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Research Article

An Approximation of Ultra-Parabolic Equations

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The first and second order of accuracy difference schemes for the approximate solution of the initial boundary value problem for ultra-parabolic equations are presented. Stability of these difference schemes is established. Theoretical results are supported by the result of numerical examples.

1. Introduction

Mathematical models that are formulated in terms of ultraparabolic equations are of great importance in various problems for instance in age-dependent population model, in the mathematical model of Brownian motion, in the theory of boundary layers, and so forth, see [1–5]. We refer also to [6–9] and the references therein for existence and uniqueness results and other properties of these models. On the other hand, Akrivis et al. [10] and Ashyralyev and Yılmaz [11, 12] developed numerical methods for ultraparabolic equations. In this paper, our interest is studying the stability of first- and second-order difference schemes for the approximate solution of the initial boundary value problem for ultraparabolic equations

$$\begin{gathered}\frac{\partial u(t,s)}{\partial t}+\frac{\partial u(t,s)}{\partial s}+Au(t,s)=f(t,s),\quad 0<t,\ s<T,\\ u(0,s)=\psi(s),\quad 0\le s\le T,\\ u(t,0)=\varphi(t),\quad 0\le t\le T,\end{gathered}\tag{1.1}$$

in an arbitrary Banach space E with a strongly positive operator A. For approximately solving problem (1.1), the first-order of accuracy difference scheme

$$\begin{gathered}\frac{u_{k,m}-u_{k-1,m}}{\tau}+\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}+Au_{k,m}=f_{k,m},\\ f_{k,m}=f(t_k,s_m),\qquad t_k=k\tau,\qquad s_m=m\tau,\quad 1\le k,\ m\le N,\ N\tau=1,\\ u_{0,m}=\psi_m,\quad 0\le m\le N,\qquad u_{k,0}=\varphi_k,\quad 0\le k\le N\end{gathered}\tag{1.2}$$

and second-order of accuracy difference scheme

$$\begin{gathered}\frac{u_{k,m}-u_{k-1,m}}{\tau}+\frac{u_{k-1,m}-u_{k-1,m-1}}{\tau}+\frac{1}{2}A(u_{k,m}+u_{k-1,m-1})=f_{k,m},\\ f_{k,m}=f\left(t_k-\frac{\tau}{2},s_m-\frac{\tau}{2}\right),\qquad t_k=k\tau,\qquad s_m=m\tau,\quad 1\le k,\ m\le N,\ N\tau=1,\\ u_{0,m}=\psi_m,\quad 0\le m\le N,\qquad u_{k,0}=\varphi_k,\quad 0\le k\le N\end{gathered}\tag{1.3}$$

are presented. The stability estimates for the solution of difference schemes (1.2) and (1.3) are established. In applications, the stability in maximum norm of difference shemes for multidimensional ultraparabolic equations with Dirichlet condition is established. Applying the difference schemes, the numerical methods are proposed for solving one-dimensional ultraparabolic equations.

Theorem 1.1. *For the solution of* (1.2), *we have the following stability inequality:*

$$\max_{1\le k\le N}\max_{1\le m\le N}\|u_{k,m}\|_E\le C\left(\max_{0\le m\le N}\|\psi_m\|_E+\max_{0\le k\le N}\|\varphi_k\|_E+\max_{1\le k\le N}\max_{1\le m\le N}\|f_{k,m}\|_E\right),\tag{1.4}$$

where C is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

Proof. Using (1.2), we get

$$\frac{u_{k,m}-u_{k-1,m-1}}{\tau}+Au_{k,m}=f_{k,m}.\tag{1.5}$$

From that it follows

$$u_{k,m}=Ru_{k-1,m-1}+\tau Rf_{k,m},\tag{1.6}$$

where $R=(I+\tau A)^{-1}$. By the mathematical induction, we will prove that

$$u_{k,m}=R^n u_{k-n,m-n}+\sum_{j=1}^{n}\tau R^{n-j+1}f_{k-n+j,m-n+j}\tag{1.7}$$

is true for all positive integers n. It is obvious that for $n = 1, 2$ formula (1.7) is true. Assume that for $n = r$

