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Research Article

Nearly Radical Quadratic Functional Equations in *p*-2-Normed Spaces

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We establish some stability results in 2-normed spaces for the radical quadratic functional equation $f(\sqrt{\sum_{i=1}^{n}(x_i+y_i)^2})+f(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}) = 2\sum_{i=1}^{n}(f(x_i)+f(y_i))$ and then use subadditive functions to prove its stability in p-2-normed spaces.

1. Introduction and Preliminaries

The story of the stability of functional equations dates back to 1925 when a stability result appeared in the celebrated book by Póolya and Szeg [1]. In 1940, Ulam [2, 3] posed the famous Ulam stability problem which was partially solved by Hyers [4] in the framework of Banach spaces. Later Aoki [5] considered the stability problem with unbounded Cauchy differences. In 1978, Rassias [6] provided a generalization of Hyers' theorem by proving the existence of unique linear mappings near approximate additive mappings. Găvruţa [7] obtained the generalized result of T. M. Rassias' theorem which allows the Cauchy difference to be controlled by a general unbounded function. On the other hand, Rassias, Găvruţa, and several authors proved the Ulam-Gavruta-Rassias stability of several functional equations. For more details about the results concerning such problems, the reader is referred to [8–30].

Gajda and Ger [31] showed that one can get analogous stability results, for subadditive multifunctions. For further results see [32–42], among others.

The most famous functional equation is the Cauchy equation $f(x+y) = f(x) + f(y)$ any solution of which is called additive. It is easy to see that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx^2$ with c an arbitrary constant is a solution of the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

So, it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known [43, 44] that a function $f : X \to Y$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B_1 : X \times X \to Y$ such that $f(x) = B_1(x,x)$ for all $x \in X$. The $B_1(x,y) = (1/4)(f(x+y) - f(x-y))$ for all $x, y \in X$.

We briefly recall some definitions and results used later on in the paper. For more details, the reader is referred to [45–49]. The theory of 2-normed spaces was first developed by Gähler [46] in the mid-1960s, while that of 2-Banach spaces was studied later by Gähler and White [47, 49].

Definition 1.1 (see [45]). Let $\mathcal{X}$ be a real linear space over $\mathbb{R}$ with $\dim \mathcal{X} > 1$ and $\|\cdot,\cdot\| : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ a function.

Then $(\mathcal{X}, \|\cdot,\cdot\|)$ is called a linear 2-normed space if

($^2N_1$) $\|x,y\| > 0$ and $\|x,y\| = 0$ if and only if x and y are linearly dependent,

($^2N_2$) $\|x,y\| = \|y,x\|$,

($^2N_3$) $\|\alpha x, y\| = |\alpha| \|x,y\|$, for any $\alpha \in \mathbb{R}$,

($^2N_4$) $\|x, y+z\| \le \|x,y\| + \|x,z\|$,

for all $x, y, z \in \mathcal{X}$. The function $\|\cdot,\cdot\|$ is called the 2-norm on $\mathcal{X}$.

Let $(\mathcal{X}, \|\cdot,\cdot\|)$ be a linear 2-normed space. If $x \in \mathcal{X}$ and $\|x,y\| = 0$, for all $y \in \mathcal{X}$, then $x = 0$. Moreover, for a linear 2-normed space $(\mathcal{X}, \|\cdot,\cdot\|)$, the functions $x \to \|x,y\|$ are continuous functions of $\mathcal{X}$ into $\mathbb{R}$ for each fixed $y \in \mathcal{X}$ (see [48]).

A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a Cauchy sequence if there are two points $y, z \in \mathcal{X}$ such that y and z are linearly independent, $\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0$ and $\lim_{n,m\to\infty} \|x_n - x_m, z\| = 0$.

A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a convergent sequence if there is an $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$, for all $y \in \mathcal{X}$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.

A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space. For a convergent sequence $\{x_n\}$ in a 2-normed space $\mathcal{X}$, $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$, for all $y \in \mathcal{X}$ (see [48]).

