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Review Article

Some Properties and Identities of Bernoulli and Euler Polynomials Associated with p-adic Integral on $\mathbb{Z}_p$

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We investigate some properties and identities of Bernoulli and Euler polynomials. Further, we give some formulae on Bernoulli and Euler polynomials by using p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

For $f \in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ in the bosonic sense is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)d\mu(x) \tag{1.1}$$

(see [1, 2]). The fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \tag{1.2}$$

(see [3]). As is well known, Bernoulli polynomials are defined by

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}, \tag{1.3}$$

with the usual convention about replacing $B^n(x)$ by $B_n(x)$, symbolically (see [1–19]). In the special case $x=0$, $B_n(0)=B_n$ is called the nth Bernoulli number.

The Euler polynomials are also defined by the generating function as follows:

$$\frac{2}{e^t+1}e^{xt}=e^{E(x)t}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{1.4}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$, symbolically (see [1–19]). In the special case $x=0$, $E_n(0)=E_n$ is called the n-th Euler number.

By (1.3) and (1.4), we easily see that

$$\begin{aligned}B_n(x)&=(B+x)^n=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_l,\\ E_n(x)&=(E+x)^n=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}E_l,\end{aligned} \tag{1.5}$$

where $\binom{n}{l}=n!/(n-l)!l!=n(n-1)(n-2)\cdots(n-l+1)/l!$ (see [14, 16, 19]).

The following properties of Bernoulli numbers and polynomials are well known (see [10, 11]).

For $n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$,

$$\sum_{j=0}^{n}\binom{n}{j}y^{n-j}\frac{B_{j+1}(x)}{j+1}=\frac{B_{n+1}(x+y)-y^{n+1}}{n+1}, \tag{1.6}$$

$$\sum_{j=0}^{[n/2]}\binom{n}{2j}y^{n-2j}\frac{B_{2j+1}(x)}{2j+1}=\frac{B_{n+1}(x+y)+(-1)^nB_{n+1}(x-y)}{2n+2}, \tag{1.7}$$

$$\sum_{j=1}^{[(n+1)/2]}\binom{n}{2j-1}y^{n+1-2j}\frac{B_{2j}(x)}{2j}=\frac{B_{n+1}(x+y)+(-1)^{n-1}B_{n+1}(x-y)-2y^{n+1}}{2n+2}, \tag{1.8}$$

where $[\cdot]$ is Gauss' symbol.

First, we investigate some identities of Euler polynomials corresponding to (1.6), (1.7) and (1.8). From those identities, we derive some interesting identities and properties by using p-adic integral on $\mathbb{Z}_p$.

2. Some Identities of Bernoulli and Euler Polynomials

By (1.4), we get

$$E_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} y^{k-j} E_j(x), \quad \text{for} \in \mathbb{Z}_+. \tag{2.1}$$

From (2.1), we note that

$$\begin{aligned} E_k(x+y) &= \sum_{j=0}^{k} \binom{k}{j} y^{k-j} E_j(x) \\ &= y^k + \sum_{j=1}^{k} \frac{k}{j} \binom{k-1}{j-1} y^{k-j} E_j(x). \end{aligned} \tag{2.2}$$

Thus, we have

$$\sum_{j=0}^{k-1} \binom{k-1}{j} y^{k-1-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_k(x+y) - y^k}{k}. \tag{2.3}$$

Replacing k by $k+1$ in (2.3), we obtain the following proposition.

Proposition 2.1. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{k} \binom{k}{j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x+y) - y^{k+1}}{k+1}. \tag{2.4}$$

Let us replace y by $-y$ in Proposition 2.1. Then we have

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{E_{k+1}(x-y) - (-1)^{k+1} y^{k+1}}{k+1}. \tag{2.5}$$

Thus, we see that

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^{j} y^{k-j} \frac{E_{j+1}(x)}{j+1} = \frac{(-1)^k E_{k+1}(x-y) + y^{k+1}}{k+1}. \tag{2.6}$$

Therefore, adding (2.4) and (2.6), we obtain the following proposition.

