Research Article

The Uniqueness of Analytic Functions on Annuli Sharing Some Values

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The purpose of this paper is to deal with the shared values and uniqueness of analytic functions on annulus. Two theorems about analytic functions on annulus sharing four distinct values are obtained, and these theorems are improvement of the results given by Cao and Yi.

1. Introduction

In this paper, we will study the uniqueness problem of analytic functions in the field of complex analysis and adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained (see [1–3]).

We use \( \mathbb{C} \) to denote the open complex plane, \( \overline{\mathbb{C}} \) to denote the extended complex plane, and \( \mathbb{X} \) to denote the subset of \( \mathbb{C} \). For \( a \in \overline{\mathbb{C}} \), we say that \( f(z) - a \) and \( g(z) - a \) have the same zeros with the same multiplicities (ignoring multiplicities) in \( \mathbb{X} \) (or \( \mathbb{C} \)) if two meromorphic functions \( f \) and \( g \) share the value \( a \) CM (IM) in \( \mathbb{X} \) (or \( \mathbb{C} \)). In addition, we also use \( f = a \equiv g = a \) in \( \mathbb{X} \) (or \( \mathbb{C} \)) to express that \( f \) and \( g \) share the value \( a \) CM in \( \mathbb{X} \) (or \( \mathbb{C} \)), \( f = a \iff g = a \) in \( \mathbb{X} \) (or \( \mathbb{C} \)) to express that \( f \) and \( g \) share the value \( a \) IM in \( \mathbb{X} \) (or \( \mathbb{C} \)), and \( f = a \Rightarrow g = a \) in \( \mathbb{X} \) (or \( \mathbb{C} \)) to express that \( f = a \) implies \( g = a \) in \( \mathbb{X} \) (or \( \mathbb{C} \)).

In 1929, Nevanlinna (see [4]) proved the following well-known theorem.

**Theorem 1.1** (see [4]). If \( f \) and \( g \) are two nonconstant meromorphic functions that share five distinct values \( a_1, a_2, a_3, a_4, \text{ and } a_5 \) IM in \( \mathbb{C} \), then \( f(z) \equiv g(z) \).
After his theorem, the uniqueness theory of meromorphic functions sharing values in the whole complex plane attracted many investigations (see [2]). In 2003, Zheng [5] studied the uniqueness problem under the condition that five values are shared in some angular domain in $\mathbb{C}$. There were many results in the field of the uniqueness with shared values in the complex plane and angular domain, see ([5–12]). The whole complex plane $\mathbb{C}$ and angular domain all can be regarded as simply connected region. Thus, it is interesting to consider the uniqueness theory of meromorphic functions in the multiply connected region. Here, we will mainly study the uniqueness of meromorphic functions in doubly connected domains of complex plane $\mathbb{C}$. By the doubly connected mapping theorem [13] each doubly connected domain is conformally equivalent to the annulus $\{ z : r < |z| < R \}$, $0 \leq r < R \leq +\infty$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case the homothety $z \mapsto z/\sqrt{rR}$ reduces the given domain to the annulus $\{ z : 1/R_0 < |z| < R_0 \}$, where $R_0 = \sqrt{Rr}$. Thus, every annulus is invariant with respect to the inversion $z \mapsto 1/z$ in two cases.

In 2005, Krystiyanyn and Kondratyuk [14, 15] proposed the Nevanlinna theory for meromorphic functions on annuli (see also [16]). We will show the basic notions of the Nevanlinna theory on annuli in the next section. In 2009, Cao et al. [17, 18] investigated the uniqueness of meromorphic functions on annuli sharing some values and some sets and obtained an analog of Nevanlinna’s famous five-value theorem as follows.

**Theorem 1.2** (see [18, Theorem 3.2]). Let $f_1$ and $f_2$ be two transcendental or admissible meromorphic functions on the annulus $\mathcal{A} = \{ z : 1/R_0 < |z| < R_0 \}$, where $1 < R_0 \leq +\infty$. Let $a_j$ ($j = 1, 2, 3, 4, 5$) be five distinct complex numbers in $\mathbb{C}$. If $f_1, f_2$ share $a_j$ IM for $j = 1, 2, 3, 4, 5$, then $f_1(z) \equiv f_2(z)$.

**Remark 1.3.** For the case $R_0 = +\infty$, the assertion was proved by Kondratyuk and Laine [16].

From Theorem 1.2, we can get the following results easily.

