial x^2}. \tag{4.3}$$

It is well known that  $A$  has a discrete spectrum with eigenvalues of the form  $n^2, n \in \mathbb{N}$ , and corresponding normalized eigenfunctions given by  $z_n = \sqrt{(2/\pi)} \sin(nx)$ . Moreover,  $-A$  generates a compact analytic semigroup  $S(t)$  ( $t \geq 0$ ) in  $X$ , and

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n. \tag{4.4}$$

It is not difficult to verify that  $\|S(t)\|_{\mathcal{L}(X)} \leq e^{-t}$  for all  $t \geq 0$ . Hence, we take  $M = 1$ .

The following results are also well known.

(I) The operator  $A$  can be written as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n \tag{4.5}$$

for every  $u \in D(A)$ .

(II) The operator  $A^{1/2}$  is given by

$$A^{1/2}u = \sum_{n=1}^{\infty} n \langle u, z_n \rangle z_n \tag{4.6}$$

for each  $u \in D(A^{1/2}) := \{v \in X : \sum_{n=1}^{\infty} n < v, z_n > z_n \in X\}$  and  $\|A^{-(1/2)}\|_{\mathcal{L}(X)} = 1$ .

**Lemma 4.1** (see [28]). *If  $m \in D(A^{1/2})$ , then  $m$  is absolutely continuous,  $m' \in X$  and  $\|m'\|_X = \|A^{1/2}m\|_X$ .*

Let  $X_{1/2} = (D(A^{1/2}), \|\cdot\|_{1/2})$ , where  $\|x\|_{1/2} := \|A^{1/2}x\|_X$  for all  $x \in D(A^{1/2})$ . Assume that  $g : [0, \pi] \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies the following conditions.

- (i) For each  $(x, t) \in [0, \pi] \times [0, T]$ , the function  $g(x, t, \cdot, \cdot)$  is continuous.
- (ii) For each  $(\xi, \eta) \in \mathbb{R}^2$ , the function  $g(\cdot, \cdot, \xi, \eta)$  is measurable.

(iii) For each  $t \in [0, T]$  and  $\xi, \eta \in \mathbb{R}$ ,  $g(\cdot, t, \xi, \eta)$  is differentiable, and  $(\partial/\partial x)g(x, t, \xi, \eta) \in X$ .

(iv)  $g(0, \cdot, \cdot, \cdot) = g(\pi, \cdot, \cdot, \cdot) = 0$ .

(v) There exist the functions  $\ell_1, \ell_0 \in L^\infty([0, T], \mathbb{R}^+)$  such that

$$\left| \frac{\partial}{\partial x} g(x, t, \xi, \eta) \right| \leq \ell_1(t)(|\xi| + |\eta|) + \ell_0(t) \quad (4.7)$$

for all  $(x, t, \xi, \eta) \in [0, \pi] \times [0, T] \times \mathbb{R} \times \mathbb{R}$ .

Define  $f(t, u(t), Gu(t))(x) = g(x, t, u(x, t), \int_0^t K(t, s)u(x, s)ds)$ . Then, for each  $\phi \in X_{1/2}$ , from assumptions (iii) and (iv), we have

$$\begin{aligned} \langle f(t, \phi, G\phi), z_n \rangle &= \int_0^\pi g\left(x, t, \phi(x, t), \int_0^t K(t, s)\phi(x, s)ds\right) \cdot \sqrt{\frac{2}{\pi}} \sin(nx) dx \\ &= \frac{1}{n} \int_0^\pi \left( \frac{\partial}{\partial x} g\left(x, t, \phi(x, t), \int_0^t K(t, s)\phi(x, s)ds\right) \right) \cdot \sqrt{\frac{2}{\pi}} \cos(nx) dx. \end{aligned} \quad (4.8)$$

This implies from (II) that  $f : [0, T] \times X_{1/2} \times X_{1/2} \rightarrow X_{1/2}$ . Moreover, for any  $r > 0$ , by Minkowski inequality, assumption (v) and Lemma 4.1, we have

$$\begin{aligned} \sup_{\|\phi\|_{1/2} \leq r} \|f(t, \phi, G\phi)\|_{1/2} &= \sup_{\|\phi\|_{1/2} \leq r} \left\| \frac{\partial}{\partial x} g\left(x, t, \phi(x, t), \int_0^t K(t, s)\phi(x, s)ds\right) \right\|_X \\ &= \sup_{\|\phi\|_{1/2} \leq r} \left( \int_0^\pi \left| \frac{\partial}{\partial x} g\left(x, t, \phi(x, t), \int_0^t K(t, s)\phi(x, s)ds\right) \right|^2 dx \right)^{1/2} \\ &\leq \sup_{\|\phi\|_{1/2} \leq r} \left( \int_0^\pi \left[ \ell_1(t) \left( |\phi(x, t)| + \left| \int_0^t K(t, s)\phi(x, s)ds \right| \right) \right. \right. \\ &\quad \left. \left. + \ell_0(t) \right]^2 dx \right)^{1/2} \\ &\leq \sup_{\|\phi\|_{1/2} \leq r} [\ell_1(t)(\|\phi\|_X + K^*T\|\phi\|_X) + \ell_0(t)] \\ &= \sup_{\|\phi\|_{1/2} \leq r} \left[ (1 + K^*T)\ell_1(t) \|A^{-(1/2)} \cdot A^{1/2}\phi\|_X + \ell_0(t) \right] \\ &\leq (1 + K^*T)r\ell_1(t) + \ell_0(t) \triangleq g_r(t). \end{aligned} \quad (4.9)$$