Proposition 2.2. *For $k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{[k/2]} \binom{k}{2j} y^{k-2j} \frac{E_{2j+1}(x)}{2j+1} = \frac{E_{k+1}(x+y) + (-1)^k E_{k+1}(x-y)}{2k+2}. \tag{2.7}$$

From (2.2), we note that

$$\sum_{j=1}^{k}\frac{1}{j}\binom{k-1}{j-1}y^{k-j}(-1)^{j}E_{j}(x)=\frac{(-1)^{k}E_{k}(x-y)-y^{k}}{k}. \tag{2.8}$$

By (2.3) and (2.8), we get

$$\sum_{j=1}^{[k/2]}\frac{1}{2j}\binom{k-1}{2j-1}y^{k-2j}E_{2j}(x)=\frac{E_{k}(x+y)+(-1)^{k}E_{k}(x-y)-2y^{k}}{2k}. \tag{2.9}$$

Therefore, replacing k by $k+1$, we obtain the following proposition.

Proposition 2.3. *For $k\in\mathbb{N}$, one has*

$$\sum_{j=1}^{[(k+1)/2]}\binom{k}{2j-1}y^{k+1-2j}\frac{E_{2j}(x)}{2j}=\frac{E_{k+1}(x+y)+(-1)^{k+1}E_{k+1}(x-y)-2y^{k+1}}{2k+2}. \tag{2.10}$$

Letting $y=1$ in Proposition 2.1, we have

$$\sum_{j=0}^{k}\binom{k}{j}\frac{E_{j+1}(x)}{j+1}=\frac{E_{k+1}(x+1)-1}{k+1}, \tag{2.11}$$

$$\begin{aligned}E_{k+1}(x+1)&=\sum_{l=0}^{k+1}\binom{k+1}{l}(E+1)^{l}x^{k+1-l}\\&=(2-E_{0})x^{k+1}-\sum_{l=1}^{k+1}\binom{k+1}{l}E_{l}x^{k+1-l}\\&=2x^{k+1}-\sum_{l=0}^{k+1}\binom{k+1}{l}E_{l}x^{k+1-l}=2x^{k+1}-E_{k+1}(x).\end{aligned} \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following corollary.

Corollary 2.4. *For $k\in\mathbb{Z}_{+}$, one has*

$$\sum_{j=0}^{k}\binom{k}{j}\frac{E_{j+1}(x)}{j+1}=-\frac{E_{k+1}(x)}{k+1}+\frac{2x^{k+1}-1}{k+1}. \tag{2.13}$$

Replacing y by 1 and k by $2k$ in Proposition 2.2, we have

$$\begin{aligned}\sum_{j=0}^{k}\binom{2k}{2j}\frac{E_{2j+1}(x)}{2j+1} &= \frac{E_{2k+1}(x+1)+E_{2k+1}(x-1)}{4k+2}\\ &= \frac{E_{2k+1}(x+1)+E_{2k+1}(x)+E_{2k+1}(x)+E_{2k+1}(x-1)}{4k+2}-\frac{2E_{2k+1}(x)}{4k+2}\\ &= \frac{2x^{2k+1}+2(x-1)^{2k+1}}{4k+2}-\frac{E_{2k+1}(x)}{2k+1}.\end{aligned}\tag{2.14}$$

Therefore, by (2.14), we obtain the following corollary.

Corollary 2.5. *For $k\in\mathbb{Z}_{+}$, one has*

$$\sum_{j=0}^{k}\binom{2k}{2j}\frac{E_{2j+1}(x)}{2j+1} = -\frac{E_{2k+1}(x)}{2k+1}+\frac{x^{2k+1}+(x-1)^{2k+1}}{2k+1}.\tag{2.15}$$

Replacing y by 1 and k by $2k$ in Proposition 2.3, we have

$$\begin{aligned}\sum_{j=1}^{k}\binom{2k}{2j-1}\frac{E_{2j}(x)}{2j} &= \frac{E_{2k+1}(x+1)-E_{2k+1}(x-1)-2}{4k+2}\\ &= \frac{(E_{2k+1}(x+1)+E_{2k+1}(x))-(E_{2k+1}(x)+E_{2k+1}(x-1))}{4k+2}-\frac{1}{2k+1}\\ &= \frac{2x^{2k+1}-2(x-1)^{2k+1}}{4k+2}-\frac{1}{2k+1}\\ &= \frac{x^{2k+1}-(x-1)^{2k+1}}{2k+1}-\frac{1}{2k+1}.\end{aligned}\tag{2.16}$$

Therefore, by (2.16), we obtain the following corollary.

