Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 870831, 25 pages
doi:10.1155/2012/870831

Research Article

On Almost Automorphic Mild Solutions for Nonautonomous Stochastic Evolution Equations

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Received 1 March 2012; Accepted 26 May 2012

Academic Editor: Toka Diagana

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We consider a class of nonautonomous stochastic evolution equations in real separable Hilbert spaces. We establish a new composition theorem for square-mean almost automorphic functions under non-Lipschitz conditions. We apply this new composition theorem as well as intermediate space techniques, Krasnoselskii fixed point theorem, and Banach fixed point theorem to investigate the existence of square-mean almost automorphic mild solutions. Some known results are generalized and improved.

1. Introduction

The concept of almost periodicity is of great importance in probability for investigating stochastic processes [1–3]. The basic results on the almost periodic functions and their applications to deterministic differential equations may refer to [4, 5] and references therein. The concept of almost automorphy introduced initially by Bochner [6] is an important generalization of the classical almost periodicity. Since then, there has been an intense interest in studying several extensions of this concept such as asymptotic almost automorphy, p-almost automorphy, and Stepanov-like almost automorphy (see [5, 7–9] and references therein). Much of the motivation has come from mathematical physics, mathematical biology, and various fields of science and engineering [10–12].

Besides, it should be pointed out that noise or stochastic perturbation is unavoidable and omnipresent in nature as well as that in man-made systems. Therefore, we must import the stochastic effects into the investigation of differential systems. In fact, the existence of almost periodic solutions for stochastic differential systems has been thoroughly investigated (see [13–17] and reference therein) while the existence of almost automorphic solutions for stochastic version has been in growing state. More precisely, in [18], the concept of square-mean almost automorphic process was introduced and investigated. Particularly, such a concept was utilized to study the existence and stability of square-mean almost automorphic mild solutions for a class of stochastic differential equations of the form

$$dx(t) = Ax(t) + f(t, x(t))dt + g(t, x(t))dW(t), \quad t \in \mathbb{R}, \tag{1.1}$$

in a Hilbert space; Chang et al. [19] extended the results in [18] to nonautonomous stochastic differential equations in Hilbert spaces; in [20], the square-mean pseudo almost automorphic process and its application to (1.2) were investigated; in [21], existence and exponential stability of almost automorphic mild solutions were considered to a class of stochastic differential equations with finite delay of the form

$$dx(t) = Ax(t) + f(t, x(t), x_t)dt + g(t, x(t), x_t)dW(t), \quad t \in \mathbb{R}; \tag{1.2}$$

one can also see [22, 23] for the existence of square-mean almost automorphic mild solutions of stochastic differential equations.

In this paper, we consider a general setting; that is, we make extensive use of intermediate space techniques to investigate the existence of square-mean almost automorphic mild solutions to the class of abstract nonautonomous neutral stochastic evolution equations of the form

$$d\left[x(t) + g(t, B_1x(t))\right] = \left[A(t)x(t) + b(t, B_2x(t))\right]dt + \sigma(t, B_3x(t))dW(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $A(t) : \mathfrak{D}(A(t)) \subset L^2(P, H) \rightarrow L^2(P, H)$ is a family of closed linear operators whose corresponding analytic semigroup is exponential dichotomy, B_i, $i = 1, 2, 3$, are bounded operators, $W(t)$ is a Q-Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathcal{F}_t = \sigma\{(W(u) - W(v)) : u, v \leq t\}$, and $g : L^2(P, H) \rightarrow L^2(P, H_\beta)$, $b : L^2(P, H) \rightarrow L^2(P, H)$, and $\sigma : L^2(P, H) \rightarrow L^2(P, \mathcal{L}_2^0(K, H))$ are jointly continuous functions to be specified later.

The rest of this paper is organized as follows. In Section 2 we present some basic notations and preliminary results. Section 3 is devoted to the study of existence of almost automorphic mild solutions for systems (1.3).

2. Preliminaries

For more details on this section, we refer to Da Prato et al. [24], Diagana [8], and Fu-Liu [18]. Throughout this paper, we assume that $(H, \|\cdot\|)$, $(K, \|\cdot\|_K)$ are real separable Hilbert spaces and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is supposed to be a filtered complete probability space. Denote by $L^2(P, H)$ the Banach space of all H-valued random variables x such that $E\|x\|^2 = \int_\Omega \|x\|^2 dP < +\infty$ endowed with the norm $\|x\|_2 = (\mathbb{E}\|x\|^2)^{1/2}$. If K_1, K_2 are Banach spaces, we denote by $\mathcal{L}(K_1, K_2)$ the Banach spaces of bounded linear operators from K_1 to K_2 equipped with natural operator norm; when $K_1 = K_2$, this is simply denoted by $\mathcal{L}(K_1)$. Furthermore, $\mathcal{L}_2^0(K, H)$ denotes the space of all Q-Hilbert-Schmidt operators from K to H with the norm

$$|\xi|^2_{\mathcal{L}_2^0} := \operatorname{tr}(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}(K, H). \tag{2.1}$$

For $t \in \mathbb{R}$, $A(t) : \mathfrak{D}(A(t)) \subset L^2(P, H) \to L^2(P, H)$ is a family of closed linear operators (not necessarily densely defined) satisfying the so-called Acquistapace-Terreni conditions (ATCs for short; see Lemma 2.3). If L is a linear operator on H, then the symbols $\mathfrak{D}(L), \rho(L), \sigma(L), N(L), R(L)$ stand, respectively, for the domain, resolvent set, spectrum, kernel, and range of L. We also set $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$ and $Q = I - P$ for a projection P.

Definition 2.1. A family of bounded linear operators $\{U(t,s) : t \geq s, t, s \in \mathbb{R}\}$ on $L^2(P, H)$ associated with $A(t)$ is said to be an evolution family of operators if the following conditions hold:

(i) $U(t,s)U(s,r) = U(t,r)$ for all $t, s, r \in \mathbb{R}$, such that $t \geq s \geq r$;

(ii) $U(t,t) = I$, for $t \in \mathbb{R}$;

(iii) $(t,s) \to U(t,s) \in \mathcal{L}(L^2(P,H))$ is strongly continuous, for $t > s$;

(iv) $U(\cdot, s) \in C'((s,\infty), \mathcal{L}(H))$ and

$$\frac{\partial}{\partial t} U(t,s) = A(t)U(t,s). \tag{2.2}$$

Definition 2.2 (see [8]). One says that an evolution family $\{U(t,s) : t \geq s, t, s \in \mathbb{R}\}$ is exponential dichotomy (or hyperbolic) if there are projections $P(t)$, $t \in \mathbb{R}$, being uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(1) $U(t,s)P(s) = P(t)U(t,s)$;

(2) the restriction $U_Q(t,s) : Q(s)H \to Q(t)H$ of $U(t,s)$ is invertible, and we set $U_Q(s,t) := U_Q(t,s)^{-1}$;

(3) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}\|$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$, for $t \geq s, t, s \in \mathbb{R}$.

If U has an exponential dichotomy, then the operator family

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s, t, s \in \mathbb{R}; \\ -U_Q(t,s)Q(s), & t < s, t, s \in \mathbb{R} \end{cases} \tag{2.3}$$

is called Green's function corresponding to U and $P(\cdot)$. If $P(t) = I$ for $t \in \mathbb{R}$, then U is exponentially stable.

The following lemma holds by [25].

Lemma 2.3. *If $A(t)$ satisfy the ATCs; that is, there exists a positive constant λ_0 such that the operator $A(t),\ t \in \mathbb{R}$, satisfying*

$$\begin{gathered}\Sigma_\theta \cup \{0\} \subseteq \rho(A(t) - \lambda_0), \qquad \|R(\lambda, A(t) - \lambda_0)\| \le \frac{K}{1+|\lambda|}, \\ \|(A(t) - \lambda_0)R(\lambda_0, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \le L|t-s|^\mu |\lambda|^{-\nu}\end{gathered} \tag{2.4}$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C} - \{0\} : |\arg \lambda| \le \theta\}$ and constants $\theta \in (\pi/2, \pi)$, $L, K \ge 0$, $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$, then there exists a unique evolution family $\{U(t,s) : t \ge s, t, s \in \mathbb{R}\}$ on $L^2(P, H)$.

Definition 2.4 (see [26]). A linear operator $A : \mathfrak{D}(A) \subset H \to H$ (not necessarily densely defined) is said to be sectorial if the following hold.

