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Research Article

Set-Valued Fixed-Point Theorems for Generalized Contractive Mappings on Fuzzy Metric Spaces

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The purpose of this paper is to introduce new types of asymptotically (g, φ)-contractions which generalize the Binayak S. Choudhury type contraction on fuzzy metric spaces and prove some fixed-point theorems for single- and multivalued mappings on fuzzy metric spaces. Hence, our results can be viewed as a generalization and improvement of many recent results.

1. Introduction and Preliminaries

The concept of fuzzy metric space was introduced in different ways by some authors (see, i.e., [1, 2]) and further,the fixed-point theory in this kind of spaces has been intensively studied (see [3–5]). Contraction mappings in probabilistic and fuzzy metric spaces have considered by many authors. Singh and Chauhan were the first to introduce contraction mapping principle in probabilistic metric space [6]. The result has been known as Seghal contraction. The structures of these spaces allow to extend the contraction mapping principle to these spaces in more than one inequivalent ways. One such concept is C-contraction which was originally introduced by O. Hadžic in [7] and subsequently studied and generalized in several works like [2, 8, 9]. In [7] O. Hadžic introduced the notion of a C-contraction in probabilistic metric space. In [10], Goleţ introduced the concept of g-contraction where g is a bijective function to generalize the Hicks-type contraction. Some other works may be noted in [4, 11–20].

In this paper we establish two coincidence point results for three mappings. For this purpose we consider fuzzy (g,φ)-contraction and fuzzy asymptotically (g,φ)-contraction, and prove several important fixed-point theorems for single- and multivalued mappings.

Definition 1.1 (see [18]). A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-norm if it is satisfies the following conditions:

(a) $*$ is associative and commutative;

(b) $*$ is continuous;

(c) $a * 1 = a$ for all $a \in [0,1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

In [8], Kramosil and Michálek gave the following definition of fuzzy metric space.

Definition 1.2. The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0,\infty)$, satisfying the following conditions. For all $x, y, z \in X$ and $t, s > 0$,

(M1) $M(x, y, 0) = 0$;

(M2) $M(x, y, t) = 1$ if and only if $x = y$;

(M3) $M(x, y, t) = M(y, x, t)$;

(M4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;

(M5) $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is left-continuous.

Example 1.3 (see [21]). Let (X, d) be a metric space. Define $a*b = a\cdot b$ (or $a*b = \min\{a, b\}$) and for all $x, y \in X$ and $\epsilon > 0$,

$$M(x, y, \epsilon) = \frac{\epsilon}{\epsilon + d(x, y)}. \tag{1.1}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric. On the other hand, note that there exists no metric on X satisfying (1.1).

Lemma 1.4. *Let $*$ be a continuous t-norm according to Definition 1.1. Then the condition*

$$a * a \geq a \quad \text{for every } a \in [0,1] \tag{1.2}$$

holds if and only if

$$a * b = \min\{a, b\} \quad \text{for every } a, b \in [0,1]. \tag{1.3}$$

Proof. Suppose (1.2) holds. Let $a, b \in [0,1]$ such that $a \leq b$. Then

$$\min\{a, b\} = a = a * 1 \geq a * b \geq a * a \geq a = \min\{a, b\}. \tag{1.4}$$

Therefore, for all $a, b \in [0,1]$, we have $a * b = \min\{a, b\}$. Suppose (1.3) holds. Then clearly

$$a * a = \min\{a, a\} = a \tag{1.5}$$

and so (1.2) holds. □

Definition 1.5 (see [8]). Let $(X, M, *)$ be a fuzzy metric space: a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted $x_n \to x$) if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \text{for each } t > 0. \tag{1.6}$$

Definition 1.6 (see [21]). Let $(X, M, *)$ be a fuzzy metric space: a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for any $\epsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \epsilon \tag{1.7}$$

for all $n, m \geq n_0$.

Definition 1.7 (see [8]). Let $(X, M, *)$ be a fuzzy metric space: a sequence $\{x_n\}$ in X is called a Cauchy sequence if

$$\lim_{n\to\infty} M\left(x_{n+p}, x_n, t\right) = 1 \quad \text{for each } t > 0,\ p > 0. \tag{1.8}$$

Definition 1.8. Let $(X, M_1, *)$ and $(Y, M_2, *)$ be two fuzzy metric spaces. A mapping $f : X \to Y$ is said to be (uniformly) continuous if for each $\epsilon > 0$, $s > 0$, there exist $\delta > 0$, $t > 0$ such that

$$M_1(x, y, s) > 1 - \delta \Longrightarrow M_2(f(x), f(y), t) > 1 - \epsilon, \tag{1.9}$$

for each $x, y \in X$.

Definition 1.9. Let $(X, M, *)$ be a fuzzy metric space and let $T : X \to X$ be a self-mapping on X. The mapping T is called asymptotically regular at $x \in X$ if

$$\lim_{n\to\infty} M\left(T^n x, T^{n+1} x, t\right) = 1. \tag{1.10}$$

2. Main Results

Let Φ be the class of all mappings $\varphi : [0,\infty) \to [0,\infty)$ with the following properties:

(i) φ is strictly increasing;

(ii) φ is right-continuous;

(iii) $\varphi(t) < t$ for all $t > 0$.