Corollary 2.6. *For $k\in\mathbb{N}$, one has*

$$\sum_{j=1}^{k}\binom{2k}{2j-1}\frac{E_{2j}(x)}{2j} = \frac{x^{2k+1}-(x-1)^{2k+1}}{2k+1}-\frac{1}{2k+1}.\tag{2.17}$$

Replacing y by $1/2$ and k by $2k$ in Proposition 2.3, we get

$$\sum_{j=1}^{k}\binom{2k}{2j-1}\left(\frac{1}{2}\right)^{2k+1-2j}\frac{E_{2j}(x)}{2j} = \frac{E_{2k+1}(x+1/2)-E_{2k+1}(x-1/2)-2^{-2k}}{4k+2}.\tag{2.18}$$

Thus, we have

$$\sum_{j=1}^{k}\binom{2k}{2j-1}2^{2j}\frac{E_{2j}(x)}{2j}=\frac{2^{2k}(E_{2k+1}(x+1/2)-E_{2k+1}(x-1/2))-1}{2k+1}, \tag{2.19}$$

$$\begin{aligned}E_{2k+1}\left(x+\frac{1}{2}\right)&=E_{2k+1}\left(x-\frac{1}{2}+1\right)=\sum_{l=0}^{2k+1}\binom{2k+1}{l}\left(x-\frac{1}{2}\right)^{2k+1-l}(E+1)^{l}\\&=2\left(x-\frac{1}{2}\right)^{2k+1}-\sum_{l=0}^{2k+1}\binom{2k+1}{l}\left(x-\frac{1}{2}\right)^{2k+1-l}E_{l}\\&=2\left(x-\frac{1}{2}\right)^{2k+1}-E_{2k+1}\left(x-\frac{1}{2}\right).\end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following corollary.

Corollary 2.7. *For $k\in\mathbb{N}$, we have*

$$\sum_{j=1}^{k}\binom{2k}{2j-1}2^{2j}\frac{E_{2j}(x)}{2j}=-\frac{2^{2k+1}E_{2k+1}(x-1/2)}{2k+1}+\frac{2^{2k+1}(x-1/2)^{2k+1}}{2k+1}-\frac{1}{2k+1}. \tag{2.21}$$

Replacing y by 1 and k by $2k+1$ in Proposition 2.2, we get

$$\begin{aligned}\sum_{j=0}^{k}\binom{2k+1}{2j}\frac{E_{2j+1}(x)}{2j+1}&=\frac{E_{2k+2}(x+1)-E_{2k+2}(x-1)}{4k+4}\\&=\frac{(E_{2k+2}(x+1)+E_{2k+2}(x))-(E_{2k+2}(x)+E_{2k+2}(x-1))}{4k+4}\\&=\frac{2x^{2k+2}-2(x-1)^{2k+2}}{4k+4}=\frac{x^{2k+2}-(x-1)^{2k+2}}{2k+2}.\end{aligned} \tag{2.22}$$

Therefore, by (2.22), we obtain the following corollary.

Corollary 2.8. *For $k\in\mathbb{Z}_{+}$, one has*

$$\sum_{j=0}^{k}\binom{2k+1}{2j}\frac{E_{2j+1}(x)}{2j+1}=\frac{x^{2k+2}-(x-1)^{2k+2}}{2k+2}. \tag{2.23}$$

Replacing k by $2k+1$ and y by 1 in Proposition 2.3, we get

$$\begin{aligned}\sum_{j=1}^{k+1}\binom{2k+1}{2j-1}\frac{E_{2j}(x)}{2j} &= \frac{E_{2k+2}(x+1)+E_{2k+2}(x-1)-2}{4k+4}\\ &= \frac{(E_{2k+2}(x+1)+E_{2k+2}(x))+(E_{2k+2}(x)+E_{2k+2}(x-1))}{4k+4}-\frac{E_{2k+2}(x)+1}{2k+2}\\ &= \frac{x^{2k+2}+(x-1)^{2k+2}}{2k+2}-\frac{E_{2k+2}(x)+1}{2k+2}.\end{aligned} \tag{2.24}$$

Therefore, by (2.24), we obtain the following corollary.