There exist constants $\zeta \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\begin{gathered}\rho(A) \supset S_{\theta,\zeta} := \{\lambda \in \mathbb{C} : \lambda \ne \zeta, |\arg(\lambda - \zeta)| < \theta\}, \\ \|R(\lambda, A)\| \le \frac{M}{|\lambda - \zeta|}, \quad \lambda \in S_{\theta,\zeta}.\end{gathered} \tag{2.5}$$

Let A be a sectorial operator on H and $\alpha \in (0,1)$. Define the real interpolation space

$$H_\alpha^A := \left\{x \in H : \|x\|_\alpha^A := \sup\|r^\alpha (A - \lambda_0) R(r, A - \lambda_0)x\| < \infty\right\}; \tag{2.6}$$

it is a Banach space endowed with the norm $\|\cdot\|_\alpha^A$. Given a family of linear operators $A(t)$, $t \in \mathbb{R}$, for $\alpha \in [0,1]$, we set $H_\alpha^t := H_\alpha^{A(t)}$ with the corresponding norms.

The following estimates for the evolution family U appeared in [8] are useful.

Lemma 2.5. *For $x \in H$, $0 \le \alpha \le 1$ and $t > s$, there exist some constants $c(\alpha)$, $m(\alpha)$ such that*

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha) e^{-(\delta/2)(t-s)} \|x\|, \tag{2.7}$$

$$\left\|\widetilde{U}_Q(s,t)Q(t)x\right\|_\alpha^s \le m(\alpha) e^{-\delta(t-s)} \|x\|. \tag{2.8}$$

Throughout the rest of this paper, we assume that the following conditions on $A(\cdot)$ and U hold:

(C_1) ATCs are satisfied and the evolution family U generated by $A(\cdot)$ has an exponential dichotomy with constants N, δ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$. Moreover, $0 \in \rho(A(t))$ for each $t \in \mathbb{R}$ and the following holds:

$$\sup_{t,s \in \mathbb{R}} \left\|A(s)A^{-1}(t)\right\|_{\mathcal{L}(H, H_\alpha)} \le c_0 \tag{2.9}$$

(C_2) there exists $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$ such that

$$H_\alpha^t = H_\alpha, \qquad H_\beta^t = H_\beta, \tag{2.10}$$

for all $t \in \mathbb{R}$, with uniform equivalent norms. And there exist constants $k(\alpha)$, k_1 such that

$$\begin{gathered} E\|x\|^2 \leq k_1 E\|x\|_\alpha^2, \quad x \in L^2(P, H_\alpha), \\ E\|x\|_\alpha^2 \leq k(\alpha) E\|x\|_\beta^2, \quad x \in L^2(P, H_\beta). \end{gathered} \tag{2.11}$$

Lemma 2.6 (see [8]). *Under the above assumptions, there exist constants $m(\alpha,\beta), n(\alpha,\beta) > 0$ such that*

$$\begin{gathered} \|A(s)U(t,s)P(s)x\|_\alpha \leq n(\alpha,\beta)(t-s)^{-\alpha} e^{-(\delta/2)(t-s)} \|x\|_\beta, \quad t > s, \\ \left\| A(s)\widetilde{U}_Q(t,s)Q(s)x \right\|_\alpha \leq m(\alpha,\beta) e^{-\delta(s-t)} \|x\|_\beta, \quad t \leq s. \end{gathered} \tag{2.12}$$

We recall some basic definitions and results of square-mean almost automorphic processes (see [18, 19]).

Let $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ be a Banach space and $L^2(P,\mathbb{B})$ its L^2-space.

Definition 2.7. A stochastic process $x : \mathbb{R} \to L^2(P,\mathbb{B})$ is said to be stochastically continuous if

$$\lim_{t \to s} E\|x(t) - x(s)\|_{\mathbb{B}}^2 = 0. \tag{2.13}$$

Definition 2.8 (see [18, 21]). A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(P,\mathbb{B})$ is said to be square-mean almost automorphic if for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m,n \to \infty} E\|x(t + s_n - s_m) - x(t)\|_{\mathbb{B}}^2 = 0. \tag{2.14}$$

This is equivalent to that there exists a stochastic process $y : \mathbb{R} \to L^2(P,\mathbb{B})$ such that, for each $t \in \mathbb{R}$,

$$\lim_{n \to \infty} E\left\|x(t + s_n) - y(t)\right\|_{\mathbb{B}}^2 = 0, \qquad \lim_{n \to \infty} E\left\|y(t - s_n) - x(t)\right\|_{\mathbb{B}}^2 = 0. \tag{2.15}$$

Denote by $\mathrm{SAA}(\mathbb{R}, L^2(P,\mathbb{B}))$ the collection of all the square-mean almost automorphic processes $x(t) : \mathbb{R} \to L^2(P,\mathbb{B})$. It is a Banach space equipped with the usual sup-norm

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \left(E\|x(t)\|_{\mathbb{B}}^2 \right)^{1/2}. \tag{2.16}$$

Lemma 2.9 (see [18]). *If x, y are square-mean almost automorphic processes, one has*

(i) *$x + y$ is square-mean almost automorphic;*

(ii) *λx is square-mean almost automorphic for every scalar λ;*

(iii) *there exists a positive constant M such that $\|x\|_\infty^2 \le M$.*

Let $(\mathbb{B}_i, \|\cdot\|_{\mathbb{B}_i})$, $i = 1, 2$ be Banach spaces, and $L^2(P, \mathbb{B}_i)$, $i = 1, 2$, their corresponding L^2-spaces, respectively.

Definition 2.10 (compare with [18, 21]). A function $f : \mathbb{R} \times L^2(P, \mathbb{B}_1) \to L^2(P, \mathbb{B}_2)$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(P, \mathbb{B}_1)$, if, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m,n\to\infty} E\|f(t + s_n - s_m, x) - f(t, x)\|_{\mathbb{B}_2}^2 = 0, \tag{2.17}$$

for each $t \in \mathbb{R}$ and $x \in L^2(P, \mathbb{B}_1)$.

This is equivalent to that there exists a function $\tilde{f} : \mathbb{R} \times L^2(P, \mathbb{B}_1) \to L^2(P, \mathbb{B}_2)$ such that, for each $t \in \mathbb{R}$ and $x \in L^2(P, \mathbb{B}_1)$,

$$\lim_{n\to\infty} E\left\|f(t + s_n, x) - \tilde{f}(t, x)\right\|_{\mathbb{B}_2}^2 = 0, \qquad \lim_{n\to\infty} E\left\|\tilde{f}(t - s_n, x) - f(t, x)\right\|_{\mathbb{B}_2}^2 = 0. \tag{2.18}$$

We need the following composition of square-mean almost automorphic processes.

Lemma 2.11. *Suppose that $f : \mathbb{R} \times L^2(P, \mathbb{B}_1) \to L^2(P, \mathbb{B}_2)$ is square-mean almost automorphic in t, and assume that $f(t, x)$ satisfies*

$$\|f(t, x) - f(t, y)\|_{\mathbb{B}_2}^2 \le \kappa\left(\|x - y\|_{\mathbb{B}_1}^2\right), \quad \text{for any } \; x, y \in L^2(P, \mathbb{B}_1),\; t \in \mathbb{R}, \tag{2.19}$$

where κ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ and $\int_{0+}(du/\kappa(u)) = +\infty$. Then for any square-mean almost automorphic process $\phi : \mathbb{R} \to L^2(P, \mathbb{B}_1)$, the stochastic process $F : \mathbb{R} \to L^2(P, \mathbb{B}_2)$ given by $F(s) := f(s, \phi(s))$ is square-mean almost automorphic.