We fix a real number p with $0 < p \le 1$, and let $\mathcal{Y}$ be a linear space. A p-2-norm is a function on $\mathcal{Y} \times \mathcal{Y}$ satisfying Definition 1.1; ($^2N_1$), ($^2N_2$), and ($^2N_4$); the following: $\|\alpha x, y\| = |\alpha|^p \|x,y\|$, for all $x, y \in \mathcal{Y}$ and all $\alpha \in \mathbb{R}$. The pair $(\mathcal{Y}, \|\cdot,\cdot\|)$ is called a p-2-normed space if $\|\cdot,\cdot\|$ is a p-2-norm on $\mathcal{Y}$. A p-2-Banach space is a complete p-2-normed space.

We recall that a subadditive function is a function $\varphi_a : A \to B$, having a domain A and a codomain $(B, \le)$ that are both closed under addition, with the following property:

$$\varphi_a(x+y) \le \varphi_a(x) + \varphi_a(y), \tag{1.2}$$

for all $x, y \in A$. Let $\ell \in \{-1, 1\}$ be fixed. If there exists a constant L with $0 < L < 1$ such that a function $\varphi_a : A \to B$ satisfies

$$\ell\varphi_a(x+y) \leq \ell L^{\ell}(\varphi_a(x) + \varphi_a(y)), \tag{1.3}$$

for all $x, y \in A$, then we say that φ_a is contractively subadditive if $\ell = 1$, and φ_a is expansively superadditive if $\ell = -1$. It follows by the last inequality that φ_a satisfies the following properties:

$$\varphi_a\left(2^{\ell}x\right) \leq 2^{\ell}L\varphi_a(x), \qquad \varphi_a\left(2^{\ell k}x\right) \leq \left(2^{\ell}L\right)^k\varphi_a(x), \tag{1.4}$$

for all $x \in A$ and integers $k \geq 1$.

Now, we consider the radical quadratic functional equation

$$f\left(\sqrt{\sum_{i=1}^{n}(x_i + y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right) = 2\sum_{i=1}^{n}(f(x_i) + f(y_i)), \tag{1.5}$$

where $n \in \mathbb{N}$ is a fixed integer and prove generalized Ulam stability, in the spirit of Găvruta (see [7]), of this functional equation in 2-normed spaces. Moreover, in this paper, we investigate new results about the generalized Ulam stability by using subadditive functions in p-2-normed spaces for the radical quadratic functional equation (1.5).

2. Main Results

In this section, let X be a linear space, and let $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of real and positive real numbers, respectively. If a mapping $f : \mathbb{R} \to X$ satisfies the functional equation (1.5), by letting $x_i = y_i = 0\,(1 \leq i \leq n)$ in (1.5), we get $f(0) = 0$. Setting $x_i = y_i = x(1 \leq i \leq n)$ in (1.5) and using $f(0) = 0$, we get

$$f\left(\sqrt{4nx^2}\right) = 4nf(x), \tag{2.1}$$

for all $x \in \mathbb{R}$. Putting $x_i = 2x, y_i = 0\,(1 \leq i \leq n)$ in (1.5) and using $f(0) = 0$, we get

$$2f\left(\sqrt{4nx^2}\right) = 2nf(2x), \tag{2.2}$$

for all $x \in \mathbb{R}$. It follows from (2.1) and (2.2) that

$$f(2^m x) = 4^m f(x), \tag{2.3}$$

for all $x \in \mathbb{R}$ and integers $m \geq 1$. Setting $y_n = -y_n$ in (1.5) and then comparing it with (1.5), we obtain $f(-y_n) = f(y_n)$, for all $y_n \in \mathbb{R}$. Letting $x_i = y_i = 0$ $(2 \leq i \leq n)$ in (1.5), we get

$$f\left(\sqrt{(x_1+y_1)^2}\right) + f\left(\sqrt{(x_1-y_1)^2}\right) = 2f(x_1) + 2f(y_1), \tag{2.4}$$

for all $x_1, y_1 \in \mathbb{R}$. It follows from (2.4) and the evenness of f that f satisfies (1.1). So we have the following lemma.