$$u_{k,m} = R^r u_{k-r,m-r} + \sum_{j=1}^{r} \tau R^{r-j+1} f_{k-r+j,m-r+j} \tag{1.8}$$

is true. In formula (1.6), replacing k and m with $k - r$ and $m - r$, respectively, we have

$$u_{k-r,m-r} = R u_{k-r-1,m-r-1} + \tau R f_{k-r,m-r}. \tag{1.9}$$

Then, using (1.8) and (1.9), we get

$$u_{k,m} = R^{r+1} u_{k-r-1,m-r-1} + \tau R^{r+1} f_{k-r,m-r} + \sum_{j=1}^{r} \tau R^{r-j+1} f_{k-r+j,m-r+j}. \tag{1.10}$$

From that it follows

$$u_{k,m} = R^{r+1} u_{k-r-1,m-r-1} + \sum_{j=1}^{r+1} \tau R^{r-j+2} f_{k-r-1+j,m-r-1+j} \tag{1.11}$$

is true for $n = r + 1$. So, formula (1.7) is proved. For $m > k$, replacing n with k in formula (1.7), we obtain that

$$u_{k,m} = R^k \varphi_{m-k} + \sum_{j=1}^{k} \tau R^{k-j+1} f_{j,m-k+j}. \tag{1.12}$$

Using estimate (see [13])

$$\left\| R^k \right\|_{E \to E} \leq M \tag{1.13}$$

and triangle inequality, we get

$$\begin{aligned} \|u_{k,m}\|_E &\leq \left\| R^k \right\|_{E \to E} \|\varphi_{m-k}\|_E + \sum_{j=1}^{k} \tau \left\| R^{k-j+1} \right\|_{E \to E} \|f_{j,m-k+j}\|_E \\ &\leq M \left[\max_{0 \leq k,m \leq N} \|\varphi_{m-k}\| + \max_{1 \leq j \leq N} \max_{1 \leq m \leq N} \|f_{j,m}\| \right] \end{aligned} \tag{1.14}$$

for any k and m. For $k > m$, replacing n with m in formula (1.7), we get

$$u_{k,m} = R^m \varphi_{k-m} + \sum_{j=1}^{m} \tau R^{m-j+1} f_{k-m+j,j}. \tag{1.15}$$

From estimate (1.13) and triangle inequality, it follows that

$$\begin{aligned}\|u_{k,m}\|_E &\le \|R^m\|_{E\to E}\|\varphi_{k-m}\|_E + \sum_{j=1}^{m}\tau\left\|R^{m-j+1}\right\|_{E\to E}\|f_{k-m+j,j}\|_E \\ &\le M\left[\max_{0\le k,m\le N}\|\varphi_{k-m}\| + \max_{1\le j\le N}\max_{1\le m\le N}\|f_{j,m}\|\right]\end{aligned} \tag{1.16}$$

for any k and m. Thus, Theorem 1.1 is proved. □

Theorem 1.2. *For the solution of* (1.3), *we have the following stability inequality:*

$$\begin{aligned}&\max_{1\le k\le N}\max_{1\le m\le N}\left\|\frac{u_{k,m}+u_{k-1,m}}{2}\right\|_E + \max_{1\le k\le N}\max_{1\le m\le N}\left\|\frac{u_{k,m}+u_{k,m-1}}{2}\right\|_E \\ &\quad\le C\left(\max_{0\le m\le N}\|\psi_m\|_E + \max_{0\le k\le N}\|\varphi_k\|_E + \max_{1\le k\le N}\max_{1\le m\le N}\|f_{k,m}\|_E\right),\end{aligned} \tag{1.17}$$

where C is independent of τ, ψ_m, φ_k, and $f_{k,m}$.