Proof. Since f and ϕ are square-mean almost automorphic processes, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}} \subset \{s'_n\}_{n\in\mathbb{N}}$ and some functions $\tilde{f}$, $\tilde{\phi}$ such that, for each $t \in \mathbb{R}$, $x \in L^2(P, \mathbb{B}_1)$,

$$\lim_{n\to\infty} E\left\|f(t + s_n, x) - \tilde{f}(t, x)\right\|_{\mathbb{B}_2}^2 = 0, \qquad \lim_{n\to\infty} E\left\|\phi(t + s_n) - \tilde{\phi}(t)\right\|_{\mathbb{B}_1}^2 = 0. \tag{2.20}$$

Let $\widetilde{F}(t) := f(t, \widetilde{\phi}(t))$. Then we have

$$\begin{aligned} E\left\| F(t+s_n) - \widetilde{F}(t) \right\|_{\mathbb{B}_2}^2 &\leq 2E\left\| f(t+s_n, \phi(t+s_n)) - f\left(t+s_n, \widetilde{\phi}(t)\right) \right\|_{\mathbb{B}_2}^2 \\ &\quad + 2E\left\| f\left(t+s_n, \widetilde{\phi}(t)\right) - \widetilde{f}\left(t, \widetilde{\phi}(t)\right) \right\|_{\mathbb{B}_2}^2 \\ &\leq 2E\kappa\left(\left\| \phi(t+s_n) - \widetilde{\phi}(t) \right\|_{\mathbb{B}_1}^2 \right) \\ &\quad + 2E\left\| f\left(t+s_n, \widetilde{\phi}(t)\right) - \widetilde{f}\left(t, \widetilde{\phi}(t)\right) \right\|_{\mathbb{B}_2}^2; \end{aligned} \tag{2.21}$$

by Jensen's inequality, it follows that

$$\begin{aligned} E\left\| F(t+s_n) - \widetilde{F}(t) \right\|_{\mathbb{B}_2}^2 &\leq 2\kappa\left(E\left\| \phi(t+s_n) - \widetilde{\phi}(t) \right\|_{\mathbb{B}_1}^2 \right) \\ &\quad + 2E\left\| f\left(t+s_n, \widetilde{\phi}(t)\right) - \widetilde{f}\left(t, \widetilde{\phi}(t)\right) \right\|_{\mathbb{B}_2}^2; \end{aligned} \tag{2.22}$$

noting that κ is concave and $\kappa(0) = 0$, we deduce that

$$\lim_{n\to\infty} E\|F(t+s_n) - \widetilde{F}(t)\|_{\mathbb{B}_2}^2 = 0, \quad \text{for each } t \in \mathbb{R}. \tag{2.23}$$

Similarly, we can prove that $\lim_{n\to\infty} E\|\widetilde{F}(t-s_n) - F(t)\|_{\mathbb{B}_2}^2 = 0$; this completes the proof. □

Lemma 2.12 (see [22]). *Let $L \in \mathcal{L}(L^2(P, \mathbb{B}_1), L^2(P, \mathbb{B}_2))$ and assume $f \in SAA(\mathbb{R}, L^2(P, \mathbb{B}_1))$. Then $Lf \in SAA(\mathbb{R}, L^2(P, \mathbb{B}_2))$.*

The consideration is mainly based on the following fixed point theorem of Krasnoselskii (see [27]).

Lemma 2.13. *Let C be a closed, bounded, and convex subset of a Banach space X. Let A and B be operators, defined on C satisfying the conditions:*

(a) *$A\phi + B\varphi \in C$ when $\phi, \varphi \in C$;*

(b) *the operator A is a contraction;*

(c) *the operator B is continuous and $B(C)$ is contained in a compact set.*

Then the equation $x = Ax + Bx$ has a solution in C.

3. Existence of Square-Mean Almost Automorphic Mild Solutions

Firstly, we present the definition of mild solution for system (1.3).

Definition 3.1. An continuous stochastic function $x : \mathbb{R} \times L^2(P, H_\alpha)$ is called a mild solution of (1.3) provided that the function

$$s \longrightarrow E\left\| A(s)U(t,s)P(s)g(s, B_1 x(s)) \right\|^2 \tag{3.1}$$

is integrable on $(-\infty, t)$,

$$s \longrightarrow E\left\| A(s)U(t,s)Q(s)g(s, B_1 x(s)) \right\|^2 \tag{3.2}$$

is integrable on (t, ∞) for each $t \in \mathbb{R}$, and $x(t)$ satisfies the following stochastic integral equation:

$$\begin{aligned} x(t) = & -g(t, B_1 x(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)g(s, B_1 x(s))ds \\ & + \int_{t}^{\infty} A(s)U(t,s)Q(s)g(s, B_1 x(s))ds + \int_{-\infty}^{t} U(t,s)P(s)b(s, B_2 x(s))ds \\ & - \int_{t}^{\infty} U(t,s)Q(s)b(s, B_2 x(s))ds + \int_{-\infty}^{t} U(t,s)P(s)\sigma(s, B_3 x(s))dW(s) \\ & - \int_{t}^{\infty} U(t,s)Q(s)\sigma(s, B_3 x(s))dW(s). \end{aligned} \tag{3.3}$$

In order to obtain our main results, we need the following assumptions.

(A_1) $B_i : L^2(P, H_\alpha) \to L^2(P, H)$, $i = 1, 2, 3$, are bounded linear operators, and we set

$$\eta := \left\{ \|B_i\|_{\mathcal{L}(L^2(P,H_\alpha), L^2(P,H))}, \ i = 1, 2, 3 \right\}. \tag{3.4}$$

(A_2) $R(\lambda_0, A(\cdot)) \in \mathcal{L}(\mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha)))$. For any sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that, for each $\epsilon > 0$, one can find $N_0 \in \mathbb{N}$ such that

$$\|A(s + s_n - s_m)U(t + s_n - s_m, s + s_n - s_m)P(s + s_n - s_m) - A(s)U(t,s)P(s)\|_\alpha \le \epsilon H(t - s) \tag{3.5}$$

whenever $n, m > N_0$, $t, s \in \mathbb{R}$, $t > s$, where $H : [0, \infty) \to [0, \infty)$ is integrable.

(A_3) Let $0 \le \alpha < 1/2 < \beta < 1$. $g : \mathbb{R} \times L^2(P, H) \to L^2(P, H_\beta)$ is square-mean almost automorphic in $t \in \mathbb{R}$, and there exists a small $K_g > 0$ such that

$$E\left\| g(t,x) - g(t,y) \right\|_\beta^2 \le K_g E\left\| x - y \right\|^2, \tag{3.6}$$

for all $t \in \mathbb{R}$ and $x, y \in L^2(P, H)$.

(A_4) $b : \mathbb{R} \times L^2(P, H) \to L^2(P, H)$, $\sigma : \mathbb{R} \times L^2(P, H) \to L^2(P, \mathcal{L}_2^0(K, H))$ are square-mean almost automorphic in $t \in \mathbb{R}$, and, for each $t \in \mathbb{R}$, $\phi, \varphi \in L^2(P, H)$,

$$\|b(t,\phi) - b(t,\varphi)\|^2 \vee \|\sigma(t,\phi) - \sigma(t,\varphi)\|_{\mathcal{L}_2^0}^2 \leq \kappa\left(\|\phi - \varphi\|^2\right), \tag{3.7}$$

where $\kappa(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a concave nondecreasing function such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+}(du/\kappa(u)) = +\infty$.

(A_5) For any $\epsilon > 0$, there exist a constant $\omega > 0$ and nondecreasing continuous functions $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $t \in \mathbb{R}$ and $x \in L^2(P, H)$ with $E\|x\|^2 > \omega$,

$$E\|b(t,x)\|^2 \vee E\|\sigma(t,x)\|_{\mathcal{L}_2^0}^2 \leq \epsilon\Lambda\left(E\|x\|^2\right). \tag{3.8}$$

Remark 3.2. Functions such as $\kappa_1(u) = u$ and

$$\kappa_2(u) = \begin{cases} u \cdot \left(\log \dfrac{1}{u}\right), & 0 \leq u \leq b < 1, \\ b \cdot \left(\log \dfrac{1}{u}\right) + \kappa_2(b-)'(u - b), & u \geq b \end{cases} \tag{3.9}$$

satisfy assumption (A_4); in particular, we see that the Lipschitz condition is a special case of the proposed assumptions.

