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Research Article

Nonlinear Stability and D-Convergence of Additive Runge-Kutta Methods for Multidelay-Integro-Differential Equations

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This paper is devoted to the stability and convergence analysis of the Additive Runge-Kutta methods with the Lagrangian interpolation (ARKLMs) for the numerical solution of multidelay-integro-differential equations (MDIDEs). GDN-stability and D-convergence are introduced and proved. It is shown that strongly algebraically stability gives D-convergence, DA- DAS- and ASI-stability give GDN-stability. A numerical example is given to illustrate the theoretical results.

1. Introduction

Delay differential equations arise in a variety of fields as biology, economy, control theory, electrodynamics (see, e.g., [1–5]). When considering the applicability of numerical methods for the solution of DDEs, it is necessary to analyze the stability of the numerical methods. In the last three decades, many works had dealt with these problems (see, e.g., [6]). For the case of nonlinear delay differential equations, this kind of methodology had been first introduced by Torelli [7] and then developed by [8–12].

In this paper, we consider the following nonlinear multidelay-integro-differential equations (MDIDEs) with m delays:

$$\begin{aligned} y'(t) = {} & f^{[1]}\left(t, y(t), y(t-\tau_1), \int_{t-\tau_1}^{t} g^{[1]}(t, s, y(s))\,ds\right) \\ & + f^{[2]}\left(t, y(t), y(t-\tau_2), \int_{t-\tau_2}^{t} g^{[2]}(t, s, y(s))\,ds\right) \end{aligned}$$

$$+\cdots+f^{[m]}\left(t,y(t),y(t-\tau_m),\int_{t-\tau_2}^{t}g^{[m]}(t,s,y(s))ds\right),\quad t\in[t_0,T],$$
$$y(t)=\varphi(t),\quad t\in[t_0-\tau,t_0],\tag{1.1}$$

where $\tau_1\le\tau_2\le\cdots\le\tau_m=\tau$, $f^{[v]}:[t_0,T]\times C^N\times C^N\times C^N\to C^N$, $g^{[v]}:[t_0,T]\times C^N\times C^N\to C^N$ $v=1,2,\ldots,m$, and $\varphi:[t_0-\tau,t_0]\to C^N$ are continuous functions such that (1.1) has a unique solution. Moreover, we assume that there exist some inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$ such that

$$\begin{aligned}&\operatorname{Re}\left\langle f^{[v]}(t,y_1,u_1,w_1)-f^{[v]}(t,y_2,u_2,w_2),y_1-y_2\right\rangle\\&\quad\le\alpha_v\left\|y_1-y_2\right\|^2+\beta_v\|u_1-u_2\|^2+\sigma_v\|w_1-w_2\|^2,\quad v=1,2,\ldots,m,\ t\ge t_0,\end{aligned}\tag{1.2}$$

$$\left\|f^{[v]}(t,y,u_1,w)-f^{[v]}(t,y,u_2,w)\right\|\le r_v\|u_1-u_2\|,\tag{1.3}$$

$$\left\|g^{[v]}(t,s,w_1)-g^{[v]}(t,s,w_2)\right\|\le\tilde{r}_v\|w_1-w_2\|,\quad (t,s)\in D\tag{1.4}$$

for all $t\in[t_0,T]$, for all $y,y_1,y_2,u,u_1,u_2,w,w_1,w_2\in C^N$, $(-\alpha_v)$, β_v, σ_v, r_v, $\tilde{r}_v$ are all nonnegative constants. Throughout this paper, we assume that the problem (1.1) has unique exact solution $y(t)$. Space discretization of some time-dependent delay partial differential equations give rises to such delay differential equations containing additive terms with different stiffness properties. In these situations, additive Runge-Kutta (ARK) methods are used. Some recent works about ARK can refer to [13, 14]. For the additive MDIDEs (1.1), similar to the proof of Theorem 2.1 in [7], it is straightforward to prove that under the conditions (1.2)~(1.4), the analytic solutions satisfy

$$\left\|y(t)-z(t)\right\|\le\max_{t_0-\tau\le t\le t_0}\left\|\varphi(t)-\psi(t)\right\|,\tag{1.5}$$

where $z(t)$ is the solution of the perturbed problem to (1.1).

To demand the discrete numerical solutions to preserve the convergence properties of the analytic solutions, Torelli [7] introduced a concept of RN-, GRN-stability for numerical methods applied to dissipative nonlinear systems of DDEs such as (1.1) when $g^{[v]}(t,s,y(s))=0$, $v=1,2,\ldots,m$, which is the straightforward generalization of the well-known concept of BN-stability of numerical methods with respect to dissipative systems of ODEs (see also [9]). More recently, one has noticed a growing interesting the analysis of delay integro-differential equations (DIDEs). This type of equations have been investigated in various fields, such as mathematical biology and control theory (see [15–17]). The theory of computational methods for delay integro-differential equations (DIDEs) has been studied by many authors, and a great deal of interesting results have been obtained (see [18–22]). Koto [23] dealt with the linear stability of Runge-Kutta (RK) methods for systems of DIDEs; Huang and Vandewalle [24] gave sufficient and necessary stability conditions for exact and discrete solutions of linear Scalar DIDEs. However, little attention has been paid to nonlinear multidelay-integro-differential equations (MDIDEs).

So, the aim of this paper is the study of stability and convergence properties for ARK methods when they are applied to nonlinear multidelay-integro-differential equations (MDIDEs) with m delays.

2. The GDN-Stability of the Additive Runge-Kutta Methods

An additive Runge-Kutta method with the Lagrangian interpolation (ARKLM) of s stages and m levels can be organized in the Butcher tableau:

$$\begin{array}{c|c|c|c|c} C & A^{[1]} & A^{[2]} & \cdots & A^{[m]} \\ \hline & b^{[1]^T} & b^{[2]^T} & \cdots & b^{[m]^T} \end{array} = \begin{array}{c|cccc|c|cccc} c_1 & a_{11}^{[1]} & a_{12}^{[1]} & \cdots & a_{1s}^{[1]} & & a_{11}^{[m]} & a_{12}^{[m1]} & \cdots & a_{1s}^{[m]} \\ c_2 & a_{21}^{[1]} & a_{22}^{[1]} & \cdots & a_{2s}^{[1]} & & a_{21}^{[m]} & a_{22}^{[m]} & \cdots & a_{2s}^{[m]} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ c_s & a_{s1}^{[1]} & a_{s2}^{[1]} & \cdots & a_{ss}^{[1]} & & a_{s1}^{[m]} & a_{s2}^{[m]} & \cdots & a_{ss}^{[m]} \\ \hline & b_1^{[1]} & b_2^{[1]} & \cdots & b_s^{[1]} & \cdots & b_1^{[m]} & b_2^{[m]} & \cdots & b_s^{[m]} \end{array}, \tag{2.1}$$

where $C = [c_1, c_2, \dots, c_s]^T$, $b^{[v]} = [b_1^{[v]}, b_2^{[v]}, \dots, b_s^{[v]}]$, and $A^{[v]} = (a_{ij}^{[v]})_{i,j=1}^{s}$.

The adoption of the method (2.1) for solving the problem (1.1) leads to

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{v=1}^{m}\sum_{j=1}^{s} b_j^{[v]} f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right), \\ y_i^{(n)} &= y_n + h\sum_{v=1}^{m}\sum_{j=1}^{s} a_{ij}^{[v]} f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right), \end{aligned} \tag{2.2}$$

where $t_n = t_0 + nh$, $t_j^{(n)} = t_n + c_j h$, y_n, and $y_j^{(n)}$, $\tilde{y}_j^{[v](n)}$ are approximations to the analytic solution $y(t_n)$, $y(t_n + c_j h)$, $y(t_n + c_j h - \tau_v)$ of (1.1), respectively, and the argument $\tilde{y}_j^{[v](n)}$ is determined by

$$\tilde{y}_j^{[v](n)} = \begin{cases} \varphi(t_n + c_j h - \tau_v) & t_n + c_j h - \tau_v \le 0 \\ \displaystyle\sum_{P_v=-d}^{r} L_{P_v}(\delta_v) y_j^{(n-m_v+P_v)} & t_n + c_j h - \tau_v > 0, \end{cases} \tag{2.3}$$

with $\tau_v = (m_v - \delta_v)h$, $\delta_v \in [0,1)$, integer $m_v \ge r+1$, $r, d \ge 0$, and

$$L_{P_v}(\delta_v) = \prod_{\substack{k=-d \\ k \ne P_v}}^{r} \left(\frac{\delta_v - k}{P_v - k}\right), \quad P_v = -d, -d+1, \dots, r. \tag{2.4}$$

We assume $m_v \geq r+1$ is to guarantee that no (unknown) values $y_j^{(i)}$ with $i \geq n$ are used in the interpolation procedure

$$w_j^{[v](n)} \text{ is an approximation to } w\left(t_j^{(n)}\right) := \int_{t_j^{(n-m_v)}}^{t_j^{(n)}} g^{[v]}\left(t_j^{(n)}, s, y(s)\right) ds, \tag{2.5}$$

which can be computed by a appropriate compound quadrature rule:

$$w_j^{[v](n)} = h \sum_{q=0}^{m_v} d_q g^{[v]}\left(t_j^{(n)}, t_j^{(n-q)}, y_j^{(n-q)}\right), \quad v = 1, 2, \ldots, m, \ j = 1, 2, \ldots, s. \tag{2.6}$$

As for the quadrature rule (2.6), we usually adopt the compound trapezoidal rule, the compound Simpsons rule or the compound Newton-Cotes rule, and so forth according to the requirement of the convergence of the method (see [19]) and denote $M = \max_{1 \leq v \leq m}\{m_v\}$ and $\eta = \max_{1 \leq v \leq m}\{\eta_v\}$ with η_v satisfing $\sum_{q=0}^{m_v} |d_q| < \eta_v$, $v = 1, 2, \ldots, m$.