Lemma 2.1. *If a mapping $f : \mathbb{R} \to X$ satisfies the functional equation* (1.5), *then f is quadratic.*

Corollary 2.2. *If a mapping $f : \mathbb{R} \to X$ satisfies the functional equation* (1.5), *then there exists a symmetric biadditive mapping $B_1 : \mathbb{R} \times \mathbb{R} \to X$ such that $f(x) = B_1(x,x)$, for all $x \in \mathbb{R}$.*

Hereafter, we will assume that $\mathcal{X}$ is a 2-Banach space. First, using an idea of Găvruţa [7], we prove the stability of (1.5) in the spirit of Ulam, Hyers, and Rassias.

Let ϕ be a function from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^+ \cup \{0\}$. A mapping $f : \mathbb{R} \to \mathcal{X}$ is called a ϕ-approximatively radical quadratic function if

$$\begin{aligned} &\left\| f\left(\sqrt{\sum_{i=1}^{n}(x_i+y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right) - 2\sum_{i=1}^{n}(f(x_i)+f(y_i)), z \right\|_{\mathcal{X}} \\ &\quad \leq \phi(x_1,\ldots,x_n,y_1,\ldots,y_n,z), \end{aligned} \tag{2.5}$$

for all $x_1,\ldots,x_n,y_1,\ldots,y_n,z \in \mathbb{R}$, where $n \in \mathbb{N}$ is a fixed integer.

Theorem 2.3. *Let $\ell \in \{-1,1\}$ be fixed, and let $f : \mathbb{R} \to \mathcal{X}$ be a ϕ- approximatively radical quadratic function with $f(0) = 0$. If the function $\phi : \mathbb{R}^{2n+1} \to \mathbb{R}^+ \cup \{0\}$ satisfies*

$$\Phi(x,z) := \sum_{j=(1-\ell)/2}^{\infty} \frac{1}{4^{\ell j}} \left(\phi\left(\overbrace{2^{\ell j}x,\ldots,2^{\ell j}x}^{2n}, z \right) + \frac{1}{2}\phi\left(\overbrace{2^{1+\ell j}x,\ldots,2^{1+\ell j}x}^{n}, \overbrace{0,\ldots,0}^{n}, z \right) \right) < \infty, \tag{2.6}$$

and $\lim_{m\to\infty}(1/4^{\ell m})\phi(2^{\ell m}x_1,\ldots,2^{\ell m}x_n,2^{\ell m}y_1,\ldots,2^{\ell m}y_n,z) = 0$, for all $x,x_1,\ldots,x_n,y_1,\ldots,y_n,z \in \mathbb{R}$, then there exists a unique quadratic mapping $\mathcal{F} : \mathbb{R} \to \mathcal{X}$, satisfies (1.5) *and the inequality*

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1}{4n}\Phi(x,y), \tag{2.7}$$

for all $x, y \in \mathbb{R}$.

Proof. Letting $x_i = x + y, y_i = x - y \ (1 \le i \le n)$ in (2.5), we get

$$\left\| f\left(\sqrt{4nx^2}\right) + f\left(\sqrt{4ny^2}\right) - 2nf(x+y) - 2nf(x-y), z \right\|_{\mathcal{X}} \le \phi\Bigg(\overbrace{x+y, \dots, x+y}^{n}, \overbrace{x-y, \dots, x-y}^{n}, z \Bigg), \tag{2.8}$$

for all $x, y, z \in \mathbb{R}$. Setting $x_i = y_i = x \ (1 \le i \le n)$ in (2.5), we get

$$\left\| f\left(\sqrt{4nx^2}\right) - 4nf(x), z \right\|_{\mathcal{X}} \le \phi\Bigg(\overbrace{x, \dots, x}^{2n}, z \Bigg), \tag{2.9}$$

for all $x, z \in \mathbb{R}$. Replacing y by x in (2.8), we obtain

$$\left\| f\left(\sqrt{4nx^2}\right) - nf(2x), z \right\|_{\mathcal{X}} \le \frac{1}{2}\phi\Bigg(\overbrace{2x, \dots, 2x}^{n}, \overbrace{0, \dots, 0}^{n}, z \Bigg), \tag{2.10}$$

for all $x, z \in \mathbb{R}$. It follows from (2.9) and (2.10) that

$$\| 4f(x) - f(2x), y \|_{\mathcal{X}} \le \frac{1}{n}\phi\Bigg(\overbrace{x, \dots, x}^{2n}, y \Bigg) + \frac{1}{2n}\phi\Bigg(\overbrace{2x, \dots, 2x}^{n}, \overbrace{0, \dots, 0}^{n}, y \Bigg), \tag{2.11}$$