**Theorem 1.4.** Under the assumptions of Theorem 1.2, if $f_1, f_2$ are two transcendental or admissible analytic functions on annulus $\mathcal{A}$ and $f_1, f_2$ share $a_j$ IM for $j = 1, 2, 3, 4$, then $f_1(z) \equiv f_2(z)$.

In fact, we will prove some general theorems on the uniqueness of analytic functions on the annuli sharing four values in this paper (see Section 3), and these theorems improve Theorem 1.4.

### 2. Basic Notions in the Nevanlinna Theory on Annuli

Let $f$ be a meromorphic function on the annulus $\mathcal{A} = \{ z : 1/R_0 < |z| < R_0 \}$, where $1 < R < R_0 \leq +\infty$. We recall the classical notations of the Nevanlinna theory as follows:

\[
N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R,
\]

\[
m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(\Re^{\theta}) \right| d\theta,
\]

\[
T(R, f) = N(R, f) + m(R, f),
\]

where $n(t, f)$ is the number of zeros of $f$ in $|z| < R$, counting multiplicity.

$N(R, f)$ is the counting function, $m(R, f)$ is the logarithmic growth function and $T(R, f)$ is the characteristic function.
where $\log^\prime x = \max\{\log x, 0\}$ and $n(t, f)$ is the counting function of poles of the function $f$ in $\{z : |z| \leq t\}$. We here show the notations of the Nevanlinna theory on annuli. Let

$$\begin{align*}
N_1(R, f) &= \int_{1/R}^1 \frac{n_1(t, f)}{t} dt, \quad N_2(R, f) = \int_1^R \frac{n_2(t, f)}{t} dt, \\
m_0(R, f) &= m(R, f) + m\left(\frac{1}{R}, f\right), \quad N_0(R, f) = N_1(R, f) + N_2(R, f),
\end{align*}$$

(2.2)

where $n_1(t, f)$ and $n_2(t, f)$ are the counting functions of poles of the function $f$ in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. The Nevanlinna characteristic of $f$ on the annulus $A$ is defined by

$$T_0(R, f) = m_0(R, f) - 2m(1, f) + N_0(R, f)$$

(2.3)

and has the following properties.

**Proposition 2.1** (see [14]). Let $f$ be a nonconstant meromorphic function on the annulus $A = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$. Then,

(i) $T_0(R, f) = T_0(R, 1/f),$

(ii) $\max\{T_0(R, f_1 \cdot f_2), T_0(R, f_1 / f_2), T_0(R, f_1 + f_2)\} \leq T_0(R, f_1) + T_0(R, f_2) + O(1).$

By Proposition 2.1, the first fundamental theorem on the annulus $A$ is immediately obtained.

**Theorem 2.2** (see [14] (the first fundamental theorem)). Let $f$ be a nonconstant meromorphic function on the annulus $A = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$. Then

$$T_0\left(R, \frac{1}{f - a}\right) = T_0(R, f) + O(1)$$

(2.4)

for every fixed $a \in \mathbb{C}$.

Khristyanyn and Kondratyuk also obtained the lemma on the logarithmic derivative on the annulus $A$.

**Theorem 2.3** (see [15] (lemma on the logarithmic derivative)). Let $f$ be a nonconstant meromorphic function on the annulus $A = \{z : 1/R_0 < |z| < R_0\}$, where $R_0 \leq +\infty$, and let $\lambda > 0$. Then,

(i) in the case $R_0 = +\infty$,

$$m_0\left(R, \frac{f^1}{f}\right) = O(\log(RT_0(R, f)))$$

(2.5)

for $R \in (1, +\infty)$ except for the set $\Delta$ such that $\int_{\Delta} R^{1-1}dR < +\infty$;
(ii) if $R_0 < +\infty$, then

$$m_0(R, f) = O \left( \frac{\log \left( \frac{T_0(R, f)}{R_0 - R} \right)}{R} \right)$$

(2.6)

for $R \in (1, R_0)$ except for the set $\Delta'_{R_0}$ such that $\int_{\Delta'_{R_0}} dR/(R_0 - R)^{\lambda-1} < +\infty$.

