Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 830585, 11 pages
doi:10.1155/2012/830585

Research Article

Inequalities between Arithmetic-Geometric, Gini, and Toader Means

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Received 24 August 2011; Accepted 20 October 2011

Academic Editor: Muhammad Aslam Noor

We find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a,b) < M(a,b) < S_{q_1}(a,b)$ and $S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals. Here $M(a,b)$, $T(a,b)$, and $S_p(a,b)$ are the arithmetic-geometric, Toader, and pth Gini means of two positive numbers a and b, respectively.

1. Introduction

For $p \in \mathbb{R}$ the pth Gini mean $S_p(a,b)$ and power mean $M_p(a,b)$ of two positive real numbers a and b are defined by

$$S_p(a,b) = \begin{cases} \left(\dfrac{a^{p-1}+b^{p-1}}{a+b}\right)^{1/(p-2)}, & p \neq 2, \\ \left(a^a b^b\right)^{1/(a+b)}, & p = 2, \end{cases} \tag{1.1}$$

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^{p}+b^{p}}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.2}$$

respectively.

It is well known that $S_p(a,b)$ and $M_p(a,b)$ are continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$. Many means are special case of these means, for example,

$$\begin{aligned} S_1(a,b) = M_1(a,b) &= \frac{a+b}{2} = A(a,b) \text{ is the arithmetic mean,} \\ S_0(a,b) = M_0(a,b) &= \sqrt{ab} = G(a,b) \text{ is the geometric mean,} \\ M_{-1}(a,b) &= \frac{2ab}{a+b} = H(a,b) \text{ is the harmonic mean.} \end{aligned} \tag{1.3}$$

Recently, the Gini and power means have been the subject of intensive research. In particular, many remarkable inequalities for these means can be found in the literature [1–7].

In [8], Toader introduced the Toader mean $T(a,b)$ of two positive numbers a and b as follows:

$$\begin{aligned} T(a,b) &= \frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \\ &= \begin{cases} \dfrac{2a\mathcal{E}\left(\sqrt{1-(b/a)^2}\right)}{\pi}, & a > b, \\ \dfrac{2b\mathcal{E}\left(\sqrt{1-(a/b)^2}\right)}{\pi}, & a < b, \\ a, & a = b, \end{cases} \end{aligned} \tag{1.4}$$

where $\mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{1/2}dt$, $r \in [0,1]$, is the complete elliptic integrals of the second kind.

The classical arithmetic-geometric mean $M(a,b)$ of two positive number a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} a_0 = a, &\qquad b_0 = b, \\ a_{n+1} = \frac{a_n + b_n}{2} = A(a_n, b_n), &\qquad b_{n+1} = \sqrt{a_n b_n} = G(a_n, b_n). \end{aligned} \tag{1.5}$$

The Gauss identity [9] shows that

$$M(1,r)\mathcal{K}\left(\sqrt{1-r^2}\right) = \frac{\pi}{2} \tag{1.6}$$

for $r \in (0,1)$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2 t)^{-1/2}dt$, $r \in [0,1)$, is the complete elliptic integrals of the first kind.

Vuorinen [10] conjectured that

$$M_{3/2}(a,b) < T(a,b) \tag{1.7}$$

for all $a, b > 0$ with $a \neq b$. This conjecture was proved by Qiu and Shen in [11] and Barnard et al. in [12], respectively.

In [13], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b) \tag{1.8}$$

for all $a, b > 0$ with $a \neq b$.

In [14–17], the authors proved that

$$M_0(a,b) = G(a,b) < M(a,b) < M_{1/2}(a,b), \tag{1.9}$$

$$L(a,b) < M(a,b) < \frac{\pi}{2}L(a,b) \tag{1.10}$$

for all $a, b > 0$ with $a \neq b$, where

$$L(a,b) = \begin{cases} \dfrac{a-b}{\log a - \log b}, & a \neq b, \\ a, & a = b, \end{cases} \tag{1.11}$$

denotes the classical logarithmic mean of two positive numbers a and b.