The proof of Theorem 1.2 is based on the following formulas:

$$\begin{aligned}u_{k,m} &= Bu_{k-1,m-1} + \tau BCf_{k,m}, \\ u_{k,m} &= B^k\psi_{m-k} + \sum_{j=1}^{k}\tau B^{k-j}Cf_{j,m-k+j}, \quad m > k, \\ u_{k,m} &= B^m\varphi_{k-m} + \sum_{j=1}^{m}\tau B^{m-j}Cf_{k-m+j,j}, \quad k > m\end{aligned} \tag{1.18}$$

for the solution of difference scheme (1.3) and the following estimate [14]:

$$\left\|B^kC\right\|_{E\to E} \le M, \tag{1.19}$$

where $B = (I-\tau A/2)(I+\tau A/2)^{-1}$ and C=$(I+\tau A/2)^{-1}$.

2. Application

Let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n$ $(0 < x_k < 1,\ 1 \le k \le n)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1]\times[0,1]\times\overline{\Omega}$, we consider the boundary-value problem for the multidimensional parabolic equation

$$\begin{gathered}\frac{\partial u(t,s,x)}{\partial t} + \frac{\partial u(t,s,x)}{\partial s} - \sum_{r=1}^{n}\alpha_r(x)\frac{\partial^2 u(t,s,x)}{\partial x_r^2} + \delta u(t,s,x) = f(t,s,x), \\ x = (x_1,\ldots,x_n)\in\Omega, \quad 0 < t,s < 1, \\ u(0,s,x) = \psi(s,x), \quad s\in[0,1], \qquad u(t,0,x) = \varphi(t,x), \quad t\in[0,1],\ x\in\overline{\Omega}, \\ u(t,s,x) = 0, \quad t,s\in[0,1],\ x\in S,\end{gathered} \tag{2.1}$$

where $\alpha_r(x) > a > 0$ $(x \in \Omega)$ and $f(t,s,x)$ $(t,s \in (0,1),\ x \in \Omega)$ are given smooth functions and $\delta > 0$ is a sufficiently large number.

We introduce the Banach spaces $C_{01}^{\beta}(\overline{\Omega})$ $(\beta = (\beta_1,\ldots,\beta_n),\ 0 < x_k < 1,\ k = 1,\ldots,n)$ of all continuous functions satisfying a Hölder condition with the indicator $\beta = (\beta_1,\ldots,\beta_n)$, $\beta_k \in (0,1)$, $1 \le k \le n$, and with weight $x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}$, $0 \le x_k < x_k + h_k \le 1$, $1 \le k \le n$, which is equipped with the norm

$$\begin{aligned}\|f\|_{C_{01}^{\beta}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} &+ \sup_{\substack{0\le x_k<x_k+h_k\le 1\\ 1\le k\le n}} \left|f(x_1,\ldots,x_n) - f(x_1+h_1,\ldots,x_n+h_n)\right| \\ &\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k}(1-x_k-h_k)^{\beta_k},\end{aligned} \tag{2.2}$$

where $C(\overline{\Omega})$ stands for the Banach space of all continuous functions defined on $\overline{\Omega}$, equipped with the norm

$$\|f\|_{C(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |f(x)|. \tag{2.3}$$

It is known that the differential expression

$$Av = -\sum_{r=1}^{n} \alpha_r(x) \frac{\partial^2 v(t,s,x)}{\partial x^2} + \delta v(t,s,x) \tag{2.4}$$

defines a positive operator A acting on $C_{01}^{\beta}(\overline{\Omega})$ with domain $D(A^x) \subset C_{01}^{2+\beta}(\overline{\Omega})$ and satisfying the condition $v = 0$ on S.