for all $x, y \in \mathbb{R}$. Thus,

$$\begin{gathered} \left\| f(x) - \frac{1}{4}f(2x), y \right\|_{\mathcal{X}} \le \frac{1}{4n}\phi\Bigg(\overbrace{x, \dots, x}^{2n}, y \Bigg) + \frac{1}{8n}\phi\Bigg(\overbrace{2x, \dots, 2x}^{n}, \overbrace{0, \dots, 0}^{n}, y \Bigg), \\ \left\| f(x) - 4f\left(\frac{x}{2}\right), y \right\|_{\mathcal{X}} \le \frac{1}{n}\phi\Bigg(\overbrace{\frac{x}{2}, \dots, \frac{x}{2}}^{2n}, y \Bigg) + \frac{1}{2n}\phi\Bigg(\overbrace{x, \dots, x}^{n}, \overbrace{0, \dots, 0}^{n}, y \Bigg), \end{gathered} \tag{2.12}$$

for all $x, y \in \mathbb{R}$. Hence,

$$\begin{aligned} &\left\| \frac{1}{4^{\ell k}} f\left(2^{\ell k}x\right) - \frac{1}{4^{\ell r}} f\left(2^{\ell r}x\right), y \right\|_{\mathcal{X}} \\ &\quad \le \frac{1}{4n} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{1}{4^{\ell j}} \Bigg(\phi\Bigg(\overbrace{2^{\ell j}x, \dots, 2^{\ell j}x}^{2n}, y \Bigg) + \frac{1}{2}\phi\Bigg(\overbrace{2^{1+\ell j}x, \dots, 2^{1+\ell j}x}^{n}, \overbrace{0, \dots, 0}^{n}, y \Bigg) \Bigg) \end{aligned} \tag{2.13}$$

for all $x, y \in \mathbb{R}$ and integers $r > k \geq 0$. Thus, $\{(1/4^{\ell m})f(2^{\ell m}x)\}$ is a Cauchy sequence in the 2-Banach space $\mathcal{X}$. Hence, we can define a mapping $\mathcal{F} : \mathbb{R} \to \mathcal{X}$ by $\mathcal{F}(x) := \lim_{m\to\infty}(1/4^{\ell m})f(2^{\ell m}x)$, for all $x \in \mathbb{R}$. That is,

$$\lim_{m\to\infty}\left\|\frac{1}{4^{\ell m}}f\left(2^{\ell m}x\right) - \mathcal{F}(x), y\right\|_{\mathcal{X}} = 0, \tag{2.14}$$

for all $x, y \in \mathbb{R}$. In addition, it is clear from (2.5) that the following inequality:

$$\begin{aligned}
&\left\|\mathcal{F}\left(\sqrt{\sum_{i=1}^{n}(x_i+y_i)^2}\right) + \mathcal{F}\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right) - 2\sum_{i=1}^{n}(\mathcal{F}(x_i)+\mathcal{F}(y_i)), z\right\|_{\mathcal{X}} \\
&\quad = \lim_{m\to\infty}\frac{1}{4^{\ell m}}\left\|f\left(\sqrt{4^{\ell m}\sum_{i=1}^{n}(x_i+y_i)^2}\right) + f\left(\sqrt{4^{\ell m}\sum_{i=1}^{n}(x_i-y_i)^2}\right)\right. \\
&\qquad\qquad \left. -2\sum_{i=1}^{n}\left(f\left(2^{\ell m}x_i\right)+f\left(2^{\ell m}y_i\right)\right), z\right\|_{\mathcal{X}} \\
&\quad \leq \lim_{m\to\infty}\frac{1}{4^{\ell m}}\phi\left(2^{\ell m}x_1,\ldots,2^{\ell m}x_n,2^{\ell m}y_1,\ldots,2^{\ell m}y_n,z\right) = 0
\end{aligned} \tag{2.15}$$