Very recently, Chu and Wang [18] and Guo and Qi [19] proved that

$$L_0(a,b) < T(a,b) < L_{1/4}(a,b) \tag{1.12}$$

for all $a, b > 0$ with $a \neq b$, and $L_0(a,b)$ and $L_{1/4}(a,b)$ are the best possible lower and upper Lehmer mean bounds for the Toader mean $T(a,b)$, respectively. Here, the pth Lehmer mean $L_p(a,b)$ of two positive numbers a and b is defined by $L_p(a,b) = (a^{p+1} + b^{p+1})/(a^p + b^p)$.

The main purpose of this paper is to find the greatest values p_1, p_2 and least values q_1, q_2 such that the double inequalities $S_{p_1}(a,b) < M(a,b) < S_{q_1}(a,b)$ and $S_{p_2}(a,b) < T(a,b) < S_{q_2}(a,b)$ hold for all $a, b > 0$ with $a \neq b$ and present some new bounds for the complete elliptic integrals.

2. Preliminary Knowledge

Throughout this paper, we denote $r' = \sqrt{1-r^2}$ for $r \in [0,1]$.

For $0 < r < 1$, the following derivative formulas were presented in [9, Appendix E, pages 474–475]:

$$\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, \qquad \frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r},$$
$$\frac{d\left[\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]}{dr} = r\mathcal{K}(r), \qquad \frac{d[\mathcal{K}(r) - \mathcal{E}(r)]}{dr} = \frac{r\mathcal{E}(r)}{r'^2}. \tag{2.1}$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), \tag{2.2}$$

$$\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}(r) - r'^2\mathcal{K}(r)}{1+r}. \tag{2.3}$$

Lemma 2.1 can be found in [9, Theorem 3.21(7), (8), and (10), and Exercise 3.43(13) and (46)].

Lemma 2.1. *(1) $r'^c\mathcal{K}(r)$ is strictly decreasing from $[0,1)$ onto $(0,\pi/2]$ for $c \in [1/2,\infty)$;*

(2) $r'^c\mathcal{E}(r)$ is strictly increasing on $(0,1)$ if and only if $c \leq -1/2$ and strictly decreasing if and only if $c > 0$;

(3) $\mathcal{K}(r)/\log(4/r')$ is strictly decreasing from $(0,1)$ onto $(1,\pi/\log 16)$;

(4) $2\mathcal{E}(r) - r'^2\mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$;

(5) $[\mathcal{E}(r) - r'^2\mathcal{K}(r)]/[r^2\mathcal{K}(r)]$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$.

3. Main Results

Theorem 3.1. *Inequality $S_{1/2}(a,b) < M(a,b) < S_1(a,b)$ holds for all $a,b > 0$ with $a \neq b$, and $S_{1/2}(a,b)$ and $S_1(a,b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a,b)$.*

Proof. From (1.1) and (1.5) we clearly see that both $S_p(a,b)$ and $M(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a = 1 > b$. Let $t = b$ and $r = (1-t)/(1+t)$. Then from (1.1) and (1.6) together with (2.2) we clearly see that

$$\begin{aligned} M(a,b) - S_{1/2}(a,b) &= \frac{\pi}{2\mathcal{K}\left(\sqrt{1-t^2}\right)} - \left[\frac{(1+t)\sqrt{t}}{1+\sqrt{t}}\right]^{2/3} \\ &= \frac{\pi}{2(1+r)\mathcal{K}(r)} - \left[\frac{2\sqrt{1-r}}{(1+r)\left(\sqrt{1+r}+\sqrt{1-r}\right)}\right]^{2/3} \\ &= \frac{1}{1+r}\left[\frac{\pi}{2\mathcal{K}(r)} - \left(\frac{2r'}{\sqrt{1+r}+\sqrt{1-r}}\right)^{2/3}\right]. \end{aligned} \tag{3.1}$$