The discretization of problem (2.1) is carried out in two steps. In the first step, let us define the grid sets

$$\begin{aligned}\widetilde{\Omega}_h = \{&x = x_m = (h_1 m_1,\ldots,h_n m_n),\ m = (m_1,\ldots,m_n), \\ &0 \le m_r \le N_r,\ h_r N_r = L,\ r = 1,\ldots,n\}, \\ \Omega_h = \widetilde{\Omega}_h \cap \Omega, \qquad &S_h = \widetilde{\Omega}_h \cap S.\end{aligned} \tag{2.5}$$

We introduce the Banach spaces $C_h = C_h(\widetilde{\Omega}_h)$, $C_h^{\beta} = C_{01}^{\beta}(\widetilde{\Omega}_h)$ of grid functions $\varphi^h(x) = \{\varphi(h_1 m_1,\ldots,h_n m_n)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\begin{aligned}\left\|\varphi^h\right\|_{C(\overline{\Omega}_h)} &= \max_{x\in\overline{\Omega}_h} \left|\varphi^h(x)\right|, \\ \left\|\varphi^h\right\|_{C_{01}^{\beta}(\overline{\Omega}_h)} &= \left\|\varphi^h\right\|_{C(\overline{\Omega_h})} + \sup_{\substack{0\le x_k<x_k+h_k\le 1\\ 1\le k\le n}} \left|\varphi^h(x_1,\ldots,x_n) - \varphi^h(x_1+h_1,\ldots,x_n+h_n)\right| \\ &\quad\times \prod_{k=1}^{n} h_k^{-\beta_k} x_k^{\beta_k}(1-x_k-h_k)^{\beta_k}.\end{aligned} \tag{2.6}$$

To the differential operator A generated by problem (2.1), we assign the difference operator A_h^x by the formula

$$A_h^x u_x^h = -\sum_{r=1}^{n} a_r(x)\left(u_{-\overline{x_r}}^h\right)_{x_r, j_r} \tag{2.7}$$

acting in the space of grid functions $u^h(x)$, satisfying the condition $u^h(x) = 0$ for all $x \in S_h$.

With the help of A_h^x, we arrive at the initial boundary-value problem

$$\begin{gathered}\frac{\partial u^h(t,s,x)}{\partial t} + \frac{\partial u^h(t,s,x)}{\partial s} + A_h^x u^h(t,s,x) = f^h(t,s,x), \quad 0 < t,s < 1,\ x \in \Omega_h, \\ u^h(0,s,x) = \psi^h(s,x), \quad 0 \le s \le 1, \qquad u^h(t,0,x) = \varphi^h(t,x), \quad 0 \le t \le 1,\ x \in \tilde{\Omega}_h\end{gathered} \tag{2.8}$$

for an infinite system of ordinary differential equations.

In the second step, we replace problem (2.8) by difference scheme(1.2)

$$\begin{gathered}\frac{u_{k,m}^h - u_{k-1,m}^h}{\tau} + \frac{u_{k-1,m}^h - u_{k-1,m-1}^h}{\tau} + A_h^x u_{k,m}^h = f_{k,m}^h(x), \quad x \in \Omega_h, \\ f_{k,m}^h(x) = f^h(t_k, s_m, x), \quad t_k = k\tau,\ s_m = m\tau,\ 1 \le k,m \le N,\ x \in \tilde{\Omega}_h, \\ u_{0,m}^h = \psi_m^h, \quad 0 \le m \le N, \qquad u_{k,0}^h = \varphi_k^h, \quad 0 \le k \le N\end{gathered} \tag{2.9}$$

and by difference scheme(1.3)

$$\begin{gathered}\frac{u_{k,m}^h - u_{k-1,m}^h}{\tau} + \frac{u_{k-1,m}^h - u_{k-1,m-1}^h}{\tau} + \frac{1}{2} A_h^x (u_{k,m} + u_{k-1,m-1}) = f_{k,m}^h(x), \quad x \in \Omega_h, \\ f_{k,m}^h(x) = f^h\left(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x\right), \quad t_k = k\tau,\ s_m = m\tau,\ 1 \le k,\ m \le N,\ x \in \tilde{\Omega}_h, \\ u_{0,m}^h = \psi_m^h, \quad 0 \le m \le N, \qquad u_{k,0}^h = \varphi_k^h, \quad 0 \le k \le N.\end{gathered} \tag{2.10}$$

It is known that A_h^x is a positive operator in $C(\tilde{\Omega}_h)$ and $C_{01}^{\beta}(\overline{\Omega}_h)$. Let us give a number of corollaries of Theorems 1.1 and 1.2.