holds for all $x_1,\ldots,x_n,y_1,\ldots,y_n,z \in \mathbb{R}$, and so by Lemma 2.1, the mapping $\mathcal{F} : \mathbb{R} \to \mathcal{X}$ is quadratic. Taking the limit $r \to \infty$ in (2.13) with $k = 0$, we find that the mapping $\mathcal{F}$ is quadratic mapping satisfying the inequality (2.7) near the approximate mapping $f : \mathbb{R} \to \mathcal{X}$ of (1.5). To prove the aforementioned uniqueness, we assume now that there is another quadratic mapping $\mathcal{G} : \mathbb{R} \to \mathcal{X}$ which satisfies (1.5) and the inequality (2.7). Since the mapping $\mathcal{G} : \mathbb{R} \to \mathcal{X}$ satisfies (1.5), then

$$\mathcal{G}\left(2^{\ell}x\right) = 4^{\ell}\mathcal{G}(x), \qquad \mathcal{G}\left(2^{\ell m}x\right) = 4^{\ell m}\mathcal{G}(x) \tag{2.16}$$

for all $x \in \mathbb{R}$ and integers $m \geq 1$. Thus, one proves by the last equality and (2.7) that

$$\left\|\frac{1}{4^{\ell m}}f(2^{\ell m}x) - \mathcal{G}(x), y\right\|_{\mathcal{X}} = \frac{1}{4^{\ell m}}\left\|f(2^{\ell m}x) - \mathcal{G}(2^{\ell m}x), y\right\|_{\mathcal{X}} \leq \frac{1}{4^{m\ell+1}n}\,\Phi\left(2^{\ell m}x, y\right), \tag{2.17}$$

for all $x, y \in \mathbb{R}$ and integers $m \geq 1$. Therefore, from $m \to \infty$, one establishes $\mathcal{F}(x) - \mathcal{G}(x) = 0$ for all $x \in \mathbb{R}$. □

Corollary 2.4. *Let $\ell \in \{-1, 1\}$ be fixed. If there exist nonnegative real numbers p_i, q_i, q with $\ell\sum_{i=1}^{n}(p_i+q_i) < 2\ell$ such that a mapping $f : \mathbb{R} \to \mathcal{X}$ satisfies the inequality*

$$\begin{aligned}
&\left\|f\left(\sqrt{\sum_{i=1}^{n}(x_i+y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}\right) - 2\sum_{i=1}^{n}(f(x_i)+f(y_i)), z\right\|_{\mathcal{X}} \\
&\quad \leq \theta\prod_{i=1}^{n}|x_i|^{p_i}\left|y_i\right|^{q_i}|z|^{q},
\end{aligned} \tag{2.18}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{R}$ *and some* $\theta \geq 0$*, then there exists a unique quadratic mapping* $\mathcal{F} : \mathbb{R} \to \mathcal{X}$*, satisfies* (1.5) *and the inequality*

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1}{\ell n(4 - 2^{\lambda})}\theta |x|^{\lambda}|y|^{q}, \tag{2.19}$$

for all $x, y \in \mathbb{R}$*, where* $\lambda := \sum_{i=1}^{n}(p_i + q_i)$.

Corollary 2.5. *Let* $\ell \in \{-1, 1\}$ *be fixed. If there exist nonnegative real numbers* s, t *with* $\ell s < 2\ell$ *such that a mapping* $f : \mathbb{R} \to \mathcal{X}$ *satisfies the inequality*

$$\begin{aligned} &\left\| f\left(\sqrt{\sum_{i=1}^{n}(x_i + y_i)^2}\right) + f\left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\right) - 2\sum_{i=1}^{n}(f(x_i) + f(y_i)), z \right\|_{\mathcal{X}} \\ &\quad \leq \theta \sum_{i=1}^{n}(|x_i|^{s} + |y_i|^{s})|z|^{t}, \end{aligned} \tag{2.20}$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{R}$ *and some* $\theta \geq 0$*, then there exists a unique quadratic mapping* $\mathcal{F} : \mathbb{R} \to \mathcal{X}$ *satisfies* (1.5) *and the inequality*

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{X}} \leq \frac{1 + 2^{s-2}}{\ell(2 - 2^{s-1})}\theta |x|^{s}|y|^{t}, \tag{2.21}$$

for all $x, y \in \mathbb{R}$.