Let

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)}\right]^3 - \left(\frac{2r'}{\sqrt{1+r}+\sqrt{1-r}}\right)^2. \tag{3.2}$$

Then $F(r)$ can be rewritten as

$$F(r) = \left[\frac{\pi}{2\mathcal{K}(r)}\right]^3 - \frac{2r'^2}{1+r'} = \frac{2r'^2}{1+r'}F_1(r), \tag{3.3}$$

where

$$F_1(r) = \left(\frac{\pi}{2}\right)^3 \frac{1+r'}{2r'^2\mathcal{K}(r)^3} - 1. \tag{3.4}$$

It is well known that the function $r \to \sqrt{r} + 1/\sqrt{r}$ is positive and strictly decreasing in $(0,1)$. Then (3.4) and Lemma 2.1(1) lead to the conclusion that $F_1(r)$ is strictly increasing in $(0,1)$, so that $F_1(r) > F_1(0) = 0$ for $r \in (0,1)$.

Therefore, $M(a,b) > S_{1/2}(a,b)$ follows from (3.1)–(3.3).

On the other hand, $M(a,b) < S_1(a,b) = A(a,b)$ follows directly from (1.9).

Next, we prove that $S_{1/2}(a,b)$ and $S_1(a,b)$ are the best possible lower and upper Gini mean bounds for the arithmetic-geometric mean $M(a,b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1), (1.6), and Lemma 2.1(3) we have

$$\begin{aligned}
[M(1,1-x)]^{3-2\varepsilon} - [S_{1/2+\varepsilon}(1,1-x)]^{3-2\varepsilon} &= \left[\frac{\pi}{2\int_0^{\pi/2}\left[1-(2x-x^2)\sin^2 t\right]^{-1/2} dt}\right]^{3-2\varepsilon} \\
&\quad - \left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1+(1-x)^{1/2-\varepsilon}}\right]^2,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{M(1,x)}{S_{1-\varepsilon}(1,x)} &= \lim_{x\to 0}\left[\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\mathcal{K}\left(\sqrt{1-x^2}\right)}\left(\frac{1+x^{\varepsilon}}{1+x}\right)^{1/(1+\varepsilon)}\right] \\
&= \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\mathcal{K}\left(\sqrt{1-x^2}\right)} = \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\log(4/x)}\frac{\log(4/x)}{\mathcal{K}\left(\sqrt{1-x^2}\right)} \\
&= \lim_{x\to 0}\frac{2}{\pi x^{\varepsilon/(1+\varepsilon)}\log(4/x)} = +\infty.
\end{aligned} \tag{3.6}$$

Letting $x \to 0$ and making use of the Taylor expansion, one has

$$\begin{aligned}
&\left[\frac{\pi}{2\int_0^{\pi/2}\left[1-(2x-x^2)\sin^2 t\right]^{-1/2}dt}\right]^{3-2\varepsilon}-\left[\frac{(2-x)(1-x)^{1/2-\varepsilon}}{1+(1-x)^{1/2-\varepsilon}}\right]^2\\
&=1+\left(-\frac{3}{2}+\varepsilon\right)x+\frac{(2\varepsilon-3)(4\varepsilon-3)}{16}x^2+o\left(x^2\right)\\
&\quad-\left[1+\left(-\frac{3}{2}+\varepsilon\right)x+\frac{(2\varepsilon-3)^2}{16}x^2+o\left(x^2\right)\right]\\
&=-\frac{\varepsilon(3-2\varepsilon)}{8}x^2+o\left(x^2\right).
\end{aligned} \tag{3.7}$$