In addition, we always put $y_j^{(n)} = \varphi(t_n + c_j h)$, $y_n = \varphi(t_n)$ whenever $n \leq 0$.

In order to write (2.2), (2.3), (2.5), and (2.6) in a more compact way, we introduce some notations. The $N \times N$ identity matrix will be denoted by I_N, $e = (1, 1, \ldots, 1)^T \in R^S$, $\widetilde{G} = G \otimes I_N$ is the Kronecker product of matrix G and I_N. For $u = (u_1, u_2, \ldots, u_s)^T$, $v = (v_1, v_2, \ldots, v_s)^T \in C^{NS}$, we define the inner product and the induced norm in C^{NS} as follows:

$$\langle u, v\rangle = \sum_{i=1}^{s} \langle u_i, v_i\rangle, \qquad \|u\| = \sqrt{\sum_{i=1}^{s} \|u_i\|^2}. \tag{2.7}$$

Moreover, we also adopt that

$$y^{(n)} = \begin{bmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_s^{(n)} \end{bmatrix}, \quad \widetilde{y}^{[v](n)} = \begin{bmatrix} \widetilde{y}_1^{[v](n)} \\ \widetilde{y}_2^{[v](n)} \\ \vdots \\ \widetilde{y}_s^{[v](n)} \end{bmatrix}, \quad w^{[v](n)} = \begin{bmatrix} w_1^{[v](n)} \\ w_2^{[v](n)} \\ \vdots \\ w_s^{[v](n)} \end{bmatrix}, \quad T^{(n)} = \begin{bmatrix} t_1^{(n)} \\ t_2^{(n)} \\ \vdots \\ t_s^{(n)} \end{bmatrix},$$
$$f^{[v]}\left(T^{(n)}, y^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right) = \begin{bmatrix} f^{[v]}\left(t_1^{(n)}, y_1^{(n)}, \widetilde{y}_1^{[v](n)}, w_1^{[v](n)}\right) \\ f^{[v]}\left(t_2^{(n)}, y_2^{(n)}, \widetilde{y}_2^{[v](n)}, w_2^{[v](n)}\right) \\ \vdots \\ f^{[v]}\left(t_s^{(n)}, y_s^{(n)}, \widetilde{y}_s^{[v](n)}, w_s^{[v](n)}\right) \end{bmatrix}. \tag{2.8}$$

With the above notation, method (2.2),(2.3), (2.5), and (2.6) can be written as

$$
\begin{aligned}
y_{n+1} &= y_n + h\sum_{v=1}^{m}\tilde{b}^{[v]^T} f^{[v]}\left(T^{(n)}, y^{(n)}, \tilde{y}^{[v](n)}, w^{[v](n)}\right),\\
y^{(n)} &= \tilde{e}y_n + h\sum_{v=1}^{m}\tilde{A}^{[v]} f^{[v]}\left(T^{(n)}, y^{(n)}, \tilde{y}^{[v](n)}, w^{[v](n)}\right),\\
\tilde{y}_j^{[v](n)} &= \begin{cases} \tilde{e}\varphi(t_n + c_j h - \tau_v), & t_n + c_j h - \tau_v \le t_0,\\ \displaystyle\sum_{P_v=-d}^{r} L_{P_v}(\delta_v) y_j^{(n-m_v+P_v)}, & t_n + c_j h - \tau_v > t_0, \end{cases}\\
w_j^{[v](n)} &= h\sum_{q=0}^{m_v} d_q g^{[v]}\left(t_n + c_j h, t_{n-q} + c_j h, y_j^{(n-q)}\right).
\end{aligned}
\tag{2.9}
$$

In 1997, Zhang and Zhou [25] introduced the extension of RN-stability to GDN-stability as follows.

Definition 2.1. An ARKLM (2.1) for DDEs is called GDN-stable if, numerical approximations y_n and z_n to the solution of (1.1) and its perturbed problem, respectively, satisfy

$$
\|y_n - z_n\| \le C \max_{t_0-\tau\le t<t_0} \|\varphi(t) - \psi(t)\|, \quad n \ge 0, \tag{2.10}
$$

where constant $C > 0$ depends only on the method, the parameter $\alpha_v,\ \beta_v,\ \sigma_v,\ r_v,\ \tilde{r}_v$, and the interval length $T - t_0$, $\psi(t)$ is the initial function to the perturbed problem of (1.1).

Definition 2.2. An ARKLM (2.1) is called strongly algebraically stable if matrices $M_{\gamma\mu}$ are nonnegative definite, where

$$
M_{\gamma\mu} = B^{[\gamma]}A^{[\mu]} + A^{[\gamma]^T}B^{[\mu]} - b^{[\gamma]}b^{[\mu]^T}, \qquad B^{[\gamma]} = \mathrm{diag}\left(b_1^{[\gamma]}, b_2^{[\gamma]}, \dots, b_s^{[\gamma]}\right), \tag{2.11}
$$

for $\mu, \gamma = 1, 2, \dots, m$.

Let

$$
\begin{aligned}
&\left\{y_n, y_j^{(n)}, \tilde{y}_j^{[1](n)}, \tilde{y}_j^{[2](n)}, \dots, \tilde{y}_j^{[m](n)}, w_j^{[1](n)}, w_j^{[2](n)}, \dots, w_j^{[m](n)}\right\}_{j=1}^{s},\\
&\left\{z_n, z_j^{(n)}, \tilde{z}_j^{[1](n)}, \tilde{z}_j^{[2](n)}, \dots, \tilde{z}_j^{[m](n)}, \widehat{w}_j^{[1](n)}, \widehat{w}_j^{[2](n)}, \dots, \widehat{w}_j^{[m](n)}\right\}_{j=1}^{s}
\end{aligned}
\tag{2.12}
$$

be two sequences of approximations to problems (1.1) and its perturbed problem, respectively. From method (2.1) with the same step size h, and write

$$
\begin{aligned}
U_i^{(n)} &= y_i^{(n)} - z_i^{(n)}, \qquad \tilde{U}_i^{[v](n)} = \tilde{y}_i^{[v](n)} - \tilde{z}_i^{[v](n)}, \qquad U_0^{(n)} = y_n - z_n,\\
Q_i^{[v](n)} &= h\left[f^{[v]}\left(t_i^{(n)}, y_i^{(n)}, \tilde{y}_i^{[v](n)}, w_i^{[v](n)}\right) - f^{[v]}\left(t_i^{(n)}, z_i^{(n)}, \tilde{z}_i^{[v](n)}, \widehat{w}_i^{[v](n)}\right)\right],\\
&\qquad i = 1, 2, \dots, s, \quad v = 1, 2, \dots, m.
\end{aligned}
\tag{2.13}
$$

Then (2.2) and (2.3) read

$$U_0^{(n+1)} = U_0^{(n)} + \sum_{v=1}^{m}\sum_{j=1}^{s} b_j^{[v]} Q_j^{[v](n)}, \tag{2.14}$$

$$U_i^{(n)} = U_0^{(n)} + \sum_{v=1}^{m}\sum_{j=1}^{s} a_{ij}^{[v]} Q_j^{[v](n)}. \tag{2.15}$$

Our main results about GDN-stability are contained in the following theorem.