Now, we are going to establish the modified Hyers-Ulam stability of (1.5).

Theorem 2.6. *Let* $\ell \in \{-1, 1\}$ *be fixed, let* $\mathcal{Y}$ *be a p-2-Banach space, and,* $f : \mathbb{R} \to \mathcal{Y}$ *be a* ϕ*-approximatively radical quadratic function with* $f(0) = 0$*. Assume that the map* ϕ *is contractively subadditive if* $\ell = 1$ *and is expansively superadditive if* $\ell = -1$ *with a constant L satisfying* $2^{\ell(1-3p)}L < 1$*, where* $3\ell p \leq \ell$*, then there exists a unique quadratic mapping* $\mathcal{F} : \mathbb{R} \to \mathcal{Y}$ *which satisfies* (1.5) *and the inequality*

$$\|f(x) - \mathcal{F}(x), y\|_{\mathcal{Y}} \leq \frac{1}{\ell(4^{p} - 2^{1-p}L^{\ell})}\Psi(x, y), \tag{2.22}$$

for all $x, y \in \mathbb{R}$*, where*

$$\Psi(x, y) := \frac{1}{n^{p}}\phi\left(\overbrace{x, \ldots, x}^{2n}, y\right) + \frac{1}{(2n)^{p}}\phi\left(\overbrace{2x, \ldots, 2x}^{n}, \overbrace{0, \ldots, 0}^{n}, y\right). \tag{2.23}$$

Proof. Using the same method as in the proof of Theorem 2.3, we have

$$\left\| f(x) - \frac{1}{4} f(2x), y \right\|_{\mathcal{Y}} \leq \frac{1}{4^p} \Psi(x, y), \\ \left\| f(x) - 4 f\left(\frac{x}{2}\right), y \right\|_{\mathcal{Y}} \leq 2^p \Psi\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.24}$$

for all $x, y \in \mathbb{R}$. Hence

$$\begin{aligned} \left\| \frac{1}{4^{\ell k}} f\left(2^{\ell k} x\right) - \frac{1}{4^{\ell r}} f\left(2^{\ell r} x\right), y \right\|_{\mathcal{Y}} &\leq \frac{1}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{1}{2^{3\ell p j}} \Psi\left(2^{\ell j} x, 2^{\ell j} y\right) \\ &\leq \frac{1}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \frac{\left(2^{\ell} L\right)^j}{2^{3\ell p j}} \Psi(x, y) \\ &= \frac{\Psi(x, y)}{4^p} \sum_{j=k+(1-\ell)/2}^{r-(1+\ell)/2} \left(2^{\ell(1-3p)} L\right)^j, \end{aligned} \tag{2.25}$$

for all $x, y \in \mathbb{R}$ and integers $r > k \geq 0$. Thus, $\{(1/4^{\ell m}) f(2^{\ell m} x)\}$ is a Cauchy sequence in the p-2-Banach space $\mathcal{Y}$. Hence, we can define a mapping $\mathcal{F} : \mathbb{R} \to \mathcal{Y}$ by $\mathcal{F}(x) := \lim_{n \to \infty} (1/4^{\ell n}) f(2^{\ell n} x)$, for all $x \in \mathbb{R}$. Also

$$\begin{aligned} &\left\| \mathcal{F}\left(\sqrt{\sum_{i=1}^{n} (x_i + y_i)^2}\right) + \mathcal{F}\left(\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\right) - 2\sum_{i=1}^{n} (\mathcal{F}(x_i) + \mathcal{F}(y_i)), z \right\|_{\mathcal{Y}} \\ &\quad = \lim_{m \to \infty} \left\| \frac{1}{4^{\ell m}} f\left(\sqrt{4^{\ell m} \sum_{i=1}^{n} (x_i + y_i)^2}\right) + \frac{1}{4^{\ell m}} f\left(\sqrt{4^{\ell m} \sum_{i=1}^{n} (x_i - y_i)^2}\right) \right. \\ &\qquad \left. - \frac{2}{4^{\ell m}} \sum_{i=1}^{n} \left(f\left(2^{\ell m} x_i\right) + f\left(2^{\ell m} y_i\right)\right), z \right\|_{\mathcal{Y}} \\ &\quad \leq \lim_{m \to \infty} \frac{1}{2^{3\ell p m}} \phi\left(2^{\ell m} x_1, \ldots, 2^{\ell m} x_n, 2^{\ell m} y_1, \ldots, 2^{\ell m} y_n, 2^{\ell m} z\right) \\ &\quad \leq \lim_{m \to \infty} \left(2^{\ell(1-3p)} L\right)^m \phi(x_1, \ldots, x_n, y_1, \ldots, y_n, z) = 0 \end{aligned} \tag{2.26}$$