Equations (3.5)–(3.7) imply that for any $1<\varepsilon<1/2$ there exist $\delta_1=\delta_1(\varepsilon)\in(0,1)$ and $\delta_2=\delta_2(\varepsilon)\in(0,1)$, such that $M(1,1-x)<S_{1/2+\varepsilon}(1,1-x)$ for $x\in(0,\delta_1)$ and $M(1,x)>S_{1-\varepsilon}(1,x)$ for $x\in(0,\delta_2)$. □

Theorem 3.2. *Inequality $S_1(a,b)<T(a,b)<S_{3/2}(a,b)$ holds for all $a,b>0$ with $a\neq b$, and $S_1(a,b)$ and $S_{3/2}(a,b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a,b)$.*

Proof. From (1.1) and (1.4) we clearly see that both $S_p(a,b)$ and $T(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $a=1>b$. Let $t=b$ and $r=(1-t)/(1+t)$. Then from (1.1), (1.4), and (2.3) we have

$$\begin{aligned}
\frac{T(a,b)}{S_{3/2}(a,b)}&=\frac{2}{\pi}\mathcal{E}\left(\sqrt{1-t^2}\right)\cdot\left(\frac{1+\sqrt{t}}{1+t}\right)^2\\
&=\frac{2}{\pi}\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)\cdot(1+r)\cdot\left(\frac{\sqrt{1+r}+\sqrt{1-r}}{2}\right)^2\\
&=\frac{2}{\pi}\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right]\cdot\left(\frac{\sqrt{1+r}+\sqrt{1-r}}{2}\right)^2\\
&=\frac{1}{\pi}\left(1+r'\right)\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right].
\end{aligned} \tag{3.8}$$

Let

$$G(r)=\frac{1}{\pi}\left(1+r'\right)\left[2\mathcal{E}(r)-r'^2\mathcal{K}(r)\right]. \tag{3.9}$$

Then simple computations lead to

$$G(0) = 1, \tag{3.10}$$

$$\begin{aligned} G'(r) &= \frac{1}{\pi}\left[\left(-\frac{r}{r'}\right)\left(2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right) + (1+r')\left(\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r}\right)\right] \\ &= \frac{r'(1+r')\left[\mathcal{E}(r) - r'^2\mathcal{K}(r)\right] - r^2\left[2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]}{\pi r r'} \\ &= \frac{r}{\pi r'} G_1(r), \end{aligned} \tag{3.11}$$

where

$$G_1(r) = (1+r')r'\mathcal{K}(r)\left[\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{r^2\mathcal{K}(r)}\right] - \left[2\mathcal{E}(r) - r'^2\mathcal{K}(r)\right]. \tag{3.12}$$

It follows from (3.12) and Lemma 2.1(1), (4), and (5) that $G_1(r)$ is strictly decreasing from $(0,1)$ onto $(-2,0)$. Then (3.11) leads to the conclusion that $G'(r) < 0$ for $r \in (0,1)$. Hence $G(r)$ is strictly decreasing in $(0,1)$.

Therefore, $T(a,b) < S_{3/2}(a,b)$ follows from (3.8)–(3.10) together with the monotonicity of $G(r)$.

On the other hand, $T(a,b) > S_1(a,b) = A(a,b)$ follows directly from (1.7).

Next, we prove that $S_1(a,b)$ and $S_{3/2}(a,b)$ are the best possible lower and upper Gini mean bounds for the Toader mean $T(a,b)$.

For any $0 < \varepsilon < 1/2$ and $0 < x < 1$, from (1.1) and (1.4) one has

$$\begin{aligned} [T(1,1-x)]^{1+2\varepsilon} - [S_{3/2-\varepsilon}(1,1-x)]^{1+2\varepsilon} &= \left[\frac{2}{\pi}\int_0^{\pi/2}\left[1-\left(2x-x^2\right)\sin^2 t\right]^{1/2} dt\right]^{1+2\varepsilon} \\ &\quad - \left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}}\right]^2, \end{aligned} \tag{3.13}$$

$$\lim_{x\to 0}\frac{T(1,x)}{S_{1+\varepsilon}(1,x)} = \lim_{x\to 0}\left[\frac{2}{\pi}\mathcal{E}\left(\sqrt{1-x^2}\right)\left(\frac{1+x^{\varepsilon}}{1+x}\right)^{1/(1-\varepsilon)}\right] = \frac{2}{\pi} < 1. \tag{3.14}$$