Theorem 2.1. *For the solution of difference scheme* (2.9), *we have the following stability inequality:*

$$\begin{aligned}&\max_{1\le k\le N} \max_{1\le m\le N} \left\| u_{k,m}^h \right\|_{C(\tilde{\Omega}_h)} \\ &\quad \le C_1 \left(\max_{0\le m\le N} \left\| \psi_m^h \right\|_{C(\tilde{\Omega}_h)} + \max_{0\le k\le N} \left\| \varphi_k^h \right\|_{C(\tilde{\Omega}_h)} + \max_{1\le k\le N} \max_{1\le m\le N} \left\| f_{k,m}^h \right\|_{C(\tilde{\Omega}_h)} \right),\end{aligned} \tag{2.11}$$

where C_1 is independent of τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

Theorem 2.2. *For the solution of difference scheme* (2.10)*, we have the following stability inequality:*

$$\begin{aligned}&\max_{1\le k\le N}\max_{1\le m\le N}\left\|\frac{u_{k,m}^h+u_{k-1,m-1}^h}{2}\right\|_{C(\tilde{\Omega}_h)}\\&\quad\le C_1\left(\max_{0\le m\le N}\left\|\varphi_m^h\right\|_{C(\tilde{\Omega}_h)}+\max_{0\le k\le N}\left\|\varphi_k^h\right\|_{C(\tilde{\Omega}_h)}+\max_{1\le k\le N}\max_{1\le m\le N}\left\|f_{k,m}^h\right\|_{C(\tilde{\Omega}_h)}\right),\end{aligned}\tag{2.12}$$

where C_1 does not depend on τ, ψ_m^h, φ_k^h, and $f_{k,m}^h$.

3. Numerical Analysis

In this section, the initial boundary value problem

$$\begin{gathered}\frac{\partial u(s,t,x)}{\partial t}+\frac{\partial u(s,t,x)}{\partial s}-\frac{\partial^2 u(s,t,x)}{\partial x^2}+2u(t,s,x)=f(t,s,x),\\ f(t,s,x)=e^{-(t+s)}\sin\pi x,\quad 0<s,t<1,\ 0<x<1,\\ u(0,t,x)=e^{-t}\sin\pi x,\quad 0<t<1,\ 0<x<1,\\ u(s,0,x)=e^{-s}\sin\pi x,\quad 0<s<1,\ 0<x<1,\\ u(s,t,0)=u(s,t,\pi)=0,\quad 0<s,t<1\end{gathered}\tag{3.1}$$

for one-dimensional ultraparabolic equations is considered.

The exact solution of problem (3.1) is

$$u(t,s,x)=e^{-(t+s)}\sin\pi x.\tag{3.2}$$

Using the first order of accuracy in t and s implicit difference scheme (2.9), we obtain the difference scheme first order of accuracy in t and s and second-order of accuracy in x

$$\begin{gathered}\frac{u_n^{k,m}-u_n^{k-1,m}}{\tau}-\frac{u_n^{k-1,m}-u_n^{k-1,m-1}}{\tau}-\frac{u_{n+1}^{k,m}-2u_n^{k,m}+u_{n-1}^{k,m}}{h^2}+2u_n^{k,m}=f_{k,m}^h,\\ f_{k,m}^h=\ f(t_k,s_m,x_n)=e^{-(t_k+s_m)}\sin x_n,\quad 1\le k,m\le N,\ 1\le n\le M-1,\\ u_n^{0,m}=e^{-s_m}\sin x_n,\quad 0\le m\le N,\ 0\le n\le M,\\ u_n^{k,0}=e^{-t_k}\sin x_n,\quad 0\le k\le N,\ 0\le n\le M,\\ u_0^{k,m}=u_M^{k,m}=0,\quad 0\le k,m\le N,\\ t_k=k\tau,\quad s_m=m\tau,\quad 1\le k,m\le N,\ N\tau=1,\\ x_n=nh,\quad 1\le n\le M,\ Mh=\pi\end{gathered}\tag{3.3}$$