Therefore,  $f$  satisfies the condition  $(H_1)$  with  $\gamma = 2T^{1/2}(1 + K^*T)\|\ell_1\|_{(L^\infty)}$ . Thus, (4.2) has at least one mild solution provided that  $\gamma < \sqrt{\pi}$  due to Theorem 3.1.

Assume furthermore that the function  $g$  satisfies the following:

(vi) for any  $r > 0$ , there exists a function  $\ell_2 \in L^\infty([0, T], \mathbb{R}^+)$  such that

$$\left| \frac{\partial}{\partial x} g(x, t, \xi_1, \eta_1) - \frac{\partial}{\partial x} g(x, t, \xi_2, \eta_2) \right| \leq \ell_2(t) (|\xi_1 - \xi_2| + |\eta_1 - \eta_2|) \quad (4.10)$$

for  $(x, t, \xi_1, \eta_1), (x, t, \xi_2, \eta_2) \in [0, \pi] \times [0, T] \times \mathbb{R} \times \mathbb{R}$  with  $|\xi_i| \leq r$  and  $|\eta_i| \leq K^*Tr, i = 1, 2$ .

Then for each  $\phi_1, \phi_2 \in X_{1/2}$ , by Lemma 4.1, we have

$$\begin{aligned} \|f(t, \phi_1, G\phi_1) - f(t, \phi_2, G\phi_2)\|_{1/2} &= \|A^{1/2}[f(t, \phi_1, G\phi_1) - f(t, \phi_2, G\phi_2)]\|_X \\ &= \left\| \frac{\partial}{\partial x} g\left(x, t, \phi_1(x, t), \int_0^t K(t, s)\phi_1(x, s)ds\right) \right. \\ &\quad \left. - \frac{\partial}{\partial x} g\left(x, t, \phi_2(x, t), \int_0^t K(t, s)\phi_2(x, s)ds\right) \right\|_X \\ &= \left( \int_0^\pi \left| \frac{\partial}{\partial x} g\left(x, t, \phi_1(x, t), \int_0^t K(t, s)\phi_1(x, s)ds\right) \right. \right. \\ &\quad \left. \left. - \frac{\partial}{\partial x} g\left(x, t, \phi_2(x, t), \int_0^t K(t, s)\phi_2(x, s)ds\right) \right|^2 dx \right)^{1/2} \\ &\leq \left( \int_0^\pi \ell_2(t) \left( |\phi_1(x, t) - \phi_2(x, t)| \right. \right. \\ &\quad \left. \left. + \left| \int_0^t K(t, s)\phi_1(x, s)ds \right. \right. \right. \\ &\quad \left. \left. \left. - \int_0^t K(t, s)\phi_2(x, s)ds \right| \right)^2 dx \right)^{1/2} \\ &\leq \ell_2(t) \left[ \left( \int_0^\pi |\phi_1(x, t) - \phi_2(x, t)|^2 dx \right)^{1/2} \right. \\ &\quad \left. + \left( \int_0^\pi \left| \int_0^t K(t, s)\phi_1(x, s)ds \right. \right. \right. \\ &\quad \left. \left. \left. - \int_0^t K(t, s)\phi_2(x, s)ds \right|^2 dx \right)^{1/2} \right] \end{aligned}$$

$$\begin{aligned}
&= \ell_2(t) (\|\phi_1 - \phi_2\|_X + \|G\phi_1 - G\phi_2\|_X) \\
&= \ell_2(t) \left( \|A^{-(1/2)} \cdot A^{1/2}(\phi_1 - \phi_2)\|_X \right. \\
&\quad \left. + \|A^{-(1/2)} \cdot A^{1/2}(G\phi_1 - G\phi_2)\|_X \right) \\
&\leq \ell_2(t) \left( \|A^{1/2}(\phi_1 - \phi_2)\|_X + \|A^{1/2}(G\phi_1 - G\phi_2)\|_X \right) \\
&= \ell_2(t) \left( \|\phi_1 - \phi_2\|_{1/2} + \|G\phi_1 - G\phi_2\|_{1/2} \right).
\end{aligned} \tag{4.11}$$

This shows that  $f$  satisfies the condition  $(H_2)$ . Hence by Theorem 3.3, the mild solution of (4.2) is unique.

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