Lemma 2.1. *For all* $t > 0, \lim_{n\to\infty}\varphi^n(t) = 0$, *where* φ^n *is the* n*-iteration of* φ.

Proof. Suppose that $\lim_{n\to\infty}\varphi^n(t_0) = l < 0$ for some $t_0 \in (0,\infty)$. By the monotonicity and right continuity of φ, we have

$$l = \lim_{n\to\infty}\varphi^{n+1}(t_0) = \varphi\left(\lim_{n\to\infty}\varphi^n(t_0)\right) = \varphi(l) < l, \tag{2.1}$$

which is a contradiction. □

Definition 2.2. Let $(X, M, *)$ be a fuzzy metric space and $\varphi \in \Phi$. We say that the mapping $f : X \to X$ is a fuzzy (g,φ)-contraction if there exists a bijective function $g : X \to X$ such that for every $x, y \in X$, $t > 0$ the following implication holds:

$$M(g(x), g(y), t) > 1 - t \Longrightarrow M(f(x), f(y), \varphi(t)) > 1 - \varphi(t). \tag{2.2}$$

Note that, if $\varphi(t) = kt$ for $k \in (0,1), t > 0$, then (2.2) is a g-contraction in the sense of Goleţ [10]. On the other hand, if g is an identity function, then (2.2) is called g-H-contraction due to Miheţ [9]. Thus, our definition (g,φ)-contraction is a generalization of the Goleţ and Miheţ's type contraction principle in the fuzzy settings.

Definition 2.3. Let $(X, M, *)$ be a fuzzy metric space and let $\varphi \in \Phi$. Let f, g, and T are three mappings defined on $(X, M, *)$ with values into itself and let one take T as asymptotically regular at $x \in X$. Then f is called fuzzy asymptotically (g,φ)-contraction with respect to T if

$$M\left(gT^n x, gT^{n+1}x, t\right) > 1 - t \Longrightarrow M\left(fT^n x, fT^{n+1}x, \varphi(t)\right) > 1 - \varphi(t). \tag{2.3}$$

Note that, if $\varphi(t) = kt$ for $k \in (0,1), t > 0$, then (2.3) is asymptotically g-contraction with respect to T in the sense of Binayak S. Choudhury.

Lemma 2.4. *Let* f *satisfy the condition given in Definition 2.2. Then* $g^{-1}f$ *is a continuous mapping on* $M(gx, gy, t)$ *with values into itself.*

Proof. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ in X under the fuzzy metric $(X, M^g, *)$; this implies that $M(g(x_n), g(x), t) = M^g(x_n, x, t) \to 1$ as $n \to \infty$ for all $t > 0$. By definition, it follows that

$$M(f(x_n), f(x), \varphi(t)) \longrightarrow 1, \tag{2.4}$$

as $n \to \infty$ for all $t > 0$, which implies that

$$M\left(gg^{-1}f(x_n), gg^{-1}f(x), \varphi(t)\right) \longrightarrow 1, \tag{2.5}$$

as $n \to \infty$ for all $t > 0$, which implies that,

$$M^g\left(g^{-1}f(x_n), g^{-1}f(x), \varphi(t)\right) \longrightarrow 1, \tag{2.6}$$

as $n \to \infty$ for all $t > 0$. This shows that $g^{-1}f$ is continuous mapping on $(X, M^g, *)$ with values into itself. □

Theorem 2.5. *Let f, g, T be three mappings defined on complete fuzzy metric space $(X, M, *)$ with values into itself where g is bijective, T is asymptotically regular at $x \in X$, and f is fuzzy asymptotically (g, φ)-contraction with respect to T with $fT = Tf$, $gT = Tg$ at $x \in X$, and $g^{-1}fT$ is continuous in X. Then $fTx = gx$.*

Proof. Since f is fuzzy asymptotically (g, φ)-contraction with respect to T, we get for $t > 0$,

$$M\left(gT^n x, gT^{n+1}x, t\right) > 1 - t \Longrightarrow M\left(fT^n x, fT^{n+1}x, \varphi(t)\right) > 1 - \varphi(t), \tag{2.7}$$

where $t > 0$, this implies that

$$M^g\left(g^{-1}fTT^{n-1}x, g^{-1}fTT^n x, \varphi(t)\right) > 1 - \varphi(t) \tag{2.8}$$

which implies that

$$M^g\left(\alpha T^{n-1}x, \alpha T^n x, \varphi(t)\right) > 1 - \varphi(t), \tag{2.9}$$

where $\alpha = g^{-1}fT$; this implies that

$$M\left(f\alpha T^{n-1}x, f\alpha T^n x, \varphi^2(t)\right) > 1 - \varphi^2(t), \tag{2.10}$$

which implies that

$$M\left(fg^{-1}fTTT^{n-2}x, fg^{-1}fTTT^{n-1}x, \varphi^2(t)\right) > 1 - \varphi^2(t), \tag{2.11}$$

which implies that

$$M\left(gg^{-1}fTg^{-1}fTT^{n-2}x, gg^{-1}fTg^{-1}fTT^{n-1}x, \varphi^2(t)\right) > 1 - \varphi^2(t), \tag{2.12}$$

which implies that

$$M^g\left(\alpha^2 T^{n-2}x, \alpha^2 T^{n-1}x, \varphi^2(t)\right) > 1 - \varphi^2(t). \tag{2.13}$$