Theorem 2.3. *Assume ARK method* (2.2) *is strongly algebraically stable, and then the corresponding ARKLM* (2.1) *is GDN-stable, and satisfies*

$$\left\|y_n - z_n\right\| \le C \max_{t_0-\tau\le t\le t_0} \left\|\varphi(t) - \psi(t)\right\|^2, \quad n \ge 0, \tag{2.16}$$

where

$$C = \exp\left[6(T-t_0)ms\sum_{v=1}^{m}\beta_v L_v^2(m_v + d + 1)\right] \max_{t_0-\tau\le t\le t_0} \left\|\varphi(t) - \psi(t)\right\|^2, \quad L_v = \max_{-d\le p_v\le r}\{L_{p_v}\}, \tag{2.17}$$

Proof. From (2.14) and (2.15) we get

$$\begin{aligned}
\left\|U_0^{(n+1)}\right\|^2 &= \left\langle U_0^{(n)} + \sum_{v=1}^{m}\sum_{i=1}^{s} b_i^{[v]} Q_i^{[v](n)}, U_0^{(n)} + \sum_{v=1}^{m}\sum_{i=1}^{s} b_i^{[v]} Q_i^{[v](n)} \right\rangle \\
&= \left\|U_0^{(n)}\right\|^2 + 2\sum_{v=1}^{m}\sum_{i=1}^{s} b_i^{[v]} \operatorname{Re}\left\langle Q_i^{[v](n)}, U_0^{(n)}\right\rangle \\
&\quad + \sum_{u,v=1}^{m}\sum_{i,j=1}^{s} b_i^{[u]} b_j^{[v]} \left\langle Q_i^{[u](n)}, Q_j^{[v](n)}\right\rangle \\
&= \left\|U_0^{(n)}\right\|^2 + 2\sum_{v=1}^{m}\sum_{i=1}^{s} b_i^{[v]} \operatorname{Re}\left\langle Q_i^{[v](n)}, U_i^{(n)} - \sum_{v=1}^{m}\sum_{j=1}^{s} a_{ij}^{[v]} Q_j^{[v](n)}\right\rangle \\
&\quad + \sum_{u,v=1}^{m}\sum_{i,j=1}^{s} b_i^{[u]} b_j^{[v]} \left\langle Q_i^{[v](n)}, Q_j^{[u](n)}\right\rangle \\
&= \left\|U_0^{(n)}\right\|^2 + 2\sum_{v=1}^{m}\sum_{i=1}^{s} b_i^{[v]} \operatorname{Re}\left\langle Q_i^{[v](n)}, U_i^{(n)}\right\rangle \\
&\quad - \sum_{u,v=1}^{m}\sum_{i,j=1}^{s} \left(b_i^{[u]} a_{ij}^{[v]} + b_j^{[v]} a_{ij}^{[u]} - b_i^{[u]} b_j^{[v]}\right) \left\langle Q_i^{[v](n)}, Q_j^{[u](n)}\right\rangle.
\end{aligned} \tag{2.18}$$

If the matrices $M_{\gamma\mu}$ are nonnegative definite, then

$$\left\|U_0^{(n+1)}\right\|^2 \le \left\|U_0^{(n)}\right\|^2 + 2\sum_{v=1}^{m}\sum_{j=1}^{s} b_j^{[v]} \operatorname{Re}\left\langle Q_j^{[v](n)}, U_j^{(n)}\right\rangle. \tag{2.19}$$

Furthermore, by conditions (1.2)~(1.4) and Schwartz inequality we have

$$\begin{aligned}
\operatorname{Re}\left\langle Q_j^{[v](n)}, U_j^{(n)} \right\rangle &= h \operatorname{Re}\left\langle f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right)\right. \\
&\qquad \left. - f^{[v]}\left(t_j^{(n)}, z_j^{(n)}, \tilde{z}_j^{[v](n)}, \widehat{w}_j^{[v](n)}\right), U_j^{(n)} \right\rangle \\
&\leq h\alpha_v \left\| U_j^{(n)} \right\|^2 + h\beta_v \left\| \tilde{U}_j^{[v](n)} \right\| + h\sigma_v \left\| w_j^{[v](n)} - \widehat{w}_j^{[v](n)} \right\|^2 \\
&\leq h\alpha_v \left\| U_j^{(n)} \right\|^2 + h\beta_v \left\| \tilde{U}_j^{[v](n)} \right\| + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
&= h\alpha_v \left\| U_j^{(n)} \right\|^2 + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2, \quad t_n + c_j h - \tau_v \leq t_0 \\
&= h\alpha_v \left\| U_j^{(n)} \right\|^2 + h\beta_v \left\| \sum_{p_v=-d}^{r} L_{p_v}(\delta_v) U_j^{(n-m_v+p_v)} \right\|^2 \\
&\quad + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2, \quad t_n + c_j h - \tau_v > t_0
\end{aligned} \tag{2.21}$$

$$\leq h\alpha_v \left\| U_j^{(n)} \right\|^2 + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2, \quad t_n + c_j h - \tau_v \leq t_0 \tag{2.22}$$

$$\begin{aligned}
&\leq h\alpha_v \left\| U_j^{(n)} \right\|^2 + 2h\beta_v L_v^2 \left\| \sum_{p_v=-d}^{r} U_j^{(n-m_v+p_v)} \right\|^2 \\
&\quad + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2, \quad t_n + c_j h - \tau_v > t_0.
\end{aligned} \tag{2.23}$$

For (2.23), we have

$$\begin{aligned}
(2.23) &\leq 2h\beta_v L_v^2 \left\| \sum_{p_v=-d}^{r} U_j^{(n-m_v+p_v)} \right\|^2 + 2h^3 \tilde{r}_v^2 \eta_v^2 \sigma_v \sum_{q=0}^{m_v} \left\| U_j^{(n-q)} \right\|^2 \\
&\leq 3h\beta_v L_v^2 \left\| \sum_{p_v=-d}^{m_v} U_j^{(n-m_v+p_v)} \right\|^2.
\end{aligned} \tag{2.24}$$

By the same way, we can also get

$$(2.22) \leq 3h\beta_v L_v^2 \left\| \sum_{P_v=-d}^{m_v} U_j^{(n-m_v+p_v)} \right\|^2. \tag{2.25}$$

Substituting (2.25) and (2.24) into (2.19), yields

$$\begin{aligned}\left\|U_0^{(n+1)}\right\|^2 &\leq \left\|U_0^{(n)}\right\|^2 + 2h\sum_{v=1}^{m}\sum_{j=1}^{s}3\beta_v L_v^2 b_j^{[v]}\left\|\sum_{p_v=-d}^{m_v}U_j^{(n-m_v+p_v)}\right\|^2 \qquad (2.26)\\ &\leq \left\|U_0^{(n)}\right\|^2 + 6h\sum_{v=1}^{m}\sum_{j=1}^{s}\beta_v L_v^2 b_j^{[v]}(m_v+d+1)\max_{-d\leq p_v\leq m_v}\left\|U_j^{(n-m_v+p_v)}\right\|^2\\ &\leq \left\|U_0^{(n)}\right\|^2 + 6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\max_{(j,p_v)\in E_v}\left\|U_j^{(n-m_v+p_v)}\right\|^2 \qquad (2.27)\\ &\leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]\max\left\{\left\|U_0^{(n)}\right\|^2, \max_{(j,p_v)\in E_v}\left\|U_j^{(n-m_v+p_v)}\right\|^2\right\},\end{aligned}$$

where $E_v = \{(j, P_v)\ 1 \leq j \leq s,\ -d \leq P_v \leq r\}$.

Similar to (2.27), the inequalities:

$$\left\|U_i^{(n)}\right\|^2 \leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]\max\left\{\left\|U_0^{(n)}\right\|^2, \max_{(j,P_v)\in E}\left\|U_j^{(n-m_v+P_v)}\right\|^2\right\} \tag{2.28}$$

follows for $i = 1, 2, \ldots, s$.

In the following, with the help of inequalities (2.27), (2.28), and induction we shall prove the inequalities:

$$\left\|U_i^{(n)}\right\|^2 \leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]^{(n+1)}\max_{t_0-\tau\leq t\leq t_0}\left\|\varphi(t)-\psi(t)\right\|^2, \tag{2.29}$$

for $n \geq 0,\ i = 1, 2, \ldots, s$.