for approximate solutions of initial boundary value problem (3.3). It can be written in the matrix form

$$\begin{gathered} Au_{n+1} + Bu_n + Cu_{n-1} = \varphi_n, \quad 1 \le n \le M-1, \\ u_0 = \vec{0}, \qquad u_M = \vec{0}. \end{gathered} \tag{3.4}$$

Here

$$A = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & b & 0 & \cdots & c & d & 0 & \cdots & 0 \\ 0 & 0 & b & 0 & \cdots & c & d & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & b & 0 & \cdots & c & d \end{bmatrix}_{(N+1)^2\times(N+1)^2}, \tag{3.5}$$

$$C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

where

$$a = \frac{-1}{h^2}, \qquad b = \frac{-1}{\tau}, \qquad c = \frac{-1}{\tau}, \qquad d = \frac{1}{\tau} + \frac{1}{\tau} + \frac{2}{h^2} + 2,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \vdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \vdots \\ \varphi_n^{N,1} \\ \vdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \vdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2\times 1}, \qquad \varphi_n^{k,m} = f(t_k, s_m, x_n) = e^{-(t_k+s_m)} \sin x_n,$$

$$u_n = \begin{bmatrix} u_n^{0,0} \\ u_n^{1,0} \\ \vdots \\ u_n^{N,0} \\ u_n^{0,1} \\ u_n^{1,1} \\ \vdots \\ u_n^{N,1} \\ \vdots \\ u_n^{0,N} \\ u_n^{1,N} \\ \vdots \\ u_n^{N,N} \end{bmatrix}_{(N+1)^2\times 1}. \tag{3.6}$$

This type system was used by Samarskii and Nikolaev [15] for difference equations. For the solution of matrix equation (3.4), we will use the modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1, \tag{3.7}$$

where $u_M = \vec{0}$, $\alpha_j (j = 1, \ldots, M-1)$ are $(N+1)^2 \times (N+1)^2$ square matrices, $\beta_j (j = 1, \ldots, M-1)$ are $(N+1)^2 \times 1$ coloumn matrices, α_1, β_1 are zero matrices, and

$$\begin{aligned} \alpha_{n+1} &= -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} &= (B + C\alpha_n)^{-1} (\varphi_n - C\beta_n), \quad n = 1, 2, 3, \ldots, (M-1). \end{aligned} \tag{3.8}$$

Using the second-order of accuracy in t and s implicit difference scheme (2.10), we obtain the difference scheme second-order of accuracy in t and s and second-order of accuracy in x

$$\begin{aligned} &\frac{u_n^{k,m} - u_n^{k-1,m}}{\tau} - \frac{u_n^{k-1,m} - u_n^{k-1,m-1}}{\tau} \\ &\quad - \frac{1}{2}\left[\frac{u_{n+1}^{k,m} - 2u_n^{k,m} + u_{n-1}^{k,m}}{h^2} + 2u_n^{k,m} + \frac{u_{n+1}^{k-1,m-1} - 2u_n^{k-1,m-1} + u_{n-1}^{k-1,m-1}}{h^2} + 2u_n^{k-1,m-1}\right] = f_{k,m}^h, \\ &f_{k,m}^h = f(t_k, s_m, x_n) = e^{-(t_k + s_m - \tau)} \sin x_n, \quad 1 \le k, m \le N,\ 1 \le n \le M-1, \\ &u_n^{0,m} = e^{-s_m} \sin x_n, \quad 0 \le m \le N,\ 0 \le n \le M, \\ &u_n^{k,0} = e^{-t_k} \sin x_n, \quad 0 \le k \le N,\ 0 \le n \le M, \\ &u_0^{k,m} = u_M^{k,m} = 0, \quad 0 \le k, m \le N, \\ &t_k = k\tau, \quad s_m = m\tau, \quad 1 \le k, m \le N, \quad N\tau = 1, \\ &x_n = nh, \quad 1 \le n \le M,\ Mh = \pi \end{aligned} \tag{3.9}$$