Throughout the rest of this paper, we denote by

$$\|x\|_{\alpha,\infty} := \sup_{t\in\mathbb{R}}\left(E\|x(t)\|_\alpha^2\right)^{1/2} \tag{3.10}$$

the sup-norm of the space $\mathrm{SAA}(R, L^2(P, H_\alpha))$. Let Π_i, $i = 1, \ldots, 6$, be the operators defined by

$$\begin{aligned}
\Pi_1 x(t) &= \int_{-\infty}^{t} A(s)U(t,s)P(s)g(s, B_1x(s))ds, \\
\Pi_2 x(t) &= \int_{t}^{\infty} A(s)U(t,s)Q(s)g(s, B_1x(s))ds, \\
\Pi_3 x(t) &= \int_{-\infty}^{t} U(t,s)P(s)b(s, B_2x(s))ds, \\
\Pi_4 x(t) &= \int_{t}^{\infty} U(t,s)Q(s)b(s, B_2x(s))ds, \\
\Pi_5 x(t) &= \int_{-\infty}^{t} U(t,s)P(s)\sigma(s, B_3x(s))dW(s), \\
\Pi_6 x(t) &= \int_{t}^{\infty} U(t,s)Q(s)\sigma(s, B_3x(s))dW(s).
\end{aligned} \tag{3.11}$$

Lemma 3.3. *Under assumptions (A_1)–(A_3), the operators Π_i, $i = 1,2$, defined above map $SAA(\mathbb{R}, L^2(P, H_\alpha))$ into itself.*

Proof. Let $x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. By Lemma 2.12, $s \to B_i x(s) \in \mathrm{SAA}(\mathbb{R}, L^2(P, H))$ as $B_i \in \mathcal{L}(L^2(P, H_\alpha), L^2(P, H))$, $i = 1,2,3$. And hence, $G(s) := g(s, B_1 x(s)) \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\beta))$ by Lemma 2.11. In particular, $\sup_{t\in\mathbb{R}} E\|G(t)\|_\beta^2 < \infty$. Let us show that $\Pi_1 x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. Indeed, since $G(s) \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\beta))$, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\lim_{m,n\to\infty} E\|G(t + s_n - s_m) - G(t)\|_\beta^2 = 0. \tag{3.12}$$

From (A_2), for any $\epsilon > 0$, one can find $N_0 \in \mathbb{N}$ such that

$$\|A(s + s_n - s_m)U(t + s_n - s_m, s + s_n - s_m)P(s + s_n - s_m) - A(s)U(t,s)P(s)\|_\alpha \le \epsilon H(t - s), \tag{3.13}$$

whenever $n, m > N_0$, $t, s \in \mathbb{R}$, $t > s$. Thus,

$$\begin{aligned}
&E\|(\Pi_1 x)(t + s_n - s_m) - (\Pi_1 x)(t)\|_\alpha^2 \\
&\quad = E\left\|\int_{-\infty}^{t+s_n-s_m} A(s)U(t + s_n - s_m, s)P(s)G(s)ds - \int_{-\infty}^{t} A(s)U(t,s)P(s)G(s)ds\right\|_\alpha^2 \\
&\quad = E\left\|\int_{-\infty}^{t} A(s + s_n - s_m)U(t + s_n - s_m, s + s_n - s_m)P(s + s_n - s_m)\right. \\
&\qquad\qquad \left.\cdot G(s + s_n - s_m)ds - \int_{-\infty}^{t} A(s)U(t,s)P(s)G(s)ds\right\|_\alpha^2 \\
&\quad \le 2E\left\|\int_{-\infty}^{t} [A(s + s_n - s_m)U(t + s_n - s_m, s + s_n - s_m)\right. \\
&\qquad\qquad\quad \left.\cdot P(s + s_n - s_m) - A(s)U(t,s)P(s)]G(s)ds\right\|_\alpha^2 \\
&\qquad + 2E\left\|\int_{-\infty}^{t} A(s + s_n - s_m)U(t + s_n - s_m, s + s_n - s_m)\right. \\
&\qquad\qquad \left.\cdot P(s + s_n - s_m)[G(s + s_n - s_m) - G(s)]ds\right\|_\alpha^2 \\
&\quad = 2(L_1 + L_2).
\end{aligned} \tag{3.14}$$

Using (3.13) and condition (C_1), one has

$$\begin{aligned} L_1 &\le E\left(\int_{-\infty}^{t} \|(A(s+s_n-s_m)U(t+s_n-s_m,s+s_n-s_m)\right.\\ &\qquad\left.\cdot P(s+s_n-s_m)-A(s)U(t,s)P(s))G(s)\|_\alpha ds\right)^2\\ &\le \epsilon^2\int_{-\infty}^{t} H(t-s)ds\int_{-\infty}^{t} H(t-s)E\|G(s)\|_\alpha^2 ds\\ &\le \left(\int_{-\infty}^{t} H(t-s)ds\right)^2 k(\alpha)\sup_{t\in\mathbb{R}} E\|G(t)\|_\beta^2\\ &\le \epsilon^2 k(\alpha)\|H\|_{L^1}^2\sup_{t\in\mathbb{R}} E\|G(t)\|_\beta^2. \end{aligned} \tag{3.15}$$

For L_2, we use Lemma 2.6 to get

$$\begin{aligned} L_2 &\le E\left(\int_{-\infty}^{t} \|A(s+s_n-s_m)U(t+s_n-s_m,s+s_n-s_m)\right.\\ &\qquad\left.\cdot P(s+s_n-s_m)[G(s+s_n-s_m)-G(s)]\|_\alpha ds\right)^2\\ &\le n^2(\alpha,\beta)E\left[\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-(\delta/2)(t-s)}\|G(s+s_n-s_m)-G(s)\|_\beta ds\right]^2\\ &\le n^2(\alpha,\beta)\Gamma^2(1-\alpha)\sup_{t\in\mathbb{R}} E\|G(s+s_n-s_m)-G(s)\|_\beta^2. \end{aligned} \tag{3.16}$$

Combing this estimates with (3.12), one obtains

$$\lim_{m,n\to\infty} E\|(\Pi_1 x)(t+s_n-s_m)-(\Pi_1 x)(t)\|_\alpha^2 = 0, \tag{3.17}$$

which implies that $\Pi_1 x\in \mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$. By a similar argument, we can show that $\Pi_2 x\in \mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$. □

Lemma 3.4. *Under assumptions (A_1), (A_2), and (A_4), the operators Π_i, $i=3,4$, defined above map $SAA(\mathbb{R}, L^2(P,H_\alpha))$ into itself.*

Proof. Let $x \in \mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$. By Lemmas 2.11 and 2.12 it follows that $\Xi(s) := b(s,B_2x(s)) \in \mathrm{SAA}(\mathbb{R}, L^2(P,H))$. Particularly, $\sup_{t\in\mathbb{R}} E\|\Xi(t)\|^2 < \infty$. We now show that

$\Pi_3 x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. Since $\Xi(s) := b(s, B_2 x(s)) \in \mathrm{SAA}(\mathbb{R}, L^2(P, H))$, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\lim_{m,n\to\infty} E\|\Xi(t + s_n - s_m) - \Xi(t)\|^2 = 0. \tag{3.18}$$

For any $\epsilon > 0$, by making changes of variables we have

$$\begin{aligned}
&E\|(\Pi_3 x)(t + s_n - s_m) - (\Pi_3 x)(t)\|_\alpha^2 \\
&\quad = E\left\|\int_{-\infty}^{t+s_n-s_m} U(t + s_n - s_m, s)P(s)\Xi(s)ds - \int_{-\infty}^{t} U(t, s)P(s)\Xi(s)ds\right\|_\alpha^2 \\
&\quad = E\left\|\int_{-\infty}^{t} U(t + s_n - s_m, s + s_n - s_m)P(s + s_n - s_m)\Xi(s + s_n - s_m)ds\right. \\
&\qquad \left. - \int_{-\infty}^{t} U(t, s)P(s)\Xi(s)ds\right\|_\alpha^2 \\
&\quad = E\left\|\int_{0}^{\infty} U(t + s_n - s_m, t - s + s_n - s_m)P(t - s + s_n - s_m)\Xi(t - s + s_n - s_m)ds\right. \\
&\qquad \left. - \int_{-\infty}^{t} U(t, t - s)P(t - s)\Xi(t - s)ds\right\|_\alpha^2;
\end{aligned} \tag{3.19}$$

an elementary inequality shows that

$$\begin{aligned}
&E\|(\Pi_3 x)(t + s_n - s_m) - (\Pi_3 x)(t)\|_\alpha^2 \\
&\quad \le 3E\left\|\int_{0}^{\infty} U(t + s_n - s_m, t - s + s_n - s_m)P(t - s + s_n - s_m)\right. \\
&\qquad\qquad \left. \times[\Xi(t - s + s_n - s_m) - \Xi(t - s)]ds\right\|_\alpha^2 \\
&\qquad + 3E\left\|\int_{0}^{\epsilon} [U(t + s_n - s_m, t - s + s_n - s_m)P(t - s + s_n - s_m)\right. \\
&\qquad\qquad \left. -U(t, t - s)P(t - s)]\Xi(t - s)ds\right\|_\alpha^2 \\
&\qquad + 3E\left\|\int_{\epsilon}^{\infty} [U(t + s_n - s_m, t - s + s_n - s_m)P(t - s + s_n - s_m)\right. \\
&\qquad\qquad \left. -U(t, t - s)P(t - s)]\Xi(t - s)ds\right\|_\alpha^2 \\
&\quad = 3(I_1 + I_2 + I_3).
\end{aligned} \tag{3.20}$$

Using Lemma 2.5, one has

$$\begin{aligned}
I_1 &\le c^2(\alpha)E\left[\int_0^t s^{-\alpha}e^{-(\delta/2)s}\|\Xi(t-s+s_n-s_m)-\Xi(t-s)\|ds\right]^2 \\
&\le c^2(\alpha)\left(\frac{\delta}{2}\right)^{\alpha-1}\Gamma^2(1-\alpha)\sup_{t\in\mathbb{R}}E\|\Xi(t+s_n-s_m)-\Xi(t)\|^2;
\end{aligned}\tag{3.21}$$

a straightforward computation yields

$$\begin{aligned}
I_2 &\le E\left[\int_0^\epsilon \|[U(t+s_n-s_m,t-s+s_n-s_m)P(t-s+s_n-s_m)-U(t,t-s)P(t-s)]\Xi(t-s)\|_\alpha ds\right]^2 \\
&\le 4\epsilon^2M^2\sup_{t\in\mathbb{R}}E\|\Xi(t)\|^2,
\end{aligned}\tag{3.22}$$

where M is a constant satisfying $\sup_{t,s}\|U(t,s)P(s)\|\le M$.