In fact, it is clear from (2.27), (2.28), and $m_v \geq r + 1$ such that

$$\left\|U_i^{(0)}\right\|^2 \leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]\max_{t_0-\tau\leq t\leq t_0}\left\|\varphi(t)-\psi(t)\right\|^2, \quad i = 0, 1, 2, \ldots, s. \tag{2.30}$$

Suppose for $n \leq k\ (k \geq 0)$ that

$$\left\|U_i^{(n)}\right\|^2 \leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]^{(n+1)}\max_{t_0-\tau\leq t\leq t_0}\left\|\varphi(t)-\psi(t)\right\|^2, \quad i = 0, 1, 2, \ldots, s. \tag{2.31}$$

Then from (2.27) and (2.28), $m_v \geq r+1$ and $1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1) > 1$, we conclude that

$$\left\|U_i^{(k+1)}\right\|^2 \leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]^{(k+2)} \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|^2, \quad i=0,1,2,\ldots,s. \tag{2.32}$$

This completes the proof of inequalities (2.29). In view of (2.29), we get for $n \geq 0$ that

$$\begin{aligned}\left\|U_0^{(n)}\right\|^2 &\leq \left[1+6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right]^{(n+1)} \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|^2 \\ &\leq \exp\left[(n+1)6hms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right] \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|^2 \\ &\leq \exp\left[6(T-t_0)ms\sum_{v=1}^{m}\beta_v L_v^2(m_v+d+1)\right] \max_{t_0-\tau\leq t\leq t_0}\|\varphi(t)-\psi(t)\|^2.\end{aligned} \tag{2.33}$$

As a result, we know that method (2.1) is GDN-stable. □

3. D-Convergence

In order to study the convergence of numerical methods for MDIDEs, we have to mention the concept of the convergence for stiff ODEs.

In 1981, Frank et al. [26] introduced the important concept of B-convergence for numerical methods applied to nonlinear stiff initial value problems of ordinary differential equations. Later, there have been rapid developments in the study of B-convergence, and a significant number of important results have already been found for Runge-Kutta methods. In fact, B-convergence result is nothing but a realistic global error estimate based on one-sided Lipschitz constant [27]. In this section, we start discussing the convergence of ARKLM (2.1) for MDIDEs (1.1) with conditions (1.2)–(1.4). The approach to the derivation of these estimates is similar to that used in [25]. We assume the analytic solution $y(t)$ of (1.1) is smooth enough, and its derivatives used later are bounded by

$$\left\|D^{(i)}y(t)\right\| \leq \widetilde{M}_i, \quad t\in[t_0-\tau,T], \tag{3.1}$$

where

$$D^{(i)}y(t) = \begin{cases} y^{(i)}(t), & t\in(t_0+(j-1)h, t_0+jh), \\ y^{(i)}(t_0+jh-0), & t=t_0+jh. \end{cases} \tag{3.2}$$

If we introduce some notations

$$Y^{(n)} = \begin{bmatrix} y(t_n + c_1 h) \\ y(t_n + c_2 h) \\ \vdots \\ y(t_n + c_s h) \end{bmatrix}, \quad \widetilde{Y}^{[v](n)} = \begin{bmatrix} y(t_n + c_1 h - \tau_v) \\ y(t_n + c_2 h - \tau_v) \\ \vdots \\ y(t_n + c_s h - \tau_v) \end{bmatrix}, \quad \widetilde{w}^{[v](n)} = \begin{bmatrix} w(t_n + c_1 h - \tau_v) \\ w(t_n + c_2 h - \tau_v) \\ \vdots \\ w(t_n + c_s h - \tau_v) \end{bmatrix}. \tag{3.3}$$

With the above notations, the local errors in (2.9) can be defined as

$$y(t_{n+1}) = y(t_n) + h \sum_{v=1}^{m} \widetilde{b}^{[v]^T} f^{[v]}\left(T^{(n)}, Y^{(n)}, \widetilde{Y}^{[v](n)}, \widetilde{w}^{[v](n)}\right) + Q_n, \tag{3.4}$$

$$Y^{(n)} = \widetilde{e} y(t_n) + h \sum_{v=1}^{m} \widetilde{A}^{[v]} f^{[v]}\left(T^{(n)}, Y^{(n)}, \widetilde{Y}^{[v](n)}, \widetilde{w}^{[v](n)}\right) + r_n, \tag{3.5}$$

$$\widetilde{Y}^{[v](n)} = \left(\widetilde{Y}_1^{[v](n)}, \widetilde{Y}_2^{[v](n)}, \ldots, \widetilde{Y}_s^{[v](n)}\right)^T, \tag{3.6}$$

with

$$\widetilde{Y}_j^{[v](n)} = \begin{cases} \varphi(t_n + c_j h - \tau_v), & t_n + c_j h - \tau_v \le t_0, \\ \displaystyle\sum_{P_v=-d}^{r} L_{P_v}(\delta_v) y_j^{(n-m_v+P_v)} + \rho_j^{[v](n)}, & t_n + c_j h - \tau_v > t_0, \end{cases} \tag{3.7}$$

$$w_j^{[v](n)} = h \sum_{q=0}^{m_v} d_q g^{[v]}\left(t_n + c_j h, t_{n-q} + c_j h, y_j^{(n-q)}\right) + R_j^{[v](n)}. \tag{3.8}$$

If we take $\breve{y}_n = y(t_n)$, $\breve{y}^{(n)} = Y^{(n)}$, $\breve{y}^{[v](n)} = \widetilde{Y}^{[v](n)}$, and $\breve{\widetilde{w}}^{[v](n)} = \widetilde{w}^{[v](n)}$

Then we can get the perturbed scheme of (2.9),

$$\breve{y}_{n+1} = \breve{y}_n + h \sum_{v=1}^{m} \widetilde{b}^{[v]^T} f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \breve{\widetilde{y}}^{[v](n)}, \breve{\widetilde{w}}^{[v](n)}\right) + Q_n, \tag{3.9}$$

$$\breve{y}^{(n)} = \widetilde{e} \breve{y}_n + h \sum_{v=1}^{m} \widetilde{A}^{[v]} f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \breve{\widetilde{y}}^{[v](n)}, \breve{\widetilde{w}}^{[v](n)}\right) + r_n, \tag{3.10}$$

$$\breve{\widetilde{y}}_j^{[v](n)} = \begin{cases} \widetilde{e}\varphi(t_n + c_j h - \tau_v), & t_n + c_j h - \tau_v \le 0, \\ \displaystyle\sum_{P_v=-d}^{r} L_{P_v}(\delta_v) \breve{y}_j^{(n-m_v+P_v)} + \rho_j^{[v](n)}, & t_n + c_j h - \tau_v > 0, \end{cases} \tag{3.11}$$

$$w_j^{[v](n)} = h \sum_{q=0}^{m_v} d_q g^{[v]}\left(t_n + c_j h, t_{n-q} + c_j h, y_j^{(n-q)}\right) + R_j^{[v](n)}. \tag{3.12}$$

With perturbations, $Q_n \in C^N$, $r_n = (r_1^{(n)^T}, r_2^{(n)^T}, \ldots, r_s^{(n)^T})^T$, $R^{[v](n)} = (R_1^{[v](n)}, R_2^{[v](n)}, \ldots, R_s^{[v](n)})^T$, $\rho^{(n)} = (\rho_1^{(n)^T}, \rho_2^{(n)^T}, \ldots, \rho_s^{(n)^T}) \in C^{NS}$, according to Taylor formula and the formula in [28, pages 205–212], Q_n, r_n and ρ_n can be determined respectively, as follows:

$$Q_n = \sum_{l=1}^{P} \frac{h^l}{(l-1)!} \left(\frac{1}{l} - \sum_{v=1}^{m} \sum_{j=1}^{s} b_j^{[v]} c_j^{l-1} \right) D^{(l)} y(t_n) + R_0^{(n)}, \tag{3.13}$$

$$r_i^{(n)} = \sum_{l=1}^{P} \frac{h^l}{(l-1)!} \left(\frac{1}{l} c_i^l - \sum_{v=1}^{m} \sum_{j=1}^{s} a_{ij}^{[v]} c_j^{l-1} \right) D^{(l)} y(t_n) + R_i^{(n)}, \tag{3.14}$$

$$\rho_i^{(n)} = \frac{h^{q+1}}{(q+1)!} \sum_{v=1}^{m} \prod_{P_v=-d}^{r} (\delta_v - P_v) D^{(q+1)} y\left(\xi_i^{(n)}\right), \quad \xi_i^{(n)} \in (t_{n-m_v-d} + c_i h, t_{n-m_v+r} + c_i h), \tag{3.15}$$

where $q = d + r$, $R_i^{(n)}$, and $\xi_i^{(n)}$ satisfy $\|R_i^{(n)}\| \le \widehat{M}_i h^{i+1}$, $i = 0, 1, 2, \ldots, s$, $h \in (0, h_0]$, h_0 depends only on the method, and $\widehat{M}_i$ $(i = 0, 1, 2, \ldots, s)$ depends only on the method and some $\widetilde{M}_i$ in (3.2).