Letting $x \to 0$ and making use of the Taylor expansion, we get

$$\begin{aligned}
&\left[\frac{2}{\pi}\int_0^{\pi/2}\left[1-\left(2x-x^2\right)\sin^2 t\right]^{1/2}dt\right]^{1+2\varepsilon} - \left[\frac{2-x}{1+(1-x)^{1/2-\varepsilon}}\right]^2 \\
&= 1-\left(\frac{1}{2}+\varepsilon\right)x+\frac{(2\varepsilon+1)(4\varepsilon+1)}{16}x^2+o\left(x^2\right) \\
&\quad -\left[1-\left(\frac{1}{2}+\varepsilon\right)x+\frac{(2\varepsilon+1)^2}{16}x^2+o\left(x^2\right)\right] \\
&= \frac{\varepsilon(2\varepsilon+1)}{8}x^2+o\left(x^2\right).
\end{aligned} \tag{3.15}$$

Equations (3.13)–(3.15) imply that for any $0<\varepsilon<1/2$ there exist $\delta_3=\delta_3(\varepsilon)\in(0,1)$ and $\delta_4=\delta_4(\varepsilon)\in(0,1)$, such that $T(1,1-x) > S_{3/2-\varepsilon}(1,1-x)$ for $x\in(0,\delta_3)$ and $T(1,x) < S_{1+\varepsilon}(1,x)$ for $x\in(0,\delta_4)$. □

4. Remarks and Corollaries

Remark 4.1. From (3.9) and Lemma 2.1(4) we clearly see that $G(1^-)=2/\pi$. Then (3.8) and (3.9) together with the monotonicity of $G(r)$ lead to the conclusion that

$$\frac{2}{\pi}S_{3/2}(a,b) < T(a,b) \tag{4.1}$$

for all $a,b>0$ with $a\neq b$.

Remark 4.2. We find that the lower bound $L(a,b)$ in (1.10) and the best possible lower Gini mean bound $S_{1/2}(a,b)$ in Theorem 3.1 are not comparable. In fact, from (1.1) and (1.11) we have

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{S_{1/2}(1,x)}{L(1,x)} = \lim_{x\to+\infty}\left[\frac{1+x^{-1}}{1+x^{-1/2}}\right]^{2/3}\frac{x^{2/3}\log x}{x-1} = \lim_{x\to+\infty}\frac{\log x}{x^{1/3}-x^{-2/3}} = 0, \\
&S_{1/2}(1,1+x)-L(1,1+x) = 1+\frac{1}{2}x-\frac{1}{16}x^2+o\left(x^2\right)-\left[1+\frac{1}{2}x-\frac{1}{12}x^2+o\left(x^2\right)\right] \\
&\qquad = \frac{1}{48}x^2+o\left(x^2\right) \quad (x\longrightarrow 0).
\end{aligned} \tag{4.2}$$

Table 1: Comparison of $\mathcal{K}(r)$ with $H(r)$ for some $r \in (0,1)$.

r	$\mathcal{K}(r)$	$H(r)$
0.1	1.574745561517···	1.574745561518···
0.2	1.586867847···	1.586867848···
0.3	1.608048620···	1.608048634···
0.4	1.639999866···	1.640000021···
0.5	1.685750355···	1.685751528···
0.6	1.750753803···	1.750760840···
0.7	1.845693998···	1.845732233···
0.8	1.995302778···	1.995519211···