We denote the deficiency of $a \in \mathbb{C} = \mathbb{C} \cup \{\infty\}$ with respect to a meromorphic function $f$ on the annulus $A$ by

$$\delta_0(a, f) = \delta_0(0, f - a) = \liminf_{r \to R_0} \frac{m_0(r, 1/(f - a))}{T_0(r, f)} = 1 - \limsup_{r \to R_0} \frac{N_0(r, 1/(f - a))}{T_0(r, f)}$$

(2.7)

and denote the reduced deficiency by

$$\Theta_0(a, f) = \Theta_0(0, f - a) = 1 - \limsup_{r \to R_0} \frac{N_0(r, 1/(f - a))}{T_0(r, f)},$$

(2.8)

where

$$N_0 \left( r, 1 \over f - a \right) = N_1 \left( r, 1 \over f - a \right) + N_2 \left( r, 1 \over f - a \right)$$

$$= \int_{1/R}^{1} \frac{\overline{N}_1(t, 1/(f - a))}{t} \, dt + \int_{1}^{R} \frac{\overline{N}_2(t, 1/(f - a))}{t} \, dt$$

(2.9)

in which each zero of the function $f - a$ is counted only once. In addition, we use $\overline{N}_1^k(t, 1/(f - a))$ (or $\overline{N}_2^k(t, 1/(f - a))$) to denote the counting function of poles of the function $1/(f - a)$ with multiplicities $\leq k$ (or $> k$) in $|z| = 1$, each point counted only once. Similarly, we can give the notations $\overline{N}_1^k(t, f), \overline{N}_2^k(t, f), \overline{N}_2^k(t, f), \overline{N}_1^k(t, f), \overline{N}_2^k(t, f)$, and $\overline{N}_0^k(t, f)$.

Khristiyanyn and Kondratyuk [15] first obtained the second fundamental theorem on the annulus $A$. Later, Cao et al. [18] introduced other forms of the second fundamental theorem on annuli as follows.

**Theorem 2.4** (see [18, Theorem 2.3] (the second fundamental theorem)). Let $f$ be a nonconstant meromorphic function on the annulus $A = \{z : 1/R_0 < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. Let $a_1, a_2, \ldots, a_q$ be $q$ distinct complex numbers in the extended complex plane $\overline{\mathbb{C}}$. Let $\lambda \geq 0$. Then,

(i) $(q - 2)T_0(R, f) < \sum_{j=1}^{q} N_0(R, 1/(f - a_j)) - N_0^{(1)}(R, f) + S(R, f)$,

(ii) $(q - 2)T_0(R, f) < \sum_{j=1}^{q} N_0(R, 1/(f - a_j)) + S(R, f),

where

$$N_0^{(1)}(R, f) = N_0 \left( R, \frac{1}{f'} \right) + 2N_0(R, f) - N_0(R, f'),$$

(2.10)
and (i) in the case \( R_0 = +\infty \),

\[
S(R, f) = O((\log(RT_0(R, f)))
\]

for \( R \in (1, +\infty) \) except for the set \( \Delta_R \) such that \( \int_{\Delta_R} R^{1-1}dR < +\infty \); (ii) if \( R_0 < +\infty \), then

\[
S(R, f) = O\left(\log\left(\frac{T_0(R, f)}{R_0 - R}\right)\right)
\]

for \( R \in (1, R_0) \) except for the set \( \Delta'_R \) such that \( \int_{\Delta'_R} dR / (R_0 - R)^{1-1} < +\infty \).

Definition 2.5. Let \( f(z) \) be a nonconstant meromorphic function on the annulus \( \mathbb{A} = \{ z : 1/R_0 < |z| < R_0 \} \), where \( 1 < R_0 \leq +\infty \). The function \( f \) is called a transcendental or admissible meromorphic function on the annulus \( \mathbb{A} \) provided that

\[
\limsup_{R \to \infty} \frac{T_0(R, f)}{\log R} = \infty, \quad 1 < R < R_0 = +\infty,
\]

or

\[
\limsup_{R \to R_0^-} \frac{T_0(R, f)}{- \log(R_0 - R)} = \infty, \quad 1 < R < R_0 < +\infty,
\]

respectively.

Thus, for a transcendental or admissible meromorphic function on the annulus \( \mathbb{A} \), \( S(R, f) = o(T_0(R, f)) \) holds for all \( 1 < R < R_0 \) except for the set \( \Delta_R \) or the set \( \Delta'_R \) mentioned in Theorem 2.3, respectively.

3. The Main Theorems and Some Lemmas

Now we show our main results, which improve Theorem 1.4.