Combining (2.2), (2.3), (2.5), and (2.6) with (3.9), (3.10), (3.11), and (3.12) yields the following recursion scheme for the $\varepsilon_0^{(n+1)} = \check{y}_{n+1} - y_{n+1}$:

$$\begin{aligned}
\varepsilon_0^{(n+1)} = \varepsilon_0^{(n)} + h \sum_{v=1}^{m} \widetilde{b}^{[v]^T} \Big\{ & f^{[v]}\left(T^{(n)}, \check{y}^{(n)}, \check{\tilde{y}}^{[v](n)}, w^{[v](n)}\right) - f^{[v]}\left(T^{(n)}, \check{y}^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right) \\
& + g_n^{[v]} \varepsilon_n + H^{[v](n)} \left(\check{\tilde{w}}^{[v](n)} - w^{[v](n)} \right) \Big\} + Qn, \\
\varepsilon_n = \widetilde{e} \varepsilon_0^{(n)} + h \sum_{v=1}^{m} \widetilde{A}^{[v]} \Big\{ & f^{[v]}\left(T^{(n)}, \check{y}^{(n)}, \check{\tilde{y}}^{[v](n)}, w^{[v](n)}\right) - f^{[v]}\left(T^{(n)}, \check{y}^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right) \\
& + g_n^{[v]} \varepsilon_n + H^{[v](n)} \left(\check{\tilde{w}}^{[v](n)} - w^{[v](n)} \right) \Big\} + r_n,
\end{aligned} \tag{3.16}$$

where $\varepsilon_0^{(n+1)} = \check{y}_{n+1} - y_{n+1}$, $\varepsilon_n = (\varepsilon_1^{(n)^T}, \varepsilon_2^{(n)^T}, \ldots, \varepsilon_s^{(n)^T})^T = \check{y}^{(n)} - y^{(n)}$,

$$\begin{aligned}
g_i^{[v](n)} &= \int_0^1 f_2\left(t_n + c_i h, y_i^{(n)} + \theta\left(\check{y}_i^{(n)} - y_i^{(n)}\right), \check{\tilde{y}}_i^{[v](n)}, \check{\tilde{w}}_i^{[v](n)}\right) d\theta, \quad i = 1, 2, \ldots, s, \\
H_i^{[v](n)} &= \int_0^1 f_4\left(t_n + c_i h, y_i^{(n)}, \check{\tilde{y}}_i^{[v](n)}, \check{\tilde{w}}_i^{[v](n)} + \theta\left(\check{\tilde{w}}_i^{[v](n)} - w_i^{[v](n)}\right)\right) d\theta,
\end{aligned} \tag{3.17}$$

here, $f_i(x_1, x_2, x_3, x_4)$ is the Jacobian matrix $(\partial f(x_1, x_2, x_3, x_4)/\partial x_i)$ $i = 1, 2, 3, 4$.

$$\begin{aligned}
H^{[v](n)}\left(\breve{\tilde{w}}^{[v](n)} - w^{[v](n)}\right) &= hH^{[v](n)} \sum_{q=1}^{m_v} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y^{(n-q)}\right)\right] \\
&\quad + hH^{[v](n)} R^{[v](n)} \\
&\quad + hH^{[v](n)} d_0\left[g^{[v]}\left(t_n, t_n, \breve{y}^{(n)}\right) - g^{[v]}\left(t_n, t_n, y^{(n)}\right)\right] \\
&= hH^{[v](n)} \sum_{q=1}^{m_v} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y^{(n-q)}\right)\right] \\
&\quad + hH^{[v](n)} R^{[v](n)} \\
&\quad + hH^{[v](n)} d_0 \int_0^1 g_3^{[v]}\left(t_n, t_n, y^{(n)} + \theta\left(\breve{y}^{(n)} - y^{(n)}\right)\right) d\theta \cdot \varepsilon_n.
\end{aligned} \tag{3.18}$$

Assume that $(\widetilde{I}_s - h\sum_{v=1}^{m} \widetilde{A}^{[v]}(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}))$ is regular, from (3.16) and (3.17), (3.18), we can get

$$\begin{aligned}
\varepsilon_0^{(n+1)} = &\left\{ I_N + h\sum_{v=1}^{m} \widetilde{b}^{[v]^T}\left[\widetilde{I}_s - h\sum_{v=1}^{m} \widetilde{A}^{[v]}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right)\right]^{-1} \right. \\
&\left. \quad \times \widetilde{e}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right) \right\} \varepsilon_0^{(n)} \\
&+ h\sum_{v=1}^{m} \widetilde{b}^{[v]^T}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right)\left[\widetilde{I}_s - h\sum_{v=1}^{m} \widetilde{A}^{[v]}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right)\right]^{-1} \\
&\times \left(r_n + h^2 \sum_{v=1}^{m} \widetilde{A}^{[v]} H^{[v](n)} R^{[v](n)}\right) \\
&+ h\sum_{v=1}^{m} \widetilde{b}^{[v]^T}\left[\widetilde{I}_s - h\sum_{v=1}^{m} \widetilde{A}^{[v]}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right)\right]^{-1} \\
&\times \left[h\sum_{v=1}^{m} \widetilde{A}^{[v]}\left(g^{[v](n)} + hH^{[v](n)} d_0 g_3^{[v](n)}\right)\right] \\
&\cdot \left\{ f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \breve{\tilde{y}}^{[v](n)}, w^{[v](n)}\right) - f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right) \right. \\
&\left. \quad + hH^{[v](n)} \sum_{q=1}^{m_v} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y^{(n-q)}\right)\right] \right\} \\
&+ Q_n + h^2 \sum_{v=1}^{m} \widetilde{b}^{[v]^T} H^{[v](n)} R^{[v](n)}.
\end{aligned} \tag{3.19}$$

Now, we introduce the concept of D-convergence from [25].

Definition 3.1. An ARKLM (2.1) with $y_n = y(t_n)$ $(n \leq 0)$, $y_i^{(n)} = y(t_n + c_i h)$ $(n < 0)$ and $\widetilde{y}_i^{[v](n)} = y(t_n + c_i h - \tau_v)$ $(n < 0)$ is called D-convergence of order p if this method, when applied to any given DIDEs (1.1) subject to (1.2)–(1.4); produce an approximation sequence y_n and the global error satisfies a bound of the form:

$$\left\| y(t_n) - y_n \right\| \leq C(t_n) h^P, \quad h \in (0, h_0], \tag{3.20}$$

where the maximum stepsize h_0 depends on characteristic parameter α_v, β_v, σ_v, r_v, $\widetilde{r}_v$ and the method, the function $C(t)$ depends only on some $\widetilde{M}_i$ in (3.2), delay τ_v, characteristic parameters α_v, β_v, σ_v, r_v, $\widetilde{r}_v$, $v = 1, 2, \ldots, m$, and the method.

Definition 3.2. The ARKLM (2.2), (2.3), (2.5), and (2.6) is said to be DA-stable if the matrix $(I_s - \sum_{\nu=1}^{m} A^{[\nu]}\xi)$ is regular for $\xi \in C^- := \{\xi \in C \mid \operatorname{Re} \xi \leq 0\}$, and $|R_i(\xi)| \leq 1$ for all $\xi \in C^-$, $i = 0, 1, \ldots, s$.

Where

$$\begin{gathered} R_i(\varepsilon_1) = 1 + \sum_{v=1}^{m} A_i^{[v]} \varepsilon_1 \left(I_s - \sum_{\nu=1}^{m} A^{[\nu]} \xi \right)^{-1} e, \\ A_0^{[v]} = b^{[v]}, \qquad A_i^{[v]} = \left(a_{i1}^{[v]}, a_{i2}^{[v]}, \ldots, a_{is}^{[v]} \right)^T, \quad i = 0, 1, \ldots, s. \end{gathered} \tag{3.21}$$

Definition 3.3. The ARKLM (2.2), (2.3), (2.5), and (2.6) is said to be ASI-stable if the matrix $(I_s - \sum_{\nu=1}^{M} A^{[\nu]}\xi)$ is regular for $\xi \in C^-$, and $(I_s - \sum_{\nu=1}^{M} A^{[\nu]}\xi)^{-1}$ is uniformly bounded for $\xi \in C^-$.

Definition 3.4. The ARKLM (2.2), (2.3), (2.5), and (2.6) is said to be DAS-stable if the matrix$(I_s - \sum_{\nu=1}^{M} A^{[\nu]}\xi)$ is regular for $\xi \in C^-$, and $\sum_{\nu=1}^{m} A_i^{[\nu]^T} \xi (I_s - \sum_{\nu=1}^{M} A^{[\nu]}\xi)^{-1}$ $(i = 0, 1, \ldots, s)$ is uniformly bounded for $\xi \in C^-$.