holds for all $x_1, \ldots, x_n, y_1, \ldots, y_n, z \in \mathbb{R}$, and so by Lemma 2.1, the mapping $\mathcal{F} : \mathbb{R} \to \mathcal{Y}$ is quadratic. Taking the limit $r \to \infty$ in (2.25) with $k = 0$, we find that the mapping $\mathcal{F}$ is quadratic mapping satisfying the inequality (2.22) near the approximate mapping $f : \mathbb{R} \to \mathcal{Y}$ of (1.5). The remaining assertion goes through in a similar way to the corresponding part of Theorem 2.3. □

References

[1] G. Póolya and G. Szeg, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, Springer, Berlin, Germany, 1925.
[2] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, NY, USA, 1960.
[3] S. M. Ulam, *Problems in Modern Mathematics*, Wiley, New York, NY, USA, 1964.
[4] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[5] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[6] T. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[7] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[8] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[9] P. Gavruta, "An answer to a question of John M. Rassias concerning the stability of Cauchy equation," in *Advances in Equations and Inequalities*, Hadronic Mathematics, pp. 67–71, 1999.
[10] J. M. Rassias and H.-M. Kim, "Generalized Hyers-Ulam stability for general additive functional equations in quasi-β-normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 302–309, 2009.
[11] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[12] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathematiques*, vol. 108, no. 4, pp. 445–446, 1984.
[13] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[14] G. Z. Eskandani, P. Gavruta, J. M. Rassias, and R. Zarghami, "Generalized Hyers-Ulam stability for a general mixed functional equation in quasi-β-normed spaces," *Mediterranean Journal of Mathematics*, vol. 8, no. 3, pp. 331–348, 2011.
[15] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces," *Journal of Mathematical Physics*, vol. 51, no. 9, Article ID 093508, 19 pages, 2010.
[16] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Intuitionistic fuzzy stability of a general mixed additive-cubic equation," *Journal of Mathematical Physics*, vol. 51, no. 6, Article ID 063519, 21 pages, 2010.
[17] T. Z. Xu, J. M. Rassias, and W. X. Xu, "On the stability of a general mixed additive-cubic functional equation in random normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 328473, 16 pages, 2010.
[18] M. Eshaghi Gordji, M. B. Ghaemi, Y. J. Cho, and H. Majani, "A general system of Euler-Lagrange-type quadratic functional equations in Menger probabilistic non-Archimedean 2-normed spaces," *Abstract and Applied Analysis*, vol. 2011, Article ID 208163, 21 pages, 2011.
[19] M. Eshaghi Gordji, "Nearly ring homomorphisms and nearly ring derivations on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2010, Article ID 393247, 12 pages, 2010.
[20] M. Eshaghi Gordji and Z. Alizadeh, "Stability and superstability of ring homomorphisms on non-Archimedean Banach algebras," *Abstract and Applied Analysis*, vol. 2011, Article ID 123656, 10 pages, 2011.
[21] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, Article ID 042303, 9 pages, 2009.
[22] C. Park and M. E. Gordji, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 51, no. 4, Article ID 044102, 7 pages, 2010.
[23] M. Eshaghi Gordji, S. Abbaszadeh, and C. Park, "On the stability of a generalized quadratic and quartic type functional equation in quasi-Banach spaces," *Journal of Inequalities and Applications*, vol. 2009, Article ID 153084, 26 pages, 2009.
[24] M. Eshaghi Gordji, A. Ebadian, and S. Zolfaghari, "Stability of a functional equation deriving from cubic and quartic functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 801904, 17 pages, 2008.
[25] F. Rahbarnia, T. M. Rassias, R. Saadati, and G. Sadeghi, "Forti's approach in fixed point theory and the stability of a functional equation on metric and ultra metric spaces," *Journal of Computational Analysis and Applications*, vol. 13, no. 3, pp. 458–462, 2011.