Theorem 3.1. Let \( f, g \) be two analytic functions on the annulus \( \mathbb{A} = \{ z : 1/R_0 < |z| < R_0 \} \), where \( 1 < R_0 \leq +\infty \), and let \( a_j \in \mathbb{C} (j = 1, 2, 3, 4) \) be four distinct values. If \( f \) and \( g \) share the two distinct values \( a_1, a_2 \) CM in \( \mathbb{A} \) and \( f = a_1 \Rightarrow g = a_3 \) in \( \mathbb{A} \) and \( f = a_4 \Rightarrow g = a_4 \) in \( \mathbb{A} \), and \( f \) is transcendental or admissible on \( \mathbb{A} \), then \( f(z) \equiv g(z) \).

Theorem 3.2. Under the assumptions of Theorem 3.1, with CM replaced by IM, we have either \( f(z) \equiv g(z) \) or

\[
f \equiv \frac{a_3g - a_1a_2}{g - a_4},
\]

and \( a_1 + a_2 = a_3 + a_4, a_3, \) and \( a_4 \) are exceptional values of \( f \) and \( g \) in \( \mathbb{A} \), respectively.
Remark 3.3. It is easily seen that Theorems 3.1 and 3.2 are improvement of Theorem 1.4.

To prove the above theorems, we need some lemmas as follows.

Lemma 3.4. Let $f, g$ be two distinct analytic functions on the annulus $\mathbb{A} = \{ z : 1/R_0 < |z| < R_0 \}$, where $1 < R_0 \leq +\infty$, and let $a_j \in \mathbb{C}$ ($j = 1, 2, 3, 4$) be four distinct complex numbers. If $f = a_j \Rightarrow g = a_j$ in $\mathbb{A}$ for $j = 1, 2, 3, 4$ and if $f$ is transcendental or admissible on $\mathbb{A}$, then $g$ is also transcendental or admissible.

Proof. By the assumption of Lemma 3.4 and applying Theorem 2.4(ii), we can get

$$3T_0(R, f) \leq \sum_{j=1}^{4} \overline{N}_0 \left( R, \frac{1}{f - a_j} \right) + S(R, f)$$

$$\leq \sum_{j=1}^{4} \overline{N}_0 \left( R, \frac{1}{g - a_j} \right) + S(R, f)$$

$$\leq 4T_0(R, g) + S(R, f).$$

Therefore

$$T_0(R, f) \leq 4T_0(R, g) + o(T_0(R, f))$$

holds for all $1 < R < R_0$ except for the set $\Delta_R$ or the set $\Delta'_R$ mentioned in Theorem 2.3, respectively. Then, from Definition 2.5, we get that $g$ is transcendental or admissible on $\mathbb{A}$. □

Lemma 3.5. Suppose that $f$ is a transcendental or admissible meromorphic function on the annulus $\mathbb{A} = \{ z : 1/R_0 < |z| < R_0 \}$, where $1 < R_0 \leq +\infty$. Let $P(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p (a_0 \neq 0)$ be a polynomial of $f$ with degree $p$, where the coefficients $a_j (j = 0, 1, \ldots, p)$ are constants, and let $b_j (j = 1, 2, \ldots, q)$ be $q (q \geq p + 1)$ distinct finite complex numbers. Then,

$$m_0 \left( R, \frac{P(f) \cdot f'}{(f - b_1)(f - b_2) \cdots (f - b_q)} \right) = S(R, f).$$

Proof. From Theorem 2.3 and the definition of $m_0(R, f)$, transcendental and admissible function, we can get this lemma by using the same argument as in Lemma 4.3 in [2]. □

Lemma 3.6. Let $f, g$ be two distinct analytic functions on the annulus $\mathbb{A} = \{ z : 1/R_0 < |z| < R_0 \}$, where $1 < R_0 \leq +\infty$. Suppose that $f$ and $g$ share $a_1, a_2, 1M$ in $\mathbb{A}$, and let $a_3 \Rightarrow g = a_3$ in $\mathbb{A}$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{A}$, and $a_j \in \mathbb{C}$ ($j = 1, 2, 3, 4$) are four distinct finite complex numbers. If $f$ is a transcendental or admissible function on $\mathbb{A}$, then $g$ is also transcendental or admissible, and

(i) $T_0(R, g) = 2T_0(R, f) + S(R),$
(ii) $T_0(R, f - g) = 3T_0(R, f) + S(R);$
(iii) $T_0(R, f) = \overline{N}_0(R, 1/(f - a_3)) + \overline{N}_0(R, 1/(f - a_4)) + S(R),$
(iv) $T_0(R, f) = \overline{N}_0(R, 1/(f - a_j)) + S(R), j = 1, 2,
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(v) $T_0(R, g) = \overline{N}_0(R, 1/(g - a_j)) + S(R), j = 3, 4,$