Lemma 3.5. *Suppose the ARKLM* (2.2), (2.3), (2.5), *and* (2.6) *is DA- DAS- and ASI-stable, then there exist positive constants h_0, γ_1, γ_2, γ_3, which depend only on the method and the parameter α_v, β_v, σ_v, r_v, $\widetilde{r}_v$ such that*

$$\begin{gathered} \left\| \widetilde{I}_s - \sum_{\nu=1}^{M} \widetilde{A}^{[\nu]} \xi \right\| \leq \gamma_1, \\ \left\| I_N + \sum_{v=1}^{m} \widetilde{A}_i^{[v]^T} \xi \left(\widetilde{I}_s - \sum_{v=1}^{m} \widetilde{A}^{[v]} \xi \right)^{-1} \widetilde{e} \right\| \leq 1 + \gamma_2 h, \\ \left\| \sum_{v=1}^{m} \widetilde{A}_i^{[v]^T} \xi \left(\widetilde{I}_s - \sum_{v=1}^{m} \widetilde{A}^{[v]} \xi \right)^{-1} v \right\| \leq \gamma_3 \|v\|, \quad v \in C^{NS}, \\ h \in (0, h_0], \quad i = 0, 1, 2 \ldots, s. \end{gathered} \tag{3.22}$$

Proof. This Lemma can be proved in the similar way as that of in [29, Lemmas 3.5–3.7]. □

Theorem 3.6. *Suppose the ARKLM* (2.2), (2.3), (2.5), *and* (2.6) *is DA- DAS- and ASI-stable, then there exist positive constants* h_0, γ_3, γ_4, γ_5, *which depend only on the method and the parameters* α_v, β_v, σ_v, r_v, $\tilde{r}_v$, *such that for* $h \in (o, h_0]$,

$$\left\|\varepsilon_i^{(n)}\right\| \leq \begin{cases} (1+h\gamma_4)\max\left\{\left\|\varepsilon_0^{(n+1)}\right\|, \max\limits_{(i,p_v)\in E}\left\|\varepsilon_i^{(n-m_v+p_v)}\right\|, \max\limits_{(i,q)\in E_q}\left\|\varepsilon_i^{(n-q)}\right\|\right\} + h\gamma_5 \max\limits_{1\leq i\leq s}\left\|\rho_i^{(n-1)}\right\| \\ \quad +\left\|\widetilde{Q}_{n-1}\right\| + \gamma_3\left\|\widetilde{\gamma}_{n-1}\right\|, \qquad i=0, \\ (1+h\gamma_4)\max\left\{\left\|\varepsilon_0^{(n+1)}\right\|, \max\limits_{(i,p_v)\in E}\left\|\varepsilon_i^{(n-m_v+p_v)}\right\|, \max\limits_{(i,q)\in E_q}\left\|\varepsilon_i^{(n-q)}\right\|\right\} + h\gamma_5 \max\limits_{1\leq i\leq s}\left\|\rho_i^{(n)}\right\| \\ \quad +\left\|\widetilde{Q}_{n}\right\| + \gamma_3\left\|\widetilde{\gamma}_{n}\right\|, \qquad i=1,2,\ldots,s, \end{cases} \tag{3.23}$$

where $\varepsilon_0^{(n)} = \breve{y}_n - y_n$, $\varepsilon_i^{(n)} = \breve{y}_i^{(n)} - y_i^{(n)}$, $E = \{(i,p_v) \mid 1 \leq i \leq s,\ -d \leq p_v \leq \gamma\}$, $E_q = \{(i,q) \mid 1 \leq i \leq s, 1 \leq q \leq m\}$, $\widetilde{Q}_n = Q_n + h^2\sum_{v=1}^{m} \widetilde{b}^{[v]^T} H^{[v](n)} R^{[v](n)}$, $\widetilde{r}_n = r_n + h^2\sum_{v=1}^{m} \widetilde{A}^{[v]} H^{[v](n)} R^{[v](n)}$.

Proof. Using (3.19) and Lemma 3.5, for $h \in (0, h_0]$, we obtain that

$$\begin{aligned} \varepsilon_0^{(n+1)} \leq{}& (1+\gamma_2 h)\left\|\varepsilon_0^{(n)}\right\| + \gamma_3\|\widetilde{r}_n\| + \left\|\widetilde{Q}_n\right\| \\ &+ h\gamma_3\left\|\sum_{v=1}^{m}\widetilde{A}^{[v]}\left\{f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \breve{\widetilde{y}}^{[v](n)}, w^{[v](n)}\right) - f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right)\right.\right. \\ &\qquad\left.\left. + hH^{[v](n)}\sum_{q=1}^{m_v} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y^{(n-q)}\right)\right]\right\}\right\| \\ &+ h\left\|\sum_{v=1}^{m}\widetilde{b}^{[v]^T}\left\{f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \breve{\widetilde{y}}^{[v](n)}, w^{[v](n)}\right) - f^{[v]}\left(T^{(n)}, \breve{y}^{(n)}, \widetilde{y}^{[v](n)}, w^{[v](n)}\right)\right.\right. \\ &\qquad\left.\left. + hH^{[v](n)}\cdot\sum_{q=1}^{m_v} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y^{(n-q)}\right)\right]\right\}\right\| \end{aligned} \tag{3.24}$$

$$\leq (1+\gamma_2 h)\left\|\varepsilon_0^{(n)}\right\| + \gamma_3\|\widetilde{r}_n\| + \left\|\widetilde{Q}_n\right\| \tag{3.25a}$$

$$\begin{aligned} &+ h\gamma_3\sum_{v=1}^{m}\left\{\sum_{i=1}^{s}\left\|\sum_{j=1}^{s} a_{ij}^{[v]}\left[f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \breve{\widetilde{y}}_j^{[v](n)}, w_j^{[v](n)}\right) - f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \widetilde{y}_j^{[v](n)}, w_j^{[v](n)}\right)\right]\right.\right. \\ &\qquad\left.\left. + hH_j^{[v](n)}\sum_{q=1}^{m_V} d_q\left[g^{[v]}\left(t_n, t_{n-q}, \breve{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right]\right\|^2\right\}^{1/2} \end{aligned} \tag{3.25b}$$

$$+ h \sum_{v=1}^{m} \left\| \sum_{j=1}^{s} b_j^{[v]} \left\{ f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \check{\tilde{y}}_j^{[v](n)}, w_j^{[v](n)}\right) - f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right) \right.\right.$$
$$\left.\left. + hH_j^{[v](n)} \sum_{q=1}^{m_V} d_q \left[g^{[v]}\left(t_n, t_{n-q}, \check{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right] \right\} \right\|. \tag{3.25c}$$

For

$$h\gamma_3 \sum_{v=1}^{m} \left\{ \sum_{i=1}^{s} \left\| \sum_{j=1}^{s} a_{ij}^{[v]} \left[f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \check{\tilde{y}}_j^{[v](n)}, w_j^{[v](n)}\right) - f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right) \right] \right.\right.$$
$$\left.\left. + hH_j^{[v](n)} \sum_{q=1}^{m_V} d_q \left[g^{[v]}\left(t_n, t_{n-q}, \check{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right] \right\|^2 \right\}^{1/2} = (3.25\text{b}). \tag{3.26}$$

Then

$$(3.25\text{b}) \le h\gamma_3 \sum_{v=1}^{m} \left\{ \sum_{i=1}^{s} 2 \sum_{j=1}^{s} \left| a_{ij}^{[v]} \right|^2 \left\{ \left\| f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \check{\tilde{y}}_j^{[v](n)}, w_j^{[v](n)}\right) \right.\right.\right.$$
$$\left. - f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right) \right\|^2$$
$$+ \left\| hH_j^{[v](n)} \sum_{q=1}^{m_V} d_q \left[g^{[v]}\left(t_n, t_{n-q}, \check{y}_j^{(n-q)}\right) \right.\right.$$
$$\left.\left.\left.\left.\left. - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right] \right\|^2 \right\} \right\}^{1/2}$$
$$\le h\gamma_3 \sum_{v=1}^{m} \left\{ \sum_{i=1}^{s} \left\{ 2 \sum_{j=1}^{s} \left| a_{ij}^{[v]} \right|^2 r_v^2 \left\| \check{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)} \right\|^2 + 2 \sum_{j=1}^{s} \left| a_{ij}^{[v]} \right|^2 h^2 H_j^{[v](n)2} 2 \sum_{q=1}^{m_v} d_q \right.\right.$$
$$\left.\left. \cdot \left\| g^{[v]}\left(t_n, t_{n-q}, \check{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right) \right\|^2 \right\} \right\}^{1/2}$$
$$\le h\gamma_3 \sum_{v=1}^{m} \sqrt{\sum_{i=1}^{s} 2} \sum_{j=1}^{s} \left| a_{ij}^{[v]} \right|^2 r_v^2 \left\| \check{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)} \right\|^2$$
$$+ 2h^2 \gamma_3 \sum_{v=1}^{m} \sqrt{\sum_{i=1}^{s} \sum_{j=1}^{s} \left| a_{ij}^{[v]} \right|^2 H_j^{[v](n)2} \sum_{q=1}^{m_v} d_q^2 \cdot \left\| g^{[v]}\left(t_n, t_{n-q}, \check{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right) \right\|^2}$$

$$
\begin{aligned}
&\le 2h\gamma_3 \sum_{v=1}^{m}\sum_{i=1}^{s}\sum_{j=1}^{s}\left|a_{ij}^{[v]}\right| r_v \left\|\breve{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| \\
&\quad + 2h^2\gamma_3 \sum_{v=1}^{m}\sum_{i=1}^{s}\sum_{j=1}^{s}\sum_{q=1}^{m_v} d_q \left|a_{ij}^{[v]}\right| \left|H_j^{[v](n)}\right| \cdot \left\|g^{[v]}\left(t_n, t_{n-q}, \breve{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right\| \\
&\le 2h\gamma_3 \sum_{v=1}^{m}\sum_{i,j=1}^{s}\left|a_{ij}^{[v]}\right| r_v \left\|\breve{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| \\
&\quad + 2h^2\gamma_3 \sum_{q=1}^{m_v} d_q \sum_{v=1}^{m}\sum_{i,j=1}^{s}\left|a_{ij}^{[v]}\right| \left|H_j^{[v](n)}\right| \tilde{r}_v \left\|\breve{y}_j^{(n-q)} - y_j^{(n-q)}\right\|.
\end{aligned}
\tag{3.27}
$$