Corollary 2.9. *For $k \in \mathbb{Z}_+$, we have*

$$\sum_{j=1}^{k+1}\binom{2k+1}{2j-1}\frac{E_{2j}(x)}{2j} = \frac{x^{2k+2}+(x-1)^{2k+2}}{2k+2}-\frac{E_{2k+2}(x)+1}{2k+2}. \tag{2.25}$$

Replacing k by $2k+1$ and y by $1/2$ in Proposition 2.2, we have

$$\sum_{j=0}^{k}\binom{2k+1}{2j}\left(\frac{1}{2}\right)^{2k+1-2j}\frac{E_{2j+1}(x)}{2j+1} = \frac{E_{2k+2}(x+1/2)-E_{2k+2}(x-1/2)}{4k+4}. \tag{2.26}$$

Thus, by multipling 2^{2k+1} on both sides, we get

$$\sum_{j=0}^{k}\binom{2k+1}{2j}2^{2j}\frac{E_{2j+1}(x)}{2j+1} = \frac{2^{2k+1}\{E_{2k+2}(x+1/2)-E_{2k+2}(x-1/2)\}}{4k+4}. \tag{2.27}$$

By (2.20) and (2.27), we see that

$$\begin{aligned}\sum_{j=0}^{k}\binom{2k+1}{2j}2^{2j}\frac{E_{2j+1}(x)}{2j+1} &= \frac{2^{2k}\left(2(x-1/2)^{2k+2}-2E_{2k+2}(x-1/2)\right)}{2k+2}\\ &= \frac{2^{2k}(x-1/2)^{2k+2}-2^{2k}E_{2k+2}(x-1/2)}{k+1}.\end{aligned} \tag{2.28}$$

Therefore, by (2.28), we obtain the following corollary.

Corollary 2.10. *For $k \in \mathbb{Z}_+$, we have*

$$\sum_{j=0}^{k}\binom{2k+1}{2j}2^{2j}\frac{E_{2j+1}(x)}{2j+1} = -\frac{2^{2k}E_{2k+2}(x-1/2)}{k+1}+\frac{2^{2k}(x-1/2)^{2k+2}}{k+1}. \tag{2.29}$$

From (1.6), we can derive the following equation:

$$\sum_{j=0}^{k-1}\binom{k}{j}\frac{B_{j+1}(x)}{j+1}=x^k-\frac{1}{k+1},\quad \text{for } k\in\mathbb{N}. \tag{2.30}$$

Let us take the p-adic integral on both sides in (2.30) as follows: for $k\in\mathbb{N}$,

$$\begin{aligned}I_1&=\sum_{j=0}^{k-1}\binom{k}{j}\int_{\mathbb{Z}_p}\frac{B_{j+1}(x)}{j+1}d\mu(x)=\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{j+1}\sum_{l=0}^{j+1}\binom{j+1}{l}B_{j+1-l}\int_{\mathbb{Z}_p}x^l d\mu(x)\\&=\sum_{j=0}^{k-1}\sum_{l=0}^{j+1}\frac{1}{j+1}\binom{k}{j}\binom{j+1}{l}B_{j+1-l}B_l.\end{aligned} \tag{2.31}$$

On the other hand,

$$I_1=\int_{\mathbb{Z}_p}x^k d\mu(x)-\frac{1}{k+1}\int_{\mathbb{Z}_p}d\mu(x)=B_k-\frac{1}{k+1}. \tag{2.32}$$

Therefore, by (2.31) and (2.32), we obtain the following theorem.

Theorem 2.11. *For $k\in\mathbb{N}$, one has*

$$\sum_{j=0}^{k-1}\sum_{l=0}^{j+1}\frac{1}{j+1}\binom{k}{j}\binom{j+1}{l}B_{j+1-l}B_l=B_k-\frac{1}{k+1}. \tag{2.33}$$

In (2.30), let us take the fermionic p-adic integral on both sides as follows:

$$\begin{aligned}I_2&=\sum_{j=0}^{k-1}\binom{k}{j}\frac{1}{j+1}\int_{\mathbb{Z}_p}B_{j+1}(x)d\mu_{-1}(x)\\&=\sum_{j=0}^{k-1}\frac{\binom{k}{j}}{j+1}\sum_{l=0}^{j+1}\binom{j+1}{l}B_{j+1-l}\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x)\\&=\sum_{j=0}^{k-1}\sum_{l=0}^{j+1}\frac{1}{j+1}\binom{k}{j}\binom{j+1}{l}B_{j+1-l}E_l.\end{aligned} \tag{2.34}$$