(vi) $T_0(R, f') = T_0(R, f) + S(R), T_0(R, g') = T_0(R, g) + S(R),$

where $S(R) := S(R, f) = S(R, g).$

Proof. By the assumption of this lemma and by Theorem 2.4(ii), we have $T_0(R, f) \leq 3T_0(R, g) + S(R, f)$ and $T_0(R, g) \leq 3T_0(R, f) + S(R, g).$ Thus, we can get $S(R, f) = S(R, g).$

Let

$$\eta := \frac{f'g'(f - g)}{(f - a_3)(f - a_4)(g - a_1)(g - a_2)}. \quad (3.5)$$

From the conditions of this lemma, we can get that $\eta$ is analytic on $\mathbb{A}$ and $\eta \neq 0$ unless $f \equiv g.$ By Lemma 3.5, we have $m_0(R, \eta) = S(R, f) + S(R, g) = S(R).$ Thus, we can get $S(R, \eta) = S(R).$

Since $f, g$ are two nonconstant analytic functions on annulus $\mathbb{A}$ and share $a_1, a_2$ IM in $\mathbb{A}$ and $f = a_3 \Rightarrow g = a_3$ and $f = a_4 \Rightarrow g = a_4$ in $\mathbb{A},$ again by Theorem 2.4, we have

$$3T_0(R, f) \leq \sum_{j=1}^{4} \overline{N}_0 \left( R, \frac{1}{f - a_j} \right) + S(R, f), \quad (3.6)$$

$$\leq \overline{N}_0 \left( R, \frac{1}{f - g} \right) + S(R, f) = T_0(R, f - g) + S(R, f), \quad (3.7)$$

$$\leq T_0(R, f) + T_0(R, g) + S(R), \quad (3.8)$$

$$T_0(R, g) \leq \overline{N}_0 \left( R, \frac{1}{g - a_1} \right) + \overline{N}_0 \left( R, \frac{1}{g - a_2} \right) + S(R, g), \quad (3.9)$$

$$= \overline{N}_0 \left( R, \frac{1}{f - a_1} \right) + \overline{N}_0 \left( R, \frac{1}{f - a_2} \right) + S(R), \quad (3.10)$$

$$\leq 2T_0(R, f) + S(R). \quad (3.11)$$

From (3.8) and (3.11), we can get (i), and from (3.7), (3.8), and (ii), we can get (iii), and from (3.6), (3.8), (3.10), (3.11), and (ii), we can get (iii). Thus, we can deduce that (iv) and (v) hold easily from (3.6)–(3.11) and (i)–(iii). Now, we will prove that (vi) holds as follows.

First, we can rewrite (3.5) as

$$f = f' = \frac{g'}{\eta(g - a_1)(g - a_2)} + \frac{f'g'(a_3f + a_4f - a_3a_4 - fg)}{\eta(f - a_3)(f - a_4)(g - a_1)(g - a_2)}. \quad (3.12)$$

From (3.12) and Lemma 3.5, we can get $m_0(R, f) \leq m_0(R, f') + S(R, f).$ Since $f$ is analytic on $\mathbb{A},$ we have

$$T_0(R, f) \leq T_0(R, f') + 2m(1, f') - 2m(1, f) + S(R, f). \quad (3.13)$$
From the fact that $f$ is transcendental or admissible, we have
\[
T_0(R, f') \leq T_0(R, f) + S(R, f) + O(1) = T_0(R, f') + S(R, f). \tag{3.14}
\]

On the other hand, since $m_0(R, f') = m_0(R, f(f'/f)) \leq m_0(R, f) + m_0(R, f'/f) + O(1)$, from Theorem 2.3, we have $m_0(R, f') \leq m_0(R, f) + S(R, f)$. Thus, we can get
\[
T_0(R, f') \leq T_0(R, f) + S(R, f) + O(1) = T_0(R, f) + S(R, f). \tag{3.15}
\]

From (3.14), (3.15) and the fact that $f$ is transcendental or admissible, we can get $T_0(R, f') = T_0(R, f) + S(R, f)$. Similarly, we can get $T_0(R, g') = T_0(R, g) + S(R, g)$.