For

$$
\begin{aligned}
h\sum_{v=1}^{m}\Bigg\| \sum_{j=1}^{s} b_j^{[v]} \Bigg\{ & f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \breve{\tilde{y}}_j^{[v](n)}, w_j^{[v](n)}\right) - f^{[v]}\left(t_j^{(n)}, y_j^{(n)}, \tilde{y}_j^{[v](n)}, w_j^{[v](n)}\right) \\
& + hH_j^{[v](n)} \sum_{q=1}^{m_V} d_q \Big[g^{[v]}\left(t_n, t_{n-q}, \breve{y}_j^{(n-q)}\right) \\
& - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\Big]\Bigg\}\Bigg\| = (3.25\text{c}),
\end{aligned}
\tag{3.28}
$$

then

$$
\begin{aligned}
(3.25\text{c}) &\le h\sum_{v=1}^{m}\sum_{j=1}^{s}\left|b_j^{(\nu)}\right| \gamma_v \left\|\breve{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| \\
&\quad + h^2 \sum_{v=1}^{m}\sum_{j=1}^{s}\left|b_j^{[v]}\right|\left|H_j^{[v](n)}\right| \left\{ \sum_{q=1}^{m_v} d_q \left\|g^{[v]}\left(t_n, t_{n-q}, \breve{y}_j^{(n-q)}\right) - g^{[v]}\left(t_n, t_{n-q}, y_j^{(n-q)}\right)\right\| \right\} \\
&\le h\sum_{v=1}^{m}\sum_{j=1}^{s}\left|b_j^{(\nu)}\right| \gamma_v \left\|\breve{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| + h^2 \sum_{v=1}^{m}\sum_{j=1}^{s}\left|b_j^{[v]}\right|\left|H_j^{[v](n)}\right| \left\{ \sum_{q=1}^{m_v} d_q \tilde{r}_v \left\|\breve{y}_j^{(n-q)} - y_j^{(n-q)}\right\| \right\}.
\end{aligned}
\tag{3.29}
$$

Combine (3.27), (3.29), and (3.25a), we have

$$
\begin{aligned}
\varepsilon_0^{(n+1)} &\le (1+\gamma_2 h)\left\|\varepsilon_0^{(n)}\right\| + \gamma_3\|\tilde{r}_n\| + \left\|\widetilde{Q}_n\right\| \\
&\quad + h\sum_{v=1}^{m}\gamma_v\left(2\gamma_3\sum_{i,j=1}^{s}\left|a_{ij}^{[v]}\right| + \sum_{j=1}^{s}\left|b_j^{[v]}\right|\right)\left\|\breve{\tilde{y}}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| \\
&\quad + h^2\sum_{v=1}^{m}\tilde{r}_v\sum_{q=1}^{m_v} d_q \sum_{j=1}^{s}\left\{2\gamma_3\sum_{i=1}^{s}\left|a_{ij}^{[v]}\right| \cdot \left|H_j^{[v](n)}\right| + \left|b_j^{[v]}\right| \cdot \left|H_j^{[v](n)}\right|\right\}\left\|\varepsilon_j^{(n-q)}\right\|.
\end{aligned}
\tag{3.30}
$$

Moreover, it follows from (2.5) and (3.11) that

$$\left\|\check{y}_j^{[v](n)} - \tilde{y}_j^{[v](n)}\right\| \le \sup_{\delta_v\in[0,1)} \sum_{P_v=-d}^{r} \left|L_{p_v}(\delta_v)\right| \max_{-d\le P_r\le r}\left\|\varepsilon_j^{(n-m_v+P_v)}\right\| + \left\|\rho_j^{[v](n)}\right\|. \tag{3.31}$$

Substituting (3.31) in (3.30), we get

$$\begin{aligned}\left\|\varepsilon_0^{(n+1)}\right\| \le{}& \left(1+\gamma_4^{(0)}h+\gamma_6^{(0)}h^2\right)\max\left\{\left\|\varepsilon_0^{(n)}\right\|, \max_{(j,p_v)\in E}\left\|\varepsilon_j^{(n-m_v+P_v)}\right\|, \max_{(j.q)\in E_q}\left\|\varepsilon_j^{(n-q)}\right\|\right\} + \left\|\tilde{Q}_n\right\| \\ &+ \gamma_3\|\tilde{r}_n\| + h\gamma_5^{(0)}\max_{(j,v)\in E_m}\left\|\rho_j^{[v](n)}\right\|, \quad h\in(0,h_0],\end{aligned} \tag{3.32}$$

where

$$\begin{gathered}\gamma_4^{(0)} = \gamma_2 + \gamma_5^{(0)}\sup_{\delta_{v\in[0,1)}}\sum_{P_v=-d}^{r}\left|L_{p_v}(\delta_v)\right|, \qquad \gamma_5^{(0)} = \sum_{v=1}^{m} r_v\left(2\gamma_3\sum_{i,j=1}^{s}\left|a_{ij}^{[v]}\right| + \sum_{j=1}^{s}\left|b_j^{[v]}\right|\right), \\ \gamma_6^{(0)} = \sum_{v=1}^{m}\tilde{\gamma}_v\sum_{q=1}^{m_v}\sum_{j=1}^{s} d_q\left\{2\gamma_3\sum_{i=1}^{s}\left|a_{ij}{}^{[v]}\cdot H_j^{[v](n)}\right| + \left|b_j^{[v]}\cdot H_j^{[v](n)}\right|\right\}, \\ E = \left\{(j,P_v)\mid 1\le j\le s,\ -d\le P_v\le\gamma\right\}, \qquad E_q = \left\{(j,q)\mid 1\le j\le s,\ 1\le q\le m_v\right\}, \\ E_m = \left\{(j,v)\mid 1\le j\le s,\ 1\le v\le m\right\}.\end{gathered} \tag{3.33}$$

By Lemma 3.5, similar to (3.32), we can obtain the inequalities:

$$\begin{aligned}\left\|\varepsilon_i^{(n)}\right\| \le{}& \left(1+h\gamma_4^{(i)}+h^2\gamma_6^{(i)}\right)\max\left\{\left\|\varepsilon_0^{(n)}\right\|, \max_{(j,P_v\in E)}\left\|\varepsilon_j^{(n-m_v+P_v)}\right\|, \max_{(j,q)\in E_q}\left\|\varepsilon_j^{(n-q)}\right\|\right\} \\ &+h\gamma_5^{(i)}\max_{(j,v)\in E_m}\left\|\rho_j^{[v](n)}\right\| + \left\|\tilde{Q}_n\right\| + \gamma_3\|\tilde{r}_n\|, \quad i=1,2,\ldots,s,\ h\in(0,h_0],\end{aligned} \tag{3.34}$$

where

$$\begin{gathered}\gamma_4^{(i)} = \gamma_2 + \gamma_5^{(i)}\sup_{\delta_{v\in[0,1)}}\sum_{P_v=-d}^{r}\left|L_{p_v}(\delta_v)\right|, \qquad \gamma_5^{(i)} = \sum_{v=1}^{m} r_v\left(2\gamma_3\sum_{i,j=1}^{s}\left|a_{ij}^{[v]}\right| + \sum_{j=1}^{s}\left|a_{ij}^{[v]}\right|\right), \\ \gamma_6^{(i)} = \sum_{v=1}^{m}\tilde{\gamma}_v\sum_{q=1}^{m_v}\sum_{j=1}^{s} d_q\left(2\gamma_3\sum_{i=1}^{s}\left|a_{ij}{}^{[v]}\cdot H_j^{[v](n)}\right| + \left|a_{ij}^{[v]}\cdot H_j^{[v](n)}\right|\right).\end{gathered} \tag{3.35}$$

Setting $\gamma_4 = \max_{0\le i\le s}\gamma_4^{(i)}$, $\gamma_5 = \max_{0\le i\le s}\gamma_5^{(i)}$, $\gamma_6 = \max_{0\le i\le s}\gamma_6^{(i)}$.