[26] R. Saadati, S. M. Vaezpour, and C. Park, "The stability of the cubic functional equation in various spaces," *Mathematical Communications*, vol. 16, no. 1, pp. 131–145, 2011.
[27] R. Saadati, M. M. Zohdi, and S. M. Vaezpour, "Nonlinear $\mathcal{L}$-random stability of an ACQ functional equation," *Journal of Inequalities and Applications*, vol. 2011, Article ID 194394, 23 pages, 2011.
[28] R. Saadati and C. Park, "Non-Archimedian $\mathcal{L}$-fuzzy normed spaces and stability of functional equations," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2488–2496, 2010.
[29] M. M. Pourpasha, J. M. Rassias, R. Saadati, and S. M. Vaezpour, "A fixed point approach to the stability of Pexider quadratic functional equation with involution," *Journal of Inequalities and Applications*, vol. 2010, Article ID 839639, 18 pages, 2010.
[30] L. Gavruta and P. Gavruta, "On a problem of John M.Rassias concerning the stability in Ulam sense of Euler-Lagrange equation," in *Functional Equations, Difference Inequalities*, pp. 47–53, Nova Sciences, 2010.
[31] Z. Gajda and R. Ger, "Subadditive multifunctions and Hyers-Ulam stability," in *General Inequalities*, vol. 5 of *International Series of Numerical Mathematics*, Birkhäuser, Basel, Switzerland, 1987.
[32] Y. J. Cho, C. Park, and R. Saadati, "Functional inequalities in non-Archimedean Banach spaces," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1238–1242, 2010.
[33] M. Eshaghi Gordji and H. Khodaei, *Stability of Functional Equations*, Lap Lambert Academic, 2010.
[34] D. H. Hyers, G. Isac, and T. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, Switzerland, 1998.
[35] S.-M. Jung and P. K. Sahoo, "Stability of a functional equation for square root spirals," *Applied Mathematics Letters*, vol. 15, no. 4, pp. 435–438, 2002.
[36] M. Eshaghi Gordji, "Stability of a functional equation deriving from quartic and additive functions," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 491–502, 2010.
[37] M. Eshaghi Gordji, "Stability of an additive-quadratic functional equation of two variables in F-spaces," *Journal of Nonlinear Science and its Applications*, vol. 2, no. 4, pp. 251–259, 2009.
[38] M. Eshaghi Gordji, M. B. Ghaemi, and H. Majani, "Generalized Hyers-Ulam-Rassias theorem in Menger probabilistic normed spaces," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 162371, 11 pages, 2010.
[39] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (n, k)-derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.
[40] M. Eshaghi Gordji, S. Kaboli Gharetapeh, J. M. Rassias, and S. Zolfaghari, "Solution and stability of a mixed type additive, quadratic, and cubic functional equation," *Advances in Difference Equations*, vol. 2009, Article ID 826130, 17 pages, 2009.
[41] M. E. Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, "Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces," *Abstract and Applied Analysis*, vol. 2009, Article ID 417473, 14 pages, 2009.
[42] C. Park and A. Najati, "Homomorphisms and derivations in C^*-algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.
[43] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, Cambridge, UK, 1989.
[44] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.
[45] Y. J. Cho, P. C. S. Lin, S. S. Kim, and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science, New York, NY, USA, 2001.
[46] S. Gähler, "Lineare 2-normierte Räume," *Mathematische Nachrichten*, vol. 28, pp. 1–43, 1964.
[47] S. Gähler, "Über 2-banach-Räume," *Mathematische Nachrichten*, vol. 42, pp. 335–347, 1969.
[48] W.-G. Park, "Approximate additive mappings in 2-banach spaces and related topics," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 1, pp. 193–202, 2011.
[49] A. G. White,, "2-banach spaces," *Mathematische Nachrichten*, vol. 42, pp. 43–60, 1969.