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Research Article

Positive Periodic Solutions of Second-Order Differential Equations with Delays

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The existence results of positive ω-periodic solutions are obtained for the second-order differential equation with delays $-u'' + a(t) = f(t, u(t-\tau_1), \ldots, u(t-\tau_n))$, where $a \in C(\mathbb{R}, (0,\infty))$ is a ω-periodic function, $f : \mathbb{R}\times[0,\infty)^n \to [0,\infty)$ is a continuous function, which is ω-periodic in t, and $\tau_1, \tau_2, \ldots, \tau_n$ are positive constants. Our discussion is based on the fixed point index theory in cones.

1. Introduction and Main Results

In this paper, we discuss the existence of positive ω-periodic solutions of the second-order differential equation with delays

$$-u''(t) + a(t)u(t) = f(t, u(t-\tau_1), \ldots, u(t-\tau_n)), \tag{1.1}$$

where $a \in C(\mathbb{R}, (0,\infty))$ is a ω-periodic function, $f : \mathbb{R} \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, which is ω-periodic in t, and $\tau_1, \tau_2, \ldots, \tau_n$ are positive constants.

In recent years, the existence of periodic solutions for second-order delay differential equations has been researched by many authors, see [1–8] and references therein. In some practice models, only positive periodic solutions are significant. In [4, 5, 7], the authors obtained the existence of positive periodic solutions for some delay second-order differential equations by using Krasnoselskii's fixed-point theorem of cone mapping. For the second-order differential equations without delay, the existence of positive periodic solutions has been discussed by more authors, see [9–14].

Motivated by the papers mentioned above, we research the existence of positive periodic solutions of (1.1) with multiple delays. We aim to obtain the essential conditions on the existence of positive periodic solutions of (1.1) via the theory of the fixed-point index in cones. The conditions concern with the relation of the coefficient function $a(t)$ and nonlinearity $f(t, x_1, \ldots, x_n)$. Let

$$m = \min_{0 \le t \le \omega} a(t), \qquad M = \max_{0 \le t \le \omega} a(t). \tag{1.2}$$

Obviously, $0 < m \le M$. Our main results are as follows.

Theorem 1.1. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$, and $f(t, x_1, \ldots, x_n)$ ω-periodic in t. If f satisfies the following conditions:*

(F1) *there exist positive constants $c_1, \ldots, c_n$ satisfying $c_1 + \cdots + c_n < m$ and $\delta > 0$ such that*

$$f(t, x_1, \ldots, x_n) \le c_1 x_1 + \cdots + c_n x_n, \tag{1.3}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \in [0, \delta]$;

(F2) *there exist positive constants $d_1, \ldots, d_n$ satisfying $d_1 + \cdots + d_n > M$ and $H > 0$ such that*

$$f(t, x_1, \ldots, x_n) \ge d_1 x_1 + \cdots + d_n x_n, \tag{1.4}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \ge H$,

then (1.1) *has at least one positive ω-periodic solution.*

Theorem 1.2. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$, and $f(t, x_1, \ldots, x_n)$ ω-periodic in t. If f satisfies the following conditions:*

(F3) *there exist positive constants $d_1, \ldots, d_n$ satisfying $d_1 + \cdots + d_n > M$ and $\delta > 0$ such that*

$$f(t, x_1, \ldots, x_n) \ge d_1 x_1 + \cdots + d_n x_n, \tag{1.5}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \in [0, \delta]$;

(F4) *there exist positive constants $c_1, \ldots, c_n$ satisfying $c_1 + \cdots + c_n < m$ and $H > 0$ such that*

$$f(t, x_1, \ldots, x_n) \le c_1 x_1 + \cdots + c_n x_n, \tag{1.6}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \ge H$,

then (1.1) *has at least one positive ω-periodic solution.*

In Theorem 1.1, the conditions (F1) and (F2) allow $f(t, x_1, \ldots, x_n)$ to be superlinear growth on $x_1, \ldots, x_n$. For example,

$$f(t, x_1, \ldots, x_n) = a_1(t) {x_1}^2 + \cdots + a_n(t) {x_n}^2 \tag{1.7}$$

satisfies (F1) and (F2), where $a_1(t), \ldots, a_n(t)$ are positive and continuous ω-periodic functions.

In Theorem 1.2, the conditions (F3) and (F4) allow $f(t, x_1, \ldots, x_n)$ to be sublinear growth on $x_1, \ldots, x_n$. For example,

$$f(t, x_1, \ldots, x_n) = b_1(t)\sqrt{|x_1|} + \cdots + b_n(t)\sqrt{|x_n|} \tag{1.8}$$

satisfies (F3) and (F4), where $b_1(t), \ldots, b_n(t)$ are positive and continuous ω-periodic functions.