Combining (3.32) with (3.34), we immediately obtain the conclusion of this theorem.

Now, we turn to study the convergence of ARKLM (2.1) for (1.1). It is always assumed that the analytic solution $y(t)$ of (1.1) is smooth enough on each internal of the form $(t_0+(j-1)h, t_0+jh)$ (j is a positive integer) as (3.2) defined. □

Theorem 3.7. *Assume ARKLM* (2.1) *with stage order* p *is DA-, DAS- and ASI-stable, then the ARKLM* (2.1) *is D-convergent of order* $\min\{p, q+1, s+1\}$, *where* $q=d+r$.

Proof. By Theorem 3.6, we have for $h\in(0,h_0]$

$$\left\|\varepsilon_i^{(n)}\right\| \le \begin{cases} (1+h\gamma_4+h^2\gamma_6)\max\left\{\left\|\varepsilon_0^{(n-1)}\right\|, \max\limits_{(i,p_v)\in E}\left\|\varepsilon_i^{(n-1-m_v+p_v)}\right\|, \max\limits_{(j,q\in E_q)}\left\|\varepsilon_j^{(n-q)}\right\|\right\} \\ \quad +T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}, \quad i=0 \\ (1+h\gamma_4+h^2\gamma_6)\max\left\{\left\|\varepsilon_0^{(n)}\right\|, \max\limits_{(i,p_v)\in E}\left\|\varepsilon_i^{(n-m_v+p_v)}\right\|, \max\limits_{(j,q)\in E_q}\left\|\varepsilon_j^{(n-q)}\right\|\right\} \\ \quad +T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}, \quad i=1,2,\ldots,s, \end{cases} \tag{3.36}$$

where

$$T_1=\widehat{M}_0+\gamma_3\sqrt{\sum_{i=1}^{s}\widehat{M}_i^2}, \qquad T_2=\frac{\gamma_s}{(q+1)!}\sum_{p_v=-d}^{r}|\delta_v-P_v|M_{q+1},$$
$$T_3=\left(\sum_{v=1}^{m}\tilde{b}^{[v]^{\mathrm{T}}}H^{[v]R}+\sum_{v=1}^{m}\tilde{A}^{[v]}H^{[v]R}\right)g^{[v](s)}(\xi). \tag{3.37}$$

It follows from an induction to (3.36) for n that

$$\left\|\varepsilon_i^{(n)}\right\| \le \begin{cases} \sum\limits_{j=0}^{n}(1+2h\gamma_4)^j\left(T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}\right), & i=0, \\ \sum\limits_{j=0}^{n+1}(1+2h\gamma_4)^j\left(T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}\right), & i=1,2,\ldots,s. \end{cases} \tag{3.38}$$

Hence, for $h\in(0,h_0]$, we arrive at

$$\begin{aligned}\|y(t_n)-y_n\| = \left\|\varepsilon_0^{(n)}\right\| &\le \sum_{j=0}^{n}(1+2h\gamma_4)^j\left(T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}\right) \\ &= \frac{(1+2h\gamma_4)^{n+1}-1}{2h\gamma_4}\left(T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}\right)\end{aligned}$$

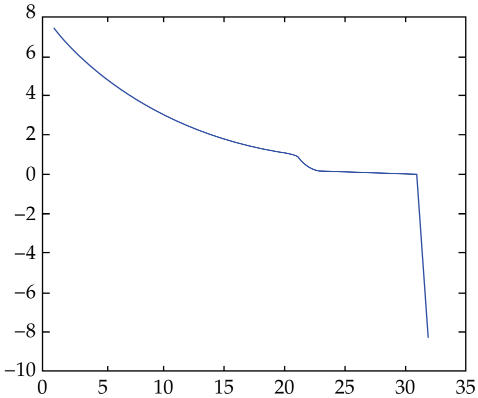

Figure 1: Values y_n with $h = 0.1$.

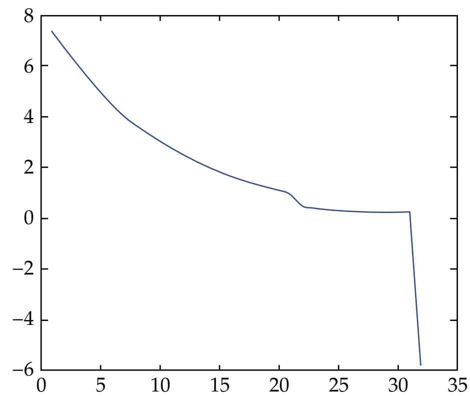

Figure 2: Values z_n with $h = 0.1$.

$$
\begin{aligned}
&\leq \frac{\exp[2(n+1)h\gamma_4]-1}{2h\gamma_4}\left(T_1h^{p+1}+T_2h^{q+2}+T_3h^{s+2}\right)\\
&\leq \frac{\exp\;[2(T-t_0)\gamma_4]\exp(2h_0\gamma_4)-1}{2\gamma_4}\left(T_1h^{p}+T_2h^{q+1}+T_3h^{s+1}\right).
\end{aligned}
\tag{3.39}
$$

Therefore, the ARKLM (2.1) is D-Convergent of order $\min\{p, q+1, s+1\}$, $(q = r + d)$. □

Table 1: Comparison between the numerical solutions for different h and the exact solution.

t	Numerical solution for $h = 0.2$	Numerical solution for $h = 0.1$	Numerical solution for $h = 0.05$	Exact solution $y(t)$
0.2	2.87569414	1.212021722	0.898712327	0.818730753
0.4	2.751727517	1.165089633	0.801604459	0.670320046
0.6	2.448183694	1.0957815	0.626189996	0.548811636

4. Some Examples

Consider the following initial value problem of multidelay-integro-differential equations:

$$\begin{aligned} y'(t) &= \left(-10^4 + 99i\right)\frac{[1+y(t)]^2}{1+[1+y(t)]^2} + 500[y(t-1) - y(t-2)] - 500\int_{t-1}^{t} y(s)ds \\ &\quad + 500\int_{t-2}^{t} y(s)ds - \exp(-t) + \left(10^4 - 99i\right) \times \frac{[1+\exp(-t)]^2}{1+[1+\exp(-t)]^2}, \quad 0 \le t \le 3, \\ y(t) &= \exp(-t), \quad -2 \le t \le 0, \end{aligned} \tag{4.1}$$

and its perturbed problem:

$$\begin{aligned} z'(t) &= \left(-10^4 + 99i\right)\frac{[1+z(t)]^2}{1+[1+z(t)]^2} + 500[z(t-1) - z(t-2)] - 500\int_{t-1}^{t} z(s)ds \\ &\quad + 500\int_{t-2}^{t} z(s)ds - \exp(-t) + \left(10^4 - 99i\right) \times \frac{[1+\exp(-t)]^2}{1+[1+\exp(-t)]^2}, \quad 0 \le t \le 3, \\ z(t) &= \exp(-t) + 0.01, \quad -2 \le t \le 0. \end{aligned} \tag{4.2}$$

It an be easily verified that $a_1 = a_2 = -9.5 \times 10^3$, $\beta_1 = \beta_2 = 250$, $\sigma_1 = \sigma_2 = 250$, and $\widetilde{r}_1 = \widetilde{r}_2 = 1$, with analytic solution $y(t) = \exp(-t)$, where the inner product is standard inner product. We apply the two-stages and two-order additive R-K method:

$$\begin{array}{c|cc|cc} 1 & 1 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ \hline & \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{array} \tag{4.3}$$

(1) to the problem (4.1) and its perturbed problem (4.2). Since the order of the method is 2, we adopt the compound trapezoidal rule for computing the integer part. According to the result of Theorems 2.3 and 3.7, the corresponding method for DIDEs is GDN-stable and D-convergent. We denote the numerical solution of problem (4.1) and (4.2) y_n and z_n, where y_n and z_n are approximations to $y(t_n)$ and $z(t_n)$, respectively. The values y_n and z_n with $h = 0.1$ are listed in Figure 1, (where the abscissa and ordinate denote variable n and y_n, resp.) and Figure 2, (where the abscissa and ordinate denote variable n and z_n, resp.). It is shown in Table 1 that the numerical solutions are toward to the exact solutions as $h \to 0$.

It is obvious that the corresponding method for MDIDEs is GDN-stable and D-convergent.

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