For I_3, applying Proposition 4.4 in [4], we have

$$\begin{aligned}
I_3 &\le E\left[\int_\epsilon^\infty \|[U(t+s_n-s_m,t-s+s_n-s_m)P(t-s+s_n-s_m)\right. \\
&\qquad \left. -U(t,t-s)P(t-s)]\Xi(t-s)\|_\alpha ds\right]^2 \\
&\le \epsilon^2\int_\epsilon^\infty e^{-(\delta/2)s}ds\int_\epsilon^\infty e^{-(\delta/2)s}E\|\Xi(t-s)\|^2ds \\
&\le 4\epsilon^2\delta^{-2}\sup_{t\in\mathbb{R}}E\|\Xi(t)\|^2.
\end{aligned}\tag{3.23}$$

Combing this estimates with (3.18), we get

$$\lim_{m,n\to\infty}E\|\Pi_3(t+s_n-s_m)-\Pi_3(t)\|_\alpha^2=0,\tag{3.24}$$

which implies that $\Pi_3x\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$ whenever $x\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$. Similarly, we can verify that $\Pi_4x\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$. □

Lemma 3.5. *Under assumptions (A_1), (A_2) and (A_4), the operators Π_i, $i=5,6$, defined above map $SAA(\mathbb{R},L^2(P,H_\alpha))$ into itself.*

Proof. Let $x\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$. Using Lemmas 2.11 and 2.12, we get $\Sigma(s):=\sigma(s,B_3x(s))\in \mathrm{SAA}(\mathbb{R},L^2(P,\mathcal{L}_2^0(K,H)))$. Particularly, $\sup_{t\in\mathbb{R}}E\|\Sigma(t)\|^2_{\mathcal{L}_2^0}<\infty$. We now show that $\Pi_5x\in$

$\mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. Since $\Sigma(s) := \sigma(s, B_3x(s)) \in \mathrm{SAA}(\mathbb{R}, L^2(P, H))$, for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ such that, for each $t \in \mathbb{R}$,

$$\lim_{m,n\to\infty} E\|\Sigma(t + s_n - s_m) - \Sigma(t)\|^2_{\mathcal{L}_2^0} = 0. \tag{3.25}$$

Note that the process $\widetilde{W}(\tau) := W(t+\tau) - W(t)$ for each $\tau \in \mathbb{R}$ is also a Brownian motion and has the same distribution as W. For any $\epsilon > 0$, similar argument as above, we have

$$\begin{aligned}
&E\|(\Pi_5 x)(t + s_n - s_m) - (\Pi_5 x)(t)\|^2_\alpha \\
&\quad= E\left\| \int_{-\infty}^{t} U(t + s_n - s_m, s + s_n - s_m) P(s + s_n - s_m)\Sigma(s + s_n - s_m) d\widetilde{W}(s) \right. \\
&\qquad\qquad \left. - \int_{-\infty}^{t} U(t,s)P(s)\Sigma(s)dW(s) \right\|^2_\alpha \\
&\quad= E\left\| \int_0^{\infty} U(t + s_n - s_m, t - s + s_n - s_m) P(t - s + s_n - s_m) \right. \\
&\qquad\qquad \left. \cdot \Sigma(t - s + s_n - s_m) dW(s) - \int_0^{\infty} U(t, t-s)P(t-s)\Sigma(t-s)dW(s) \right\|^2_\alpha \\
&\quad\le 3c^2(\alpha)\int_0^{\infty} s^{-2\alpha} e^{-\delta s} E\|\Sigma(t - s + s_n - s_m) - \Sigma(t-s)\|^2_{\mathcal{L}_2^0} ds \\
&\qquad + 12\epsilon M^2 \sup_{t\in\mathbb{R}} E\|\Sigma(t)\|^2_{\mathcal{L}_2^0} + 3\epsilon^2 \int_\epsilon^{\infty} e^{-\delta s} E\|\Sigma(t-s)\|^2_{\mathcal{L}_2^0} ds \\
&\quad\le 3c^2(\alpha)\delta^{2\alpha-1}\Gamma(2\alpha - 1)\sup_{t\in\mathbb{R}} E\|\Sigma(t + s_n - s_m) - \Sigma(t)\|^2_{\mathcal{L}_2^0} \\
&\qquad + 12\epsilon M^2 \sup_{t\in\mathbb{R}} E\|\Sigma(t)\|^2_{\mathcal{L}_2^0} + 3\delta^{-1}\epsilon^2 \sup_{t\in\mathbb{R}} E\|\Sigma(t)\|^2_{\mathcal{L}_2^0}.
\end{aligned} \tag{3.26}$$

From (3.25), we immediately get

$$\lim_{m,n\to\infty} E\|\Pi_5(t + s_n - s_m) - \Pi_5(t)\|^2_\alpha = 0, \tag{3.27}$$

which implies that $\Pi_5 x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. Similarly, we can show that $\Pi_6 x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$ whenever $x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$.

Consider the nonlinear operators Q_i, $i = 1, 2$, on $\mathrm{SAA}(R, L^2(P, H_\alpha))$ defined by

$$\begin{aligned}
Q_1 x(t) = & -g(t, B_1 x(t)) - \int_{-\infty}^{t} A(s) U(t,s) P(s) g(s, B_1 x(s)) ds \\
& + \int_{t}^{\infty} A(s) U(t,s) Q(s) g(s, B_1 x(s)) ds, \\
Q_2 x(t) = & \int_{-\infty}^{t} U(t,s) P(s) b(s, B_2 x(s)) ds - \int_{t}^{\infty} U(t,s) Q(s) b(s, B_2 x(s)) ds \\
& + \int_{-\infty}^{t} U(t,s) P(s) \sigma(s, B_3 x(s)) dW(s) - \int_{t}^{\infty} U(t,s) Q(s) \sigma(s, B_3 x(s)) dW(s),
\end{aligned} \tag{3.28}$$

for each $t \in \mathbb{R}$. In view of Lemmas 3.3, 3.4, and 3.5, it follows that Q_i, $i = 1, 2$, map $\mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$ into itself. In what follows, we will prove that Q_i, $i = 1, 2$, satisfy all the conditions in Lemma 2.13. □

Lemma 3.6. *Under assumptions* (A_1)*–*(A_3)*, the operator* Q_1 *defined above is a contraction provided that*

$$\Theta := 3K_g \eta \left(k(\alpha) + n^2(\alpha) \Gamma^2(1-\alpha) \left(\frac{2}{\delta} \right)^{2(1-\alpha)} + m^2(\alpha, \beta) \delta^{-2} \right) < 1. \tag{3.29}$$

Proof. Let $x, y \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. By using condition (C_2) and assumptions (A_1), (A_3), we have

$$\begin{aligned}
& E \left\| g(t, B_1 x(t)) - g(t, B_1 y(t)) \right\|_\alpha^2 \\
& \quad \leq k(\alpha) E \left\| g(t, B_1 x(t)) - g(t, B_1 y(t)) \right\|_\beta^2 \\
& \quad \leq k(\alpha) K_g E \| B_1 x(t) - B_1 y(t) \|^2 \\
& \quad \leq k(\alpha) K_g \eta \| x - y \|_{\alpha,\infty}^2.
\end{aligned} \tag{3.30}$$