for approximate solutions of initial boundary value problem (3.9). The matrix form (3.4) can be written. Here,

$$A = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & \cdots & a & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & 0 & \cdots & a \end{bmatrix}_{(N+1)^2\times(N+1)^2(N+1)^2\times(N+1)^2},$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ b & 0 & 0 & \cdots & 1 & c & 0 & \cdots & 0 \\ 0 & b & 0 & 0 & \cdots & 1 & c & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b & 0 & 0 & \cdots & 1 & c \end{bmatrix}_{(N+1)^2\times(N+1)^2},$$

$$C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a & 0 & \cdots & a & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & a & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & a & 0 & \cdots & a \end{bmatrix}_{(N+1)^2\times(N+1)^2}, \tag{3.10}$$

where

$$a = -\frac{1}{2h^2}, \qquad b = -\frac{1}{\tau} + \frac{1}{h^2} + 1, \qquad c = \frac{1}{\tau} + \frac{1}{h^2} + 1,$$

$$\varphi_n = \begin{bmatrix} \varphi_n^{0,0} \\ \varphi_n^{1,0} \\ \vdots \\ \varphi_n^{N,0} \\ \varphi_n^{0,1} \\ \varphi_n^{1,1} \\ \vdots \\ \varphi_n^{N,1} \\ \vdots \\ \varphi_n^{0,N} \\ \varphi_n^{1,N} \\ \vdots \\ \varphi_n^{N,N} \end{bmatrix}_{(N+1)^2\times 1}, \qquad \varphi_n^{k,m} = f\left(t_k - \frac{\tau}{2}, s_m - \frac{\tau}{2}, x_n\right) = e^{-(t_k+s_m)} \sin x_n,$$

$$u_n = \begin{bmatrix} u_n^{0,0} \\ u_n^{1,0} \\ \vdots \\ u_n^{N,0} \\ u_n^{0,1} \\ u_n^{1,1} \\ \vdots \\ u_n^{N,1} \\ \vdots \\ u_n^{0,N} \\ u_n^{1,N} \\ \vdots \\ u_n^{N,N} \end{bmatrix}_{(N+1)^2\times 1}. \tag{3.11}$$

Table 1

Difference schemes	$N = M = 10$	$N = M = 20$	$N = M = 40$	$N = M = 80$
Difference scheme (3.3)	0.028100	0.014400	0.006811	0.003225
Difference scheme (3.9)	0.000511	0.000121	0.000028	0.000006

We seek a solution of the matrix equation by the same algorithm (3.7) and (3.8).

4. Error Analysis

The errors are computed by

$$E_N^{K,M} = \max_{1\le k,m\le N, 1\le n\le M-1} \left| u(t_k, s_m, x_n) - u_n^{k,m} \right| \tag{4.1}$$

of the numerical solutions, where $u(t_k, s_m, x_n)$ represents the exact solution and $u_n^{k,m}$ represents the numerical solution at (t_k, s_m, x_n), and the results are given in Table 1.

It may be noted from Table 1 that as N, M increase, the value of the errors associated with difference scheme (3.3) decreases by a factor of approximately $1/2$ and the errors associated with difference scheme (3.9) decrease by a factor of approximately $1/4$. This confirms that difference scheme (3.3) is first order and difference scheme (3.9) is second-order as stated in Section 1. Moreover, the results show that the second-order of accuracy difference scheme (3.9) are more accurate comparing with the first order of accuracy difference scheme (3.3).

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