Our results are different from those in the references mentioned above. The conditions (F1) and (F2) in Theorem 1.1 and the conditions (F3) and (F4) in Theorem 1.2 are optimal for the existence of positive periodic solutions of (1.1). This fact can been shown from the differential equation with linear delays

$$-u''(t) + a_0 u(t) = a_1 u(t - \tau_1) + \cdots + a_n u(t - \tau_n) + h(t), \tag{1.9}$$

where $a_0, a_1, \ldots a_n$ are positive constants and $h \in C(\mathbb{R})$ is a positive ω-periodic function. If $a_1, \ldots a_n$ satisfy

$$a_1 + a_2 + \cdots + a_n = a_0. \tag{1.10}$$

Equation (1.9) has no positive ω-periodic solutions. In fact, if (1.9) has a positive ω-periodic solution, integrating the equation on $[0, \omega]$ and using the periodicity of $u(t)$, we can obtain that $\int_0^\omega h(t)dt = 0$, which contradicts to the positivity of $h(t)$. Hence, (1.9) has no positive ω-periodic solution. For $a(t) \equiv a_0$ and $f(t, x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n + h(t)$, if Condition (1.10) holds, the conditions (F1) and (F2) in Theorem 1.1 and the conditions (F3) and (F4) in Theorem 1.2 have just not been satisfied. From this, we see that the conditions in Theorems 1.1–1.2 are optimal.

The proofs of Theorems 1.1–1.2 are based on the fixed point index theory in cones, which will be given in Section 3. Some preliminaries to discuss (1.1) are presented in Section 2.

2. Preliminaries

Let $C_\omega(\mathbb{R})$ denote the Banach space of all continuous ω-periodic function $u(t)$ with norm $\|u\|_C = \max_{0\le t\le\omega}|u(t)|$. Let $C_\omega^+(\mathbb{R})$ be the cone of all nonnegative functions in $C_\omega(\mathbb{R})$. Generally, $C_\omega^m(\mathbb{R})$ denotes the mth-order continuous differentiable ω-periodic function space for $m \in \mathbb{N}$.

Let M be the positive constant defined by (1.2). For $h \in C_\omega(\mathbb{R})$, we consider the linear second-order differential equation

$$-u''(t) + Mu(t) = h(t), \quad t \in \mathbb{R}. \tag{2.1}$$

The ω-periodic solutions of (2.1) are can been expressed by the solution of the linear second-order boundary value problem

$$\begin{gathered} -u''(t) + Mu(t) = 0, \quad 0 \le t \le \omega, \\ u(0) - u(\omega) = 0, \quad \dot{u}(0) - \dot{u}(\omega) = -1, \end{gathered} \tag{2.2}$$

see [11]. Problem (2.2) has a unique solution, which is explicitly given by

$$\Phi(t) = \frac{\cosh \beta(t-\omega/2)}{2\beta \sinh(\beta\omega/2)}, \quad 0 \le t \le \omega, \tag{2.3}$$

where $\beta = \sqrt{M}$.

By a direct calculation, we easily prove the following lemma.

Lemma 2.1. *Let $M > 0$. For every $h \in C_\omega(\mathbb{R})$, the linear equation (2.1) has a unique ω-periodic solution $u(t)$, which is given by*

$$u(t) = \int_{t-\omega}^{t} \Phi(t-s)h(s)ds := Th(t), \quad t \in \mathbb{R}. \tag{2.4}$$

Moreover, $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator.

Since $\Phi(t) > 0$ for every $t \in [0,\omega]$, if $h \in C_\omega^+(\mathbb{R})$ and $h(t) \not\equiv 0$, by (2.4) the ω-periodic solution of (2.1) $u = \mathrm{Th}(t)$ is positive. Moreover, we can show that the ω-periodic solution has the following strong positivity:

$$Th(t) \ge \sigma \|Th\|_C, \quad t \in \mathbb{R},\ h \in C_\omega^+(\mathbb{R}), \tag{2.5}$$

where $\sigma = \underline{\Phi}/\overline{\Phi} = 1/\cosh(\beta\omega/2)$, in which

$$\underline{\Phi} = \min_{0\le t\le\omega} \Phi(t) = \frac{1}{2\beta\sinh(\beta\omega/2)}, \qquad \overline{\Phi} = \max_{0\le t\le\omega} \Phi(t) = \frac{\cosh(\beta\omega/2)}{2\beta\sinh(\beta\omega/2)}. \tag{2.6}$$

In fact, for $h \in C_\omega^+(\mathbb{R})$ and $t \in \mathbb{R}$, from (2.4) it follows that

$$Th(t) = \int_{t-\omega}^{t} \Phi(t-s)h(s)ds \le \overline{\Phi}\int_{t-\omega}^{t} h(s)ds = \overline{\Phi}\int_{0}^{\omega} h(s)ds, \tag{2.7}$$

and therefore,

$$\|Th\|_C \le \overline{\Phi}\int_0^\omega h(s)ds. \tag{2.8}$$

Using (2.4) and this inequality, we have that

$$\begin{aligned} Th(t) &= \int_{t-\omega}^{t} \Phi(t-s)h(s)ds \ge \underline{\Phi}\int_{t-\omega}^{t} h(s)ds = \underline{\Phi}\int_0^\omega h(s)ds \\ &= \left(\underline{\Phi}/\overline{\Phi}\right)\cdot \overline{\Phi}\int_0^\omega h(s)ds \ge \sigma\|Th\|_C. \end{aligned} \tag{2.9}$$

Hence, (2.5) holds.