Now, using Lemma 2.6 together with Hölder's inequality, we obtain

$$
\begin{aligned}
&E\left\|\int_{-\infty}^{t} A(s)U(t,s)P(s)\left[g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right]ds\right\|_{\alpha}^{2} \\
&\quad\le E\left[\int_{-\infty}^{t}\left\|A(s)U(t,s)P(s)\left[g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right]\right\|_{\alpha}ds\right] \\
&\quad\le n^2(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-(\delta/2)(t-s)}ds \\
&\qquad\times\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-(\delta/2)(t-s)}\cdot E\left\|g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right\|_{\beta}^{2}ds \\
&\quad\le n^2(\alpha)K_g\eta\Gamma^2(1-\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}\|x-y\|_{\alpha,\infty}^{2}.
\end{aligned}
\tag{3.31}
$$

Similarly,

$$
\begin{aligned}
&E\left\|\int_{t}^{\infty} A(s)U(t,s)Q(s)\left[g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right]ds\right\|_{\alpha}^{2} \\
&\quad\le E\left[\int_{t}^{\infty}\left\|A(s)U(t,s)Q(s)\left[g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right]\right\|_{\alpha}ds\right] \\
&\quad\le m^2(\alpha,\beta)\int_{t}^{\infty}e^{\delta(t-s)}ds\int_{t}^{\infty}e^{\delta(t-s)}E\left\|g\left(s,B_1x(s)\right)-g\left(s,B_1y(s)\right)\right\|_{\beta}^{2}ds \\
&\quad\le m^2(\alpha,\beta)\eta K_g\delta^{-2}\|x-y\|_{\alpha,\infty}^{2}.
\end{aligned}
\tag{3.32}
$$

Thus,

$$
\|Q_1x-Q_1y\|_{\alpha,\infty}^{2}\le\Theta\|x-y\|_{\alpha,\infty}^{2}.
\tag{3.33}
$$

The proof is completed. □

Lemma 3.7. *Under assumptions (A_1) and (A_4), the operator Q_2 defined above is continuous; moreover, its image is contained in a compact set.*

Proof. Let $V_r=\{x\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha)):\|x\|_{\alpha,\infty}^{2}\le r\}$, for some $r>0$. It is obvious that V_r is a closed bounded convex subset of $L^2(P,H_\alpha)$. We begin with the continuity of Q_2. Let $\{x^n\}\in \mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$ be a sequence with $x_n\to x$; that is, $\lim_{n\to\infty}\|x^n-x\|_{\alpha,\infty}^{2}=0$. Using the estimates in Lemma 2.5, we get

$$
\begin{aligned}
&E\left\|\int_{-\infty}^{t}U(t,s)P(s)\left[b\left(s,B_2x^n(s)\right)-b\left(s,B_2x(s)\right)\right]ds\right\|_{\alpha}^{2} \\
&\quad\le c^2(\alpha)E\left[\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-(\delta/2)(t-s)}\left\|b\left(s,B_2x^n(s)\right)-b\left(s,B_2x(s)\right)\right\|ds\right]^{2},
\end{aligned}
\tag{3.34}
$$

by the continuity of b, B_2, and Lebesgue's dominated convergence theorem, it follows that

$$E\left\|\int_{-\infty}^{t} U(t,s)P(s)[b(s,B_2x^n(s)) - b(s,B_2x(s))]ds\right\|_{\alpha}^{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

Similarly, it is easy to show that

$$E\left\|\int_{t}^{\infty} U(t,s)Q(s)[b(s,B_2x^n(s)) - b(s,B_2x(s))]ds\right\|_{\alpha}^{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.36}$$

Applying the isometry inequality, we obtain

$$\begin{aligned} &E\left\|\int_{-\infty}^{t} U(t,s)P(s)[\sigma(s,B_3x^n(s)) - \sigma(s,B_3x(s))]ds\right\|_{\alpha}^{2} \\ &\quad\leq c^2(\alpha)\int_{-\infty}^{t} (t-s)^{-2\alpha}e^{-\delta(t-s)}E\|\sigma(s,B_3x^n(s)) - \sigma(s,B_3x(s))\|_{\mathcal{L}_2^0}^{2}ds, \end{aligned} \tag{3.37}$$

by the continuity of b, B_2, and Lebesgue's dominated convergence theorem yields

$$E\left\|\int_{-\infty}^{t} U(t,s)P(s)[\sigma(s,B_3x^n(s)) - \sigma(s,B_3x(s))]dW(s)\right\|_{\alpha}^{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.38}$$

Similarly, it is easy to show that

$$E\left\|\int_{t}^{\infty} U(t,s)Q(s)[\sigma(s,B_3x^n(s)) - \sigma(s,B_3x(s))]dW(s)\right\|_{\alpha}^{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.39}$$

Thus,

$$E\|Q_2x(t) - Q_2x^n(t)\|_{\alpha}^{2} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.40}$$

which implies that $\|Q_2x - Q_2x^n\|_{\alpha,\infty}^{2} \to 0$ as $n \to \infty$.

Next, we show that $Q_2(V_r)$ is contained in a compact set. In fact, by the Ascoli-Arzela theorem, it suffices to show that Q_2 maps V_r into a equicontinuous family. Let $x \in V_r$ be arbitrary and $t_1 < t_2$.

An analogue argument as Lemma 4.8 in [13], we have

$$\begin{aligned}
&E\|\Pi_3 x(t_2) - \Pi_3 x(t_1)\|_\alpha^2 \\
&\quad \leq 2E\left\|\int_{-\infty}^{t_1} (U(t_2,s) - U(t_1,s))P(s)b(s,B_2x(s))ds\right\|_\alpha^2 \\
&\qquad + 2E\left\|\int_{t_1}^{t_2} U(t_2,s)P(s)b(s,B_2x(s))ds\right\|_\alpha^2 \\
&\quad = 2E\left\|\int_{-\infty}^{t_1}\left(\int_{t_1}^{t_2}\frac{\partial U(\tau,s)}{\partial\tau}d\tau\right)P(s)b(s,B_2x(s))ds\right\|_\alpha^2 \\
&\qquad + 2E\left\|\int_{t_1}^{t_2} U(t_2,s)P(s)b(s,B_2x(s))ds\right\|_\alpha^2 \\
&\quad \leq 2E\left[\int_{-\infty}^{t_1}\int_{t_1}^{t_2}\|A(\tau)U(\tau,s)P(s)b(s,B_2x(s))\|_\alpha d\tau\, ds\right]^2 \\
&\qquad + 2E\left[\int_{t_1}^{t_2}\|U(t_2,s)P(s)b(s,B_2x(s))\|_\alpha ds\right]^2.
\end{aligned} \tag{3.41}$$

For the first term on the right-hand side of (3.41), we have

$$\begin{aligned}
&2E\left[\int_{-\infty}^{t_1}\int_{t_1}^{t_2}\|A(\tau)U(\tau,s)P(s)b(s,B_2x(s))\|_\alpha d\tau\, ds\right]^2 \\
&\quad = 2E\left[\int_{-\infty}^{t_1}\int_{t_1}^{t_2}\left\|A(s)A^{-1}(s)A(\tau)U(\tau,s)P(s)b(s,B_2x(s))\right\|_\alpha d\tau\, ds\right]^2 \\
&\quad \leq 2c_0^2 E\left[\int_{-\infty}^{t_1}\int_{t_1}^{t_2}\|A(s)U(\tau,s)P(s)b(s,B_2x(s))\|_\alpha d\tau\, ds\right]^2 \\
&\quad \leq 2c_0^2 n^2(\alpha,\beta)E\left[\int_{-\infty}^{t_1}\int_{t_1}^{t_2}(\tau-s)^{-\alpha}e^{-(\delta/2)(\tau-s)}d\tau\|b(s,B_2x(s))\|ds\right]^2,
\end{aligned} \tag{3.42}$$

and for the second term, we get

$$\begin{aligned}
&2E\left[\int_{t_1}^{t_2}\|U(t_2,s)P(s)b(s,B_2x(s))\|_\alpha ds\right]^2 \\
&\quad \leq 2c^2(\alpha)E\left[\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-(\delta/2)(t_2-s)}\|b(s,B_2x(s))\|ds\right]^2.
\end{aligned} \tag{3.43}$$

Combing these estimates with (3.41), it follows that there exists a positive constant $k_1(\alpha,\delta)$ such that

$$E\|\Pi_3 x(t_2) - \Pi_3 x(t_1)\|_\alpha^2 \le k_1(\alpha,\delta)(t_2-t_1)^2 \sup_{t\in\mathbb{R}} E\|b(s,B_2x(s))\|^2. \tag{3.44}$$