Thus, we complete the proof of this lemma. \qed

4. The Proof of Theorem 3.1

Suppose $f \neq g$. By the assumption of Theorem 3.1, we can get the conclusions (i)–(vi) of Lemma 3.6 and that $g$ is transcendental or admissible on $\mathbb{A}$. Set
\[
\begin{align*}
q_1 & := \frac{f'(f - a_3)}{(f - a_1)(f - a_2)} - \frac{g'(g - a_3)}{(g - a_1)(g - a_2)}, \\
q_2 & := \frac{f'(f - a_4)}{(f - a_1)(f - a_2)} - \frac{g'(g - a_4)}{(g - a_1)(g - a_2)}.
\end{align*}
\tag{4.1}
\]

By Lemma 3.4, we can get
\[
m_0(R, q_i) = S(R, f) + S(R, g) = S(R), \quad i = 1, 2. \tag{4.2}
\]

Moreover, we can prove $N_0(R, q_i) = O(1)$ ($i = 1, 2$). In fact, the poles of $q_i$ in $\mathbb{A}$ only can occur at the zeros of $f - a_j$ and $g - a_j$ ($i, j = 1, 2$) in $\mathbb{A}$. Since $f, g$ share $a_1, a_2$ CM in $\mathbb{A}$, we can see that if $z_0 \in \mathbb{A}$ is a zero of $f - a_j$ with multiplicity $m(\geq 1)$, then $z_0 \in \mathbb{A}$ is a zero of $g - a_j$ ($j = 1, 2$) with multiplicity $m(\geq 1)$. Suppose that
\[
f - a_j = (z - z_0)^m \alpha_j(z), \quad g - a_j = (z - z_0)^m \beta_j(z), \tag{4.3}
\]

where $\alpha_j(z), \beta_j(z)$ are analytic functions in $\mathbb{A}$ and $\alpha_j(z_0) \neq 0, \beta_j(z_0) \neq 0$ ($j = 1, 2$); by a simple calculation, we have
\[
q_i(z_0) = K \left( \frac{\alpha_i'(z_0)}{\alpha_i(z_0)} - \frac{\beta_i'(z_0)}{\beta_i(z_0)} \right) \quad (i, j = 1, 2), \tag{4.4}
\]

where $K$ is a constant. Therefore, we can get that $q_i$ ($i = 1, 2$) are analytic in $\mathbb{A}$. Thus, from (4.2), we can get $T_0(R, q_i) = m_0(R, q_i) + O(1) = S(R)$ ($i = 1, 2$).
If \( q_i \neq 0, \ i = 1, 2 \), then we have

\[
\overline{N}_0\left( R, \frac{1}{f - a_3} \right) \leq N_0\left( R, \frac{1}{q_1} \right) \leq T_0(R, q_1) + S(R, f) + O(1) = S(R),
\]

\[
\overline{N}_0\left( R, \frac{1}{f - a_4} \right) \leq N_0\left( R, \frac{1}{q_2} \right) \leq T_0(R, q_2) + S(R, f) + O(1) = S(R).
\]  \( (4.5) \)

From (4.5) and Lemma 3.6(iv), we have \( T_0(R, f) \leq S(R) \). Thus, since \( f, g \) are transcendental or admissible functions on \( A \), that is, \( f \) and \( g \) are of unbounded characteristic, and from the definition of \( S(R) \), we can get a contradiction.

Assume that one of \( q_1 \) and \( q_2 \) is identically zero, say \( q_1 \equiv 0 \); then we have

\[
\overline{N}_0^{(2)}\left( R, \frac{1}{g - a_4} \right) = \overline{N}_0^{(2)}\left( R, \frac{1}{f - a_4} \right).
\]  \( (4.6) \)

From (3.5), we can see that \( g(z_1) = a_4 \) implies that \( f(z_1) = a_4 \) for such \( z_1 \in A \) satisfying \( \eta(z_1) \neq 0 \). Since \( T_0(R, \eta) = S(R) \), we have

\[
\overline{N}_0^{(1)}\left( R, \frac{1}{g - a_4} \right) = \overline{N}_0^{(1)}\left( R, \frac{1}{f - a_4} \right) + S(R).
\]  \( (4.7) \)

From (4.6) and (4.7), we can get

\[
\overline{N}_0\left( R, \frac{1}{g - a_4} \right) = \overline{N}_0\left( R, \frac{1}{f - a_4} \right) + S(R).
\]  \( (4.8) \)

Similarly, when \( q_2 \equiv 0 \), we can get

\[
\overline{N}_0\left( R, \frac{1}{g - a_3} \right) = \overline{N}_0\left( R, \frac{1}{f - a_3} \right) + S(R).
\]  \( (4.9) \)

From (4.8), (4.9), and Lemma 3.6(i), (v), we can get

\[
2T_0(R, f) = \overline{N}_0\left( R, \frac{1}{f - a_3} \right) + S(R)
\]  \( (4.10) \)

or

\[
2T_0(R, f) = \overline{N}_0\left( R, \frac{1}{f - a_4} \right) + S(R).
\]  \( (4.11) \)

Since \( f, g \) are transcendental or admissible functions on the annulus \( A \), we can get a contradiction again.