Now we consider the periodic solution problem of the linear differential equation with variable coefficient

$$-u''(t) + a(t)u(t) = h(t), \quad t \in \mathbb{R}. \tag{2.10}$$

Lemma 2.2. *Let $a \in C_\omega(\mathbb{R})$ be a positive ω-periodic function. For every $h \in C_\omega(\mathbb{R})$, the linear equation* (2.10) *has a unique ω-periodic solution $u := Sh$. Moreover, $S : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator and with strong positivity*

$$Sh(t) \geq \frac{m\sigma}{M}\|Sh\|_C, \quad t \in \mathbb{R},\ h \in C_\omega^+(\mathbb{R}). \tag{2.11}$$

Proof. Let M and m be the positive constants defined by (1.2). Then $0 < m \leq a(t) \leq M$, $t \in \mathbb{R}$. Let $T : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ be the ω-periodic solution operator of (2.1) given by (2.4). We rewrite (2.10) to the form of

$$-u''(t) + Mu(t) = (M - a(t))u(t) + h(t), \quad t \in \mathbb{R}. \tag{2.12}$$

Then it is easy to see that the ω-periodic solution problem of (2.10) is equivalent to the operator equation in Banach space $C_\omega(\mathbb{R})$

$$(I - T \circ B)u = Th, \tag{2.13}$$

where I is the identity operator in $C_\omega(\mathbb{R})$ and $B : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is the product operator defined by

$$Bu(t) = (M - a(t))u(t), \quad u \in C_\omega(\mathbb{R}), \tag{2.14}$$

which is a positive linear bounded operator. We prove that the norm of $T \circ B$ in $\mathcal{L}(C_\omega(\mathbb{R}), C_\omega(\mathbb{R}))$ satisfies $\|T \circ B\| < 1$.

For every $u \in C_\omega(\mathbb{R})$ and $t \in \mathbb{R}$, by the definition (2.4) of T and the positivity of Φ, we have

$$\begin{aligned}
|(T \circ B)u(t)| = |T(Bu)(t)| &= \left|\int_{t-\omega}^{t} \Phi(t-s)(M - a(s))u(s)\,ds\right| \\
&\leq \int_{t-\omega}^{t} \Phi(t-s)|(M - a(s))u(s)|\,ds \\
&\leq (M - m)\|u\|_C \int_{t-\omega}^{t} \Phi(t-s)\,ds \\
&= (M - m)\|u\|_C \int_{0}^{\omega} \Phi(s)\,ds \\
&= \left(1 - \frac{m}{M}\right)\|u\|_C.
\end{aligned} \tag{2.15}$$

Therefore, $\|(T \circ B)u\|_C \le (1-m/M)\|u\|_C$. By the arbitrariness of $u \in C_\omega(\mathbb{R})$, we have $\|T\circ B\| \le 1 - m/M < 1$.

Thus, $I - T \circ B$ has a bounded inverse operator given by the series

$$(I - T \circ B)^{-1} = \sum_{n=0}^{\infty} (T \circ B)^n \tag{2.16}$$

with the norm estimate

$$\left\|(I - T \circ B)^{-1}\right\| \le \frac{1}{1 - \|T \circ B\|} \le \frac{M}{m}. \tag{2.17}$$

Consequently, (2.13), equivalently (2.10), has a unique ω-periodic solution

$$u = (I - T \circ B)^{-1}(Th) := Sh, \tag{2.18}$$

where

$$S = (I - T \circ B)^{-1} \circ T = \sum_{n=0}^{\infty} (T \circ B)^n T. \tag{2.19}$$

By the complete continuity of T, $S : C_\omega(\mathbb{R}) \to C_\omega(\mathbb{R})$ is a completely continuous linear operator.