Continuing this process, we get

$$M^g\left(\alpha^n x, \alpha^n Tx, \varphi^n(t)\right) > 1 - \varphi^n(t). \tag{2.14}$$

For every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $\varphi^n(t) \leq \min(\epsilon, \lambda)$ whenever $n \geq n_0$. Thus, we have

$$M^g(\alpha^n x, \alpha^n Tx, \epsilon) \geq M^g(\alpha^n x, \alpha^n Tx, \varphi^n(t)) > 1 - \varphi^n(t) > 1 - \lambda, \tag{2.15}$$

which implies that $M^g(\alpha^n x, \alpha^n Tx, \epsilon) > 1 - \lambda$. Let $\{x_n\}$ be the sequence defined as $x_{n+1} = \alpha x_n$. Now, taking $Tx = x_{m-n}$ and $x = x_0$, then we get from the above inequality

$$M^g(x_n, x_m, \epsilon) > 1 - \lambda, \tag{2.16}$$

for every $n, m \geq n_0$. This implies that $\{x_n\}$ is a fuzzy Cauchy sequence in $(X, M, *)$. Since $(X, M, *)$ is complete fuzzy metric space, then also $(X, M^g, *)$ is complete fuzzy metric space, there exists $z \in X$ such that $x_n \to z$ $(n \to \infty)$ under M^g. Again, α is continuous on $(X, M^g, *)$, it follows that $\alpha x = x$, which implies that $fTx = gx$. □

References

[1] Z. K. Deng, "Fuzzy pseudo metric spaces," *Journal of Mathematics and Applications*, vol. 86, pp. 74–95, 1982.

[2] O. Kaleva and S. Seikkala, "On fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 12, no. 3, pp. 215–229, 1984.

[3] C. Di Bari and C. Vetro, "Fixed points, attractors and weak fuzzy contractive mappings in a fuzzy metric space," *Journal of Fuzzy Mathematics*, vol. 13, no. 4, pp. 973–982, 2005.

[4] M. Grabiec, "Fixed points in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 27, no. 3, pp. 385–389, 1988.

[5] V. Gregori and A. Sapena, "On fixed-point theorems in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 245–252, 2002.

[6] B. Singh and M. S. Chauhan, "Common fixed points of compatible maps in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 471–475, 2000.

[7] O. Hadžic, "Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 29, no. 1, pp. 115–125, 1989.

[8] I. Kramosil and J. Michálek, "Fuzzy metrics and statistical metric spaces," *Kybernetika*, vol. 11, no. 5, pp. 336–344, 1975.

[9] D. Miheţ, "A Banach contraction theorem in fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 144, no. 3, pp. 431–439, 2004.

[10] I. Goleţ, "On contractions in probabilistic metric spaces," *Radovi Matematički*, vol. 13, no. 1, pp. 87–92, 2004.

[11] M. A. Erceg, "Metric spaces in fuzzy set theory," *Journal of Mathematical Analysis and Applications*, vol. 69, no. 1, pp. 205–230, 1979.

[12] A. George and P. Veeramani, "On some results of analysis for fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 90, no. 3, pp. 365–368, 1997.

[13] V. Gregori and S. Romaguera, "Some properties of fuzzy metric spaces," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 485–489, 2000.

[14] I. Goleţ, "On generalized fuzzy normed spaces and coincidence point theorems," *Fuzzy Sets and Systems*, vol. 161, no. 8, pp. 1138–1144, 2010.

[15] T. L. Hicks, "Fixed point theory in probabilistic metric spaces," *Review of Research, Univerzitet u Novom Sadu*, vol. 13, pp. 63–72, 1983.

[16] E. Pap, O. Hadžić, and R. Mesiar, "A fixed point theorem in probabilistic metric spaces and an application," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 2, pp. 433–449, 1996.

[17] V. Radu, "On some contraction-type mappings in Menger spaces," *Analele Universitatii Din Timisoara*, vol. 23, no. 1-2, pp. 61–65, 1985.

[18] B. Schweizer and A. Sklar, "Probabilistic metric spaces," in *North Holland Series in Probability and Applied Mathematics*, North Holland, New York, NY, USA, 1983.

[19] V. M. Seghal and A. T. Bharucha-Ried, “Fixed points of contraction mapping on PM-spaces,” *Mathematical Systems Theory*, vol. 6, pp. 97–110, 1972.

[20] R. Vasuki, “A common fixed point theorem in a fuzzy metric space,” *Fuzzy Sets and Systems*, vol. 97, no. 3, pp. 395–397, 1998.

[21] A. George and P. Veeramani, “On some results in fuzzy metric spaces,” *Fuzzy Sets and Systems*, vol. 64, no. 3, pp. 395–399, 1994.