Thus, we complete the proof of Theorem 3.1.
5. The Proof of Theorem 3.2

Suppose that $f \neq g$. By Theorem 2.4(ii) and the fact that $f$ is transcendental or admissible on $\mathbb{A}$, we have

$$2T_0(R, f) + \overline{N}_0\left(R, \frac{1}{g - a_4}\right) \leq \overline{N}_0\left(R, \frac{1}{f - a_1}\right) + \overline{N}_0\left(R, \frac{1}{f - a_2}\right) + \overline{N}_0\left(R, \frac{1}{f - a_3}\right) + S(R, f)$$

$$\leq \overline{N}_0\left(R, \frac{1}{g - a_4}\right) + S(R, f)$$

$$\leq T_0(R, f) + T_0(R, g) + S(R, f) + S(R, g).$$

Therefore, we have

$$T_0(R, f) + \overline{N}_0\left(R, \frac{1}{g - a_4}\right) \leq T_0(R, g) + S(R, f) + S(R, g). \quad (5.2)$$

Similarly, we have

$$T_0(R, g) + \overline{N}_0\left(R, \frac{1}{f - a_3}\right) \leq T_0(R, f) + S(R, g) + S(R, f). \quad (5.3)$$

From (5.2) and (5.3), we can see that $T_0(R, f) = T_0(R, g) + S(R, f) + S(R, g), and$

$$\overline{N}_0\left(R, \frac{1}{f - a_3}\right) = S(R, f) + S(R, g), \quad \overline{N}_0\left(R, \frac{1}{g - a_4}\right) = S(R, f) + S(R, g). \quad (5.4)$$

Thus, from (5.2), (5.3), and the definition of $S(R)$, we can get that $g$ is also transcendental or admissible on $\mathbb{A}$ when $f$ is transcendental or admissible on $\mathbb{A}$.

From (5.1)–(5.4), we can also get

$$2T_0(R, f) = \overline{N}_0\left(R, \frac{1}{f - a_1}\right) + \overline{N}_0\left(R, \frac{1}{f - a_2}\right) + S(R). \quad (5.5)$$

From (5.5), we can see that “almost all” of zeros of $f - a_i$ ($i = 1, 2$) in $\mathbb{A}$ are simple. Similarly, “almost all” of zeros of $g - a_i$ ($i = 1, 2$) in $\mathbb{A}$ are simple, too. Let

$$\varphi_1 := \frac{(a_1 - a_3) f' (f - a_2)}{(f - a_1)(f - a_3)} - \frac{(a_1 - a_4) g' (g - a_2)}{(g - a_1)(g - a_4)}$$

$$\varphi_2 := \frac{(a_2 - a_3) f' (f - a_1)}{(f - a_2)(f - a_3)} - \frac{(a_2 - a_4) g' (g - a_1)}{(g - a_2)(g - a_4)}. \quad (5.6)$$
By Lemma 3.5, we can get that $m_0(R, q_1) = S(R)$ ($i = 1, 2$). Since $f, g$ share $a_1, a_2$ IM in $\mathbb{A}$ and from (5.2), we have $N_0(R, q_1) = S(R)$ ($i = 1, 2$). Therefore, we can get $T_0(R, q_1) = S(R)$, ($i = 1, 2$).