Similar computation can show that there exists a positive constant $k_2(\alpha,\delta)$ such that

$$E\|\Pi_4 x(t_2) - \Pi_4 x(t_1)\|_\alpha^2 \le k_2(\alpha,\delta)(t_2-t_1)^2 \sup_{t\in\mathbb{R}} E\|b(s,B_2x(s))\|^2. \tag{3.45}$$

As to $\Pi_5 x$, we have

$$\begin{aligned} &E\|\Pi_5 x(t_2) - \Pi_5 x(t_1)\|_\alpha^2 \\ &\quad \le 2E\left\|\int_{-\infty}^{t_1}(U(t_2,s)-U(t_1,s))P(s)\sigma(s,B_2x(s))dW(s)\right\|_\alpha^2 \\ &\quad + 2E\left\|\int_{t_1}^{t_2}U(t_2,s)P(s)\sigma(s,B_3x(s))dW(s)\right\|_\alpha^2. \end{aligned} \tag{3.46}$$

For the first term on the right-hand side of (3.46), we have

$$\begin{aligned} &2E\left\|\int_{-\infty}^{t_1}(U(t_2,s)-U(t_1,s))P(s)\sigma(s,B_2x(s))dW(s)\right\|_\alpha^2 \\ &= 2E\left\|\int_{-\infty}^{t_1}\left(\int_{t_1}^{t_2}\frac{\partial U(\tau,s)}{\partial\tau}d\tau\right)P(s)\sigma(s,B_2x(s))dW(s)\right\|_\alpha^2 \\ &= E\left\|\int_{t_1}^{t_2}A(\tau)U(\tau,t_1)\left(\int_{-\infty}^{t_1}U(t_1,s)P(s)\sigma(s,B_2x(s))dW(s)\right)d\tau\right\|_\alpha^2 \\ &\le 2E\left[\int_{t_1}^{t_2}\left\|A(\tau)U(\tau,s)U(s,t_1)\left(\int_{-\infty}^{t_1}U(t_1,s)P(s)\sigma(s,B_2x(s))dW(s)\right)\right\|_\alpha d\tau\right]^2; \end{aligned} \tag{3.47}$$

using condition (C_1), Hölder's inequality together with isometry inequality yields

$$\begin{aligned} &2E\left\|\int_{-\infty}^{t_1}(U(t_2,s)-U(t_1,s))P(s)\sigma(s,B_3x(s))dW(s)\right\|_\alpha^2 \\ &\quad \le 2c_0^2(t_2-t_1)\int_{t_1}^{t_2}E\left\|\int_{-\infty}^{t_1}A(s)U(\tau,s)P(s)\sigma(s,B_3x(s))dW(s)\right\|_\alpha^2 d\tau \\ &\quad \le 2c_0^2(t_2-t_1)n^2(\alpha,\beta)\int_{t_1}^{t_2}\int_{-\infty}^{t_1}(\tau-s)^{-2\alpha}e^{-\delta(\tau-s)}E\|\sigma(s,B_3x(s))\|_{\mathcal{L}_2^0}^2 ds\,d\tau \\ &\quad \le k_3(\alpha,\delta)(t_2-t_1)^2\sup_{t\in\mathbb{R}}E\|\sigma(t,B_3x(t))\|_{\mathcal{L}_2^0}^2, \end{aligned} \tag{3.48}$$

where $k_3(\alpha,\delta)$ is a positive constant depending on α,δ.

The second term is straightforward; we have

$$\begin{aligned}
2E\left\|\int_{t_1}^{t_2} U(t_2,s)P(s)\sigma(s,B_3x(s))dW(s)\right\|_\alpha^2
&\le c^2(\alpha)\int_{t_1}^{t_2}(t_2-s)^{-\alpha}e^{-\delta(t_2-s)}E\|\sigma(s,B_3x(s))\|_{\mathcal{L}_2^0}^2 ds \\
&\le k_4(\alpha,\delta)(t_2-t_1)\sup_{t\in\mathbb{R}} E\|\sigma(t,B_3x(t))\|_{\mathcal{L}_2^0}^2,
\end{aligned} \tag{3.49}$$

where $k_4(\alpha,\delta) > 0$.

Therefore,

$$\begin{aligned}
&E\|\Pi_5x(t_2)-\Pi_5x(t_1)\|_\alpha^2 \\
&\quad\le \Big(k_3(\alpha,\delta)(t_2-t_1)^2 + k_4(\alpha,\delta)(t_2-t_1)\Big)\sup_{t\in\mathbb{R}} E\|\sigma(t,B_3x(t))\|_{\mathcal{L}_2^0}^2.
\end{aligned} \tag{3.50}$$

Similarly, we can deduce that there exist some constants $k_5(\alpha,\delta), k_6(\alpha,\delta) > 0$ such that

$$\begin{aligned}
&E\|\Pi_6x(t_2)-\Pi_6x(t_1)\|_\alpha^2 \\
&\quad\le \Big(k_5(\alpha,\delta)(t_2-t_1)^2 + k_6(\alpha,\delta)(t_2-t_1)\Big)\sup_{t\in\mathbb{R}} E\|\sigma(t,B_3x(t))\|_{\mathcal{L}_2^0}^2.
\end{aligned} \tag{3.51}$$

Since

$$E\|Q_2x(t_2)-Q_2x(t_1)\|_\alpha^2 \le \sum_{i=3}^{6} E\|\Pi_6x(t_2)-\Pi_6x(t_1)\|_\alpha^2, \tag{3.52}$$

combing the evaluations above, we conclude that the right-hand side of (3.52) tends to zero independent of x as $t_1 \to t_2$. This completes the proof. □

Theorem 3.8. *Assume that assumptions (A_1)–(A_5) are satisfied, and one further assumes that $\Theta < 1$. Then the system* (1.3) *has a square-mean almost automorphic mild solution which can be expressed as $x = Q_1x + Q_2x$.*

Proof. Define an operator Q on $\mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$ by

$$Qx = Q_1x + Q_2x, \quad x \in \mathrm{SAA}\Big(\mathbb{R}, L^2(P,H_\alpha)\Big). \tag{3.53}$$

From Lemma 3.3 to Lemma 3.5, it is easy to see that Q maps $\mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$ into itself. To complete the proof, it suffices to show that, for some closed bounded convex subset C of $\mathrm{SAA}(\mathbb{R}, L^2(P,H_\alpha))$, we have

$$Q_1x + Q_2y \in C, \quad x, y \in C. \tag{3.54}$$

Let $\epsilon > 0$ be fixed. By (A_5) it follows that there exist a positive constant ω and nondecreasing continuous functions $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $t \in \mathbb{R}$ and $x \in L^2(P, H)$ with $E\|x\|^2 > \omega$,

$$E\|b(t,x)\|^2 \vee E\|\sigma(t,x)\|^2_{\mathcal{L}_2^0} \leq \epsilon\Lambda\left(E\|x\|^2\right). \tag{3.55}$$

Thus, for all $t \in \mathbb{R}$, $x \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$,

$$E\|b(t, B_2x)\|^2 \vee E\|\sigma(t, B_3x)\|^2_{\mathcal{L}_2^0} \leq \epsilon\Lambda\left(\eta E\|x\|^2_\alpha\right) + \bar{\omega}, \tag{3.56}$$

where $\bar{\omega} := \sup_{t\in\mathbb{R}}\{E\|b(t,x)\|^2, E\|\sigma(t,x)\|^2_{\mathcal{L}_2^0} : E\|x\|^2 \leq \omega\}$.