For every $h \in C_\omega(\mathbb{R})$, by the expression (2.19) of S, we have

$$\|Sh\|_C \le \left\|(I - T \circ B)^{-1}\right\| \cdot \|Th\|_C \le \frac{M}{m}\|\mathrm{Th}\|_C. \tag{2.20}$$

If $h \in C_\omega^+(\mathbb{R})$, by the series expression of S and the positivity of T and B, we have

$$Sh = \left(\sum_{n=0}^{\infty} (T \circ B)^n T\right) h = \sum_{n=0}^{\infty} (T \circ B)^n (Th) \ge Th. \tag{2.21}$$

Hence, form (2.5) and (2.20), it follows that

$$Sh(t) \ge Th(t) \ge \sigma\|Th\| \ge \frac{m\sigma}{M}\|Sh\|_C, \quad t \in \mathbb{R}. \tag{2.22}$$

Namely, (2.11) holds. □

Let $f \in C(\mathbb{R} \times [0,\infty)^n, [0,\infty))$. For every $u \in C_\omega^+(\mathbb{R})$, set

$$F(u)(t) := f(t, u(t - \tau_1), \dots, u(t - \tau_n)), \quad t \in \mathbb{R}. \tag{2.23}$$

Then $F : C^+_\omega(\mathbb{R}) \to C^+_\omega(\mathbb{R})$ is continuous. Define a mapping $A : C^+_\omega(\mathbb{R}) \to C^+_\omega(\mathbb{R})$ by

$$A = S \circ F. \tag{2.24}$$

By the definition of operator S, the ω-periodic solution of (1.1) is equivalent to the fixed point of A. Choose a subcone of $C^+_\omega(\mathbb{R})$ by

$$K = \left\{ u \in C^+_\omega(\mathbb{R}) \mid u(t) \geq \frac{m\sigma}{M} \|u\|_C, t \in \mathbb{R} \right\}. \tag{2.25}$$

From the strong positivity of S in Lemma 2.2 and the definition of A, we easily obtain the following lemma.

Lemma 2.3. *$A(C^+_\omega(\mathbb{R})) \subset K$, and $A : K \to K$ is completely continuous.*

Hence, the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

We recall some concepts and conclusions on the fixed point index in [15, 16]. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \neq \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \neq u$ for any $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ has definition. One important fact is that if $i(A, K \cap \Omega, K) \neq 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas are needed in our argument.

Lemma 2.4 (see [16]). *Let Ω be a bounded open subset of E with $\theta \in \Omega$ and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.5 (see [16]). *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $e \in K \setminus \{\theta\}$ such that $u - Au \neq \mu e$ for every $u \in K \cap \partial\Omega$ and $\mu \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

In next section, we will use Lemmas 2.4 and 2.5 to prove Theorems 1.1 and 1.2.

3. Proofs of Main Results

Proof of Theorem 1.1. Choose the working space $E = C_\omega(\mathbb{R})$. Let $K \subset C^+_\omega(\mathbb{R})$ be the closed convex cone in $C_\omega(\mathbb{R})$ defined by (2.25) and $A : K \to K$ the operator defined by (2.24). Then the positive ω-periodic solution of (1.1) is equivalent to the nontrivial fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in C_\omega(\mathbb{R}) \mid \|u\|_C < r\}, \qquad \Omega_2 = \{u \in C_\omega(\mathbb{R}) \mid \|u\|_C < R\}. \tag{3.1}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

Let $r \in (0, \delta)$, where δ is the positive constant in Condition (F1). We prove that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$, namely, $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_1$ and

$0 < \lambda \leq 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0 A u_0 = u_0$, then by the definition of A and Lemma 2.2, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the delay differential equation

$$-u_0''(t) + a(t)u_0(t) = \lambda_0 f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.2}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$0 \leq u_0(t-\tau_k) \leq \|u_0\|_C = r < \delta, \quad k = 1, \ldots, n, \ t \in \mathbb{R}. \tag{3.3}$$

Hence from condition (F1), it follows that

$$f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)) \leq c_1 u_0(t-\tau_1) + \cdots + c_n u_0(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.4}$$

By this and (3.2), we get that

$$-u_0''(t) + a(t)u_0(t) \leq c_1 u_0(t-\tau_1) + \cdots + c_n u_0(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.5}$$

Integrating both sides of this inequality from 0 to ω and using the periodicity of u_0, we have

$$\begin{aligned}\int_0^\omega a(t)u_0(t)dt &\leq c_1 \int_0^\omega u_0(t-\tau_1)dt + \cdots + c_n \int_0^\omega u_0(t-\tau_n)dt \\ &= (c_1 + \cdots + c_n) \int_0^\omega u_0(t)ds.\end{aligned} \tag{3.6}$$

Hence, we obtain that

$$m \int_0^\omega u_0(t)dt \leq \int_0^\omega a(t)u_0(t)dt \leq (c_1 + \cdots + c_n) \int_0^\omega u_0(t)ds. \tag{3.7}$$

By the definition of cone K, $\int_0^\omega u_0(t)dt \geq (m\sigma/M)\|u_0\|_C \cdot \omega > 0$. From (3.7), it follows that $m \leq c_1 + \cdots + c_n$, which