If $q_1 \neq 0$, then we have $\overline{N}_0(R, 1/(f - a_2)) \leq \overline{N}_0(R, 1/q_1) = S(R)$. Thus, from (5.5), we can get a contradiction easily. Similarly, when $q_2 \neq 0$, we can get a contradiction, too. Hence, $q_1, q_2$ are identically equal to 0. Then, we have $(q_1 - q_2)/(a_1 - a_2) \equiv 0$, that is,

$$\frac{f'}{f - a_3} - \frac{g'}{g - a_4} - \frac{f'}{f - a_1} + \frac{g'}{g - a_1} - \frac{f'}{f - a_2} + \frac{g'}{g - a_2} \equiv 0,$$

which implies that

$$\frac{f - a_3}{g - a_4} \cdot \frac{(g - a_1)(g - a_2)}{(f - a_1)(f - a_2)} = c,$$

where $c$ is a nonzero constant. Rewrite (5.8) as

$$g^2 - \left(a_1 + a_2 - \frac{c\gamma(f)}{f - a_3}\right)g + a_1a_2 + \frac{ca_4\gamma(f)}{f - a_3} \equiv 0,$$

where $\gamma(f) := (f - a_1)(f - a_2)$. The discriminant of (5.9) is

$$\Delta(f) = \left(a_1 + a_2 - \frac{c\gamma(f)}{f - a_3}\right)^2 - 4\left(a_1a_2 + \frac{ca_4\gamma(f)}{f - a_3}\right) := \frac{\bar{\varphi}(f)}{(f - a_3)^2},$$

where

$$\bar{\varphi}(z) := ((a_1 + a_2)(z - a_3) - c\gamma(z))^2 - 4a_1a_2(z - a_3)^2 - 4ca_4\gamma(z)(z - a_3)$$

is a polynomial of degree 4 in $z$. If $a$ is a zero of $\bar{\varphi}(z)$ in $\mathbb{A}$, obviously $a \neq a_3$. Then, from (5.9), $f(z) = a$ implies that

$$g(z) = \frac{1}{2}\left(a_1 + a_2 - \frac{c\gamma(a)}{a - a_3}\right) := b.$$

Set

$$\phi_1 := \frac{f'g'/(f - g)}{(f - a_1)(g - a_2)(f - a_3)(g - a_4)},$$

$$\phi_2 := \frac{f'g'/(f - g)}{(f - a_2)(g - a_1)(f - a_3)(g - a_4)},$$

$$\phi = \frac{\phi_2}{\phi_1} = \frac{(f - a_1)(g - a_2)}{(f - a_2)(g - a_1)}.$$
By Lemma 3.5, we can get $m_0(R, \phi_i) = S(R)$ $(i = 1, 2)$. And by a simple calculation, we can get $N_0(R, \phi_i) = S(R)$ $(i = 1, 2)$. Then we have $T_0(R, \phi_i) = S(R)$ $(i = 1, 2)$, thus we have $T_0(R, \phi) = S(R)$.

Assume that $f$ is not a Möbius transformation of $g$; then $\phi$ is a nonconstant function. Since

$$\hat{\phi}(a_1) = ((a_1 + a_2)(a_1 - a_3))^2 - 4a_1a_2(a_1 - a_3)^2 = (a_1 - a_3)^2(a_1 - a_2)^2 \neq 0,$$

$$\hat{\phi}(a_2) = ((a_1 + a_3)(a_2 - a_3))^2 - 4a_1a_2(a_2 - a_3)^2 = (a_2 - a_3)^2(a_1 - a_2)^2 \neq 0.$$ 

(5.14)

From $a \neq a_i$ $(i = 1, 2)$ and (5.8), we can get

$$\overline{N}_0\left(R, \frac{1}{f - a}\right) \leq \overline{N}_0\left(R, \frac{1}{\phi - \xi}\right) \leq T_0(R, \phi) = S(R),$$

(5.15)

where $\xi = (a - a_1)(b - a_2)/(a - a_2)(b - a_1)$. Since $f$ is transcendental or admissible analytic in $\mathbb{A}$, by Theorem 2.4(ii) and (5.4), we can get

$$T_0(R, f) \leq \overline{N}_0\left(R, \frac{1}{f - a_3}\right) + \overline{N}_0\left(R, \frac{1}{f - a}\right) + S(R) = S(R).$$

(5.16)

Since $f, g$ are transcendental or admissible functions on $\mathbb{A}$, from the above inequality, we can get a contradiction. Therefore, we can get that $f$ is a Möbius transformation of $g$ on $\mathbb{A}$. Since $f, g$ are transcendental or admissible functions on $\mathbb{A}$, by a simple calculation, we can get easily that $a_1 + a_2 = a_3 + a_4$ and

$$f \equiv \frac{a_3g - a_1a_2}{g - a_4}. $$

(5.17)

Furthermore, $a_3, a_4$ are Picard exceptional values of $f$ and $g$ in $\mathbb{A}$, respectively.

Thus, we complete the proof of Theorem 3.2.

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