Let $x, y \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. A standard computation involving assumptions $(A_1) - -(A_3)$, Lemma 2.5, and Hölder's inequality, we can deduce that

$$\begin{aligned}
&E\left\|Q_1x(t) + Q_2y(t)\right\|^2_\alpha \\
&\quad \leq 7k(\alpha)K_g E\|B_1x(t)\|^2 + 7n^2(\alpha)K_g\int_{-\infty}^t (t-s)^{-\alpha}e^{-(\delta/2)(t-s)}ds \\
&\qquad \times \int_{-\infty}^t (t-s)^{-\alpha}e^{-(\delta/2)(t-s)}E\|B_1x(s)\|^2 ds \\
&\qquad + 7m^2(\alpha,\beta)K_g\int_t^\infty e^{\delta(t-s)}ds\int_t^\infty e^{\delta(t-s)}dsE\|B_1x(s)\|^2ds \\
&\qquad + 7c^2(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-(\delta/2)(t-s)}ds\int_{-\infty}^t (t-s)^{-\alpha}e^{-(\delta/2)(t-s)}E\|b(s, B_2x(s))\|^2ds \\
&\qquad + 7m^2(\alpha)\int_t^\infty e^{\delta(t-s)}ds\int_t^\infty e^{\delta(t-s)}E\|b(s, B_2x(s))\|^2ds \\
&\qquad + 7c^2(\alpha)\int_{-\infty}^t (t-s)^{-2\alpha}e^{-\delta(t-s)}E\|\sigma(s, B_3x(s))\|^2_{\mathcal{L}_2^0}ds \\
&\qquad + 7m^2(\alpha)\int_t^\infty e^{2\delta(t-s)}E\|\sigma(s, B_3x(s))\|^2_{\mathcal{L}_2^0}ds,
\end{aligned} \tag{3.57}$$

using (A_5) and (3.56), we further derive that

$$\begin{aligned}
&E\left\|Q_1x(t) + Q_2y(t)\right\|^2_\alpha \\
&\quad \leq 7K_g\eta\left(k(\alpha) + n^2(\alpha)\Gamma^2(1-\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)} + m^2(\alpha,\beta)\delta^{-2}\right)\|x\|^2_{\alpha,\infty} \\
&\qquad + 7\left[c^2(\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}\Gamma^2(1-\alpha) + m^2(\alpha)\delta^{-2} + c^2(\alpha)\delta^{2\alpha-1}\Gamma(1-2\alpha)\right. \\
&\qquad\qquad \left. + m^2(\alpha)(2\delta)^{-1}\right]\left(\epsilon\Lambda\left(\eta\|x\|^2_{\alpha,\infty}\right) + \bar{\omega}\right).
\end{aligned} \tag{3.58}$$

Note that, for ϵ, K_g sufficiently small, we can choose $\bar{r} > 0$ such that

$$7K_g\eta\left(k(\alpha)+n^2(\alpha)\Gamma^2(1-\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}+m^2(\alpha,\beta)\delta^{-2}\right)\bar{r}$$
$$+7\left[c^2(\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}\Gamma^2(1-\alpha)+m^2(\alpha)\delta^{-2}+c^2(\alpha)\delta^{2\alpha-1}\Gamma(1-2\alpha)\right. \tag{3.59}$$
$$\left.+m^2(\alpha)(2\delta)^{-1}\right]\left(\epsilon\Lambda\left(\eta\,\bar{r}\right)+\bar{\omega}\right)\leq \bar{r}\,.$$

Let

$$C=\left\{x\in \mathrm{SAA}\left(\mathbb{R},L^2(P,H_\alpha)\right):\|x\|^2_{\alpha,\infty}\leq \bar{r}\right\}. \tag{3.60}$$

It is easy to see that C is a closed bounded convex subset of $\mathrm{SAA}(\mathbb{R},L^2(P,H_\alpha))$. Moreover, for all $x,y\in C$,

$$E\left\|Q_1x(t)+Q_2y(t)\right\|^2_\alpha\leq \bar{r}\,. \tag{3.61}$$

Therefore, $Q_1x+Q_2y\in C$. By Lemmas 3.6 and 3.7 together with Krasnoselskii fixed point theorem we conclude that there exists a square-mean almost automorphic mild solution to (1.3). This completes the proof. □

Now, we give another main result by Banach fixed point theorem. We require the following assumptions.

(A_6) $b:R\times L^2(P,H)\to L^2(P,H)$, $\sigma:R\times L^2(P,H)\to L^2(P,\mathcal{L}_2^0(K,H))$ are square-mean almost automorphic in $t\in R$ and there exist some constants $K_b,K_\sigma>0$ such that for each $t\in\mathbb{R}$, $\phi,\varphi\in L^2(P,H)$,

$$\begin{aligned}\|b(t,\phi)-b(t,\varphi)\|^2&\leq K_b\|\phi-\varphi\|^2,\\ \|\sigma(t,\phi)-\sigma(t,\varphi)\|^2_{\mathcal{L}_2^0}&\leq K_\sigma\|\phi-\varphi\|^2.\end{aligned} \tag{3.62}$$

Theorem 3.9. *Under assumptions (A_1)–(A_3) and (A_6),* (1.3) *has a unique square-mean almost automorphic mild solution $x(\cdot)\in SAA(\mathbb{R},L^2(P,H_\alpha))$ provided that*

$$L:=7K_g\eta\left(k(\alpha)+n^2(\alpha)\Gamma^2(1-\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}+m^2(\alpha,\beta)\delta^{-2}\right)$$
$$+7\eta\left[c^2(\alpha)\left(\frac{2}{\delta}\right)^{2(1-\alpha)}\Gamma^2(1-\alpha)+m^2(\alpha)\delta^{-2}+c^2(\alpha)\delta^{2\alpha-1}\Gamma(1-2\alpha)+m^2(\alpha)(2\delta)^{-1}\right]<1. \tag{3.63}$$

Proof. Let Q be the operator defined by (3.53). From Lemma 3.3 to Lemma 3.5, it is easy to see that Q maps $\mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$ into itself. To complete the proof, it suffices to show that Q is a contractive map and has a unique fixed point. To this end, let $x, y \in \mathrm{SAA}(\mathbb{R}, L^2(P, H_\alpha))$. By a similar argument as above we can deduce that

$$\begin{aligned}
&E\|Qx(t) - Qy(t)\|_\alpha^2 \\
&\quad\le 7k(\alpha)K_g E\|B_1x(t) - B_1y(t)\|^2 + 7n^2(\alpha)K_g \int_{-\infty}^t (t-s)^{-\alpha} e^{(-\delta/2)(t-s)} ds \\
&\qquad \times \int_{-\infty}^t (t-s)^{-\alpha} e^{-(\delta/2)(t-s)} E\|B_1x(s) - B_1y(s)\|^2 ds \\
&\qquad + 7m^2(\alpha,\beta)K_g \int_t^\infty e^{\delta(t-s)} ds \int_t^\infty e^{\delta(t-s)} E\|B_1x(s) - B_1y(s)\|^2 ds \\
&\qquad + 7c^2(\alpha) \int_{-\infty}^t (t-s)^{-\alpha} e^{-(\delta/2)(t-s)} ds \int_{-\infty}^t (t-s)^{-\alpha} e^{-(\delta/2)(t-s)} \\
&\qquad \times E\|b(s, B_2x(s)) - b(s, B_2y(s))\|^2 ds \\
&\qquad + 7m^2(\alpha) \int_t^\infty e^{\delta(t-s)} ds \int_t^\infty e^{\delta(t-s)} E\|b(s, B_2x(s)) - b(s, B_2y(s))\|^2 ds \\
&\qquad + 7c^2(\alpha) \int_{-\infty}^t (t-s)^{-2\alpha} e^{-\delta(t-s)} E\|\sigma(s, B_3x(s)) - \sigma(s, B_3y(s))\|_{\mathcal{L}_2^0}^2 ds \\
&\qquad + 7m^2(\alpha) \int_t^\infty e^{2\delta(t-s)} E\|\sigma(s, B_3x(s)) - \sigma(s, B_3y(s))\|_{\mathcal{L}_2^0}^2 ds \\
&\quad\le L\|x - y\|_{\alpha,\infty}^2,
\end{aligned} \tag{3.64}$$

which implies that

$$\|Qx - Qy\|_{\alpha,\infty}^2 \le L\|x - y\|_{\alpha,\infty}^2. \tag{3.65}$$

Hence, by the Banach fixed point principle, Q has a unique fixed point $x(t)$ which is obviously the square-mean almost automorphic mild solution to (1.3). The proof is completed. □

Remark 3.10. The results of Theorem 3.9 can be applied to the existence of square-mean almost automorphic mild solutions to the example in [14].

Remark 3.11. If $h \equiv 0$, $A(t)$ is densely defined and the evolution family $U(t,s)$ generated by $A(t)$ is exponentially stable (that is, $P = I$), the existence of square-mean almost automorphic mild solutions has been studied in [23] by Banach fixed point theorem; if $A(t) = A$ is the infinitesimal generator of an analytic semigroup of linear operators, the existence of square-mean almost automorphic mild solutions has been studied in [22] by Banach fixed point theorem. In other words, the results in [22, 23] have been generalized and improved.

Acknowledgment

The Project is sponsored by the NSFC (11171062), the Innovation Program of Shanghai Municipal Education Commission (12ZZ063), the Natural Science Foundation of Anhui Province (1208085MA11), and the NSF of Anhui Educational Committee (KJ2011A139).

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