On the other hand

$$I_2=\int_{\mathbb{Z}_p}x^k d\mu_{-1}(x)-\frac{1}{k+1}\int_{\mathbb{Z}_p}d\mu_{-1}(x)=E_k-\frac{1}{k+1}. \tag{2.35}$$

Therefore, by (2.34) and (2.35), we obtain the following theorem.

Theorem 2.12. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{k-1}\sum_{l=0}^{j+1}\frac{1}{j+1}\binom{k}{j}\binom{j+1}{l}B_{j+1-l}E_l = E_k - \frac{1}{k+1}. \tag{2.36}$$

From (1.7), we can easily derive the following equation:

$$\sum_{j=0}^{k-1}\binom{2k}{2j}\frac{B_{2j+1}(x)}{2j+1} = \frac{x^{2k}-(x-1)^{2k}}{2}. \tag{2.37}$$

Let us take $\int_{\mathbb{Z}_p} d\mu(x)$ on both sides in (2.37). Then we have

$$\begin{aligned} I_3 &= \sum_{j=0}^{k-1}\binom{2k}{2j}\frac{1}{2j+1}\int_{\mathbb{Z}_p} B_{2j+1}(x)d\mu(x) \\ &= \sum_{j=0}^{k-1}\binom{2k}{2j}\frac{1}{2j+1}\sum_{l=0}^{2j+1}\binom{2j+1}{l}B_{2j+1-l}\int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{j=0}^{k-1}\sum_{l=0}^{2j+1}\frac{1}{2j+1}\binom{2k}{2j}\binom{2j+1}{l}B_{2j+1-l}B_l. \end{aligned} \tag{2.38}$$

On the other hand,

$$\begin{aligned} I_3 &= \frac{1}{2}\left(\int_{\mathbb{Z}_p} x^{2k}d\mu(x) - \int_{\mathbb{Z}_p}(x-1)^{2k}d\mu(x)\right) \\ &= \frac{1}{2}(B_{2k} - B_{2k}(-1)) = \frac{1}{2}(B_{2k} - B_{2k}(2)) \\ &= \frac{1}{2}(B_{2k} - (2k + \delta_{1,2k} + B_{2k})), \end{aligned} \tag{2.39}$$

where $\delta_{n,k}$ is a Kronecker symbol.

Therefore, by (2.38) and (2.39), we obtain the following theorem.

Theorem 2.13. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{k-1}\sum_{l=0}^{2j+1}\frac{1}{2j+1}\binom{2k}{2j}\binom{2j+1}{l}B_{2j+1-l}B_l = -k. \tag{2.40}$$

Taking $\int_{\mathbb{Z}_p} d\mu_{-1}(x)$ on both sides in (2.37), we get

$$\begin{aligned} I_4 &= \sum_{j=0}^{k-1}\binom{2k}{2j}\frac{1}{2j+1}\sum_{l=0}^{2j+1}\binom{2j+1}{l}B_{2j+1-l}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{j=0}^{k-1}\sum_{l=0}^{2j+1}\frac{1}{2j+1}\binom{2k}{2j}\binom{2j+1}{l}B_{2j+1-l}E_l. \end{aligned} \tag{2.41}$$

On the other hand

$$\begin{aligned} I_4 &= \frac{1}{2}\left(\int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p}(x-1)^{2k}d\mu_{-1}(x)\right) \\ &= \frac{1}{2}\left(\int_{\mathbb{Z}_p} x^{2k} d\mu_{-1}(x) - \int_{\mathbb{Z}_p}(x+2)^{2k}d\mu_{-1}(x)\right) \\ &= \frac{1}{2}(E_{2k}-E_{2k}(2)) = \frac{1}{2}\{E_{2k}-(2+E_{2k}-2\delta_{0,2k})\} \\ &= -1+\delta_{0,k}. \end{aligned} \tag{2.42}$$

Therefore, by (2.41) and (2.42), we obtain the following theorem.