Remark 4.3. The following two equations show that the best possible upper power mean bound $M_{\log 2/\log(\pi/2)}(a,b)$ in (1.8) and the best possible upper Gini mean bound $S_{3/2}(a,b)$ in Theorem 3.2 are not comparable:

$$\lim_{x\to+\infty}\frac{S_{3/2}(1,x)}{M_{\log 2/\log(\pi/2)}(1,x)} = 2^{\log(\pi/2)/\log 2} = \frac{\pi}{2},$$

$$\begin{aligned} M_{\log 2/\log(\pi/2)}(1,1+x) - S_{3/2}(1,1+x) &= 1 + \frac{1}{2}x + \frac{1}{8}\left[\frac{\log 2}{\log(\pi/2)} - 1\right]x^2 \\ &\quad + o\left(x^2\right) - \left[1 + \frac{1}{2}x + \frac{1}{16}x^2 + o\left(x^2\right)\right] \\ &= \frac{1}{16}\left[\frac{2\log 2}{\log(\pi/2)} - 3\right]x^2 + o\left(x^2\right) \\ &= 0.00436\cdots \times x^2 + o\left(x^2\right) \quad (x \longrightarrow 0). \end{aligned} \tag{4.3}$$

From Theorem 3.1 we get an upper bound for the complete elliptic integrals of the first kind $\mathcal{K}(r)$ as follows.

Corollary 4.4. *Inequality*

$$\mathcal{K}(r) < \frac{\pi}{2}\left[\frac{1+\left(1-r^2\right)^{1/4}}{\left(1+\sqrt{1-r^2}\right)\left(1-r^2\right)^{1/4}}\right]^{2/3} \tag{4.4}$$

holds for all $r \in (0,1)$.

Remark 4.5. Computational and numerical experiments show that the upper bound in (4.4) for $\mathcal{K}(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $H(r) = \pi[1+(1-r^2)^{1/4}]^{2/3}/\{2[(1+\sqrt{1-r^2})(1-r^2)^{1/4}]^{2/3}\}$, then we have Table 1 via elementary computation.

Table 2: Comparison of $\mathcal{E}(r)$ with $J(r)$ for some $r \in (0,1)$.

r	$\mathcal{E}(r)$	$J(r)$
0.1	$1.566861942021\cdots$	$1.566861942028\cdots$
0.2	$1.554968546\cdots$	$1.554968548\cdots$
0.3	$1.534833465\cdots$	$1.534833516\cdots$
0.4	$1.505941612\cdots$	$1.505942206\cdots$
0.5	$1.467462209\cdots$	$1.467466484\cdots$
0.6	$1.418083394\cdots$	$1.418107161\cdots$
0.7	$1.355661136\cdots$	$1.355777213\cdots$
0.8	$1.276349943\cdots$	$1.276910677\cdots$

The following bounds for the complete elliptic integrals of the second kind $\mathcal{E}(r)$ follow from Theorem 3.2 and Remark 4.1.

Corollary 4.6. *Inequality*

$$\left[\frac{1+\sqrt{1-r^2}}{1+(1-r^2)^{1/4}}\right]^2 < E(r) < \frac{\pi}{2}\left[\frac{1+\sqrt{1-r^2}}{1+(1-r^2)^{1/4}}\right]^2 \tag{4.5}$$

holds for all $r \in (0,1)$.

Remark 4.7. Computational and numerical experiments show that the upper bound in (4.5) for $\mathcal{E}(r)$ is very accurate for some $r \in (0,1)$. In fact, if we let $J(r) = \pi[1+\sqrt{1-r^2}]^2 / \{2[1+(1-r^2)^{1/4}]^2\}$, then we have Table 2 via elementary computation.

Acknowledgments

This work is supported by the Natural Science Foundation of China (Grant no. 11071069), the Natural Science Foundation of Zhejiang Province (Grant no. Y7080106), and the Innovation Team Foundation of the Department of Education of Zhejiang Province (Grant no. T200924).

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