Theorem 2.14. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=0}^{k-1}\sum_{l=0}^{2j+1}\frac{1}{2j+1}\binom{2k}{2j}\binom{2j+1}{l}B_{2j+1-l}E_l = -1. \tag{2.43}$$

From (1.8), we can also derive the following equation:

$$\sum_{j=1}^{k}\binom{2k}{2j-1}\frac{B_{2j}(x)}{2j} = \frac{x^{2k}+(x-1\ )^{2k}}{2} - \frac{1}{2k+1}. \tag{2.44}$$

Let us take the bosonic p-adic integral on both sides in (2.44). Then we get

$$\begin{aligned} I_5 &= \sum_{j=1}^{k}\binom{2k}{2j-1}\frac{1}{2j}\int_{\mathbb{Z}_p} B_{2j}(x)d\mu(x) \\ &= \sum_{j=1}^{k}\binom{2k}{2j-1}\frac{1}{2j}\sum_{l=0}^{2j}\binom{2j}{l}B_{2j-l}\int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{j=1}^{k}\sum_{l=0}^{2j}\frac{1}{2j}\binom{2k}{2j-1}\binom{2j}{l}B_{2j-l}B_l. \end{aligned} \tag{2.45}$$

On the other hand,

$$\begin{aligned}
I_5 &= \frac{1}{2}\int_{\mathbb{Z}_p}\left(x^{2k}+(-1+x)^{2k}\right)d\mu(x)-\frac{1}{2k+1}\int_{\mathbb{Z}_p}d\mu(x)\\
&= \frac{1}{2}\int_{\mathbb{Z}_p}\left(x^{2k}+(x+2)^{2k}\right)d\mu(x)-\frac{1}{2k+1}\\
&= \frac{1}{2}(B_{2k}+B_{2k}(2))-\frac{1}{2k+1}\\
&= \frac{1}{2}(B_{2k}+2k+B_{2k}+\delta_{1,2k})-\frac{1}{2k+1}.
\end{aligned} \tag{2.46}$$

Therefore, by (2.45) and (2.46), we obtain the following theorem.

Theorem 2.15. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=1}^{k}\sum_{l=0}^{2j}\frac{1}{2j}\binom{2k}{2j-1}\binom{2j}{l}B_{2j-l}B_l = B_{2k}+k-\frac{1}{2k+1}. \tag{2.47}$$

Now, let us consider the fermionic p-adic integral on both sides in (2.44):

$$\begin{aligned}
I_6 &= \sum_{j=1}^{k}\binom{2k}{2j-1}\frac{1}{2j}\sum_{l=0}^{2j}\binom{2j}{l}B_{2j-l}\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x)\\
&= \sum_{j=1}^{k}\sum_{l=0}^{2j}\frac{1}{2j}\binom{2k}{2j-1}\binom{2j}{l}B_{2j-l}E_l.
\end{aligned} \tag{2.48}$$

On the other hand,

$$\begin{aligned}
I_6 &= \frac{1}{2}\int_{\mathbb{Z}_p}\left(x^{2k}+(x-1)^{2k}\right)d\mu_{-1}(x)-\frac{1}{2k+1}\int_{\mathbb{Z}_p}d\mu_{-1}(x)\\
&= \frac{1}{2}\int_{\mathbb{Z}_p}\left(x^{2k}+(x+2)^{2k}\right)d\mu_{-1}(x)-\frac{1}{2k+1}\\
&= \frac{1}{2}(E_{2k}+E_{2k}(2))-\frac{1}{2k+1}\\
&= \frac{1}{2}(E_{2k}+(2+E_{2k}-2\delta_{0,2k}))-\frac{1}{2k+1}\\
&= E_{2k}+1-\delta_{0,2k}-\frac{1}{2k+1}=\frac{2k}{2k+1}.
\end{aligned} \tag{2.49}$$

Therefore, by (2.48) and (2.49), we obtain the following theorem.

Theorem 2.16. *For $k \in \mathbb{N}$, one has*

$$\sum_{j=1}^{k} \sum_{l=0}^{2j} \frac{1}{2j} \binom{2k}{2j-1} \binom{2j}{l} B_{2j-l} E_l = \frac{2k}{2k+1}. \tag{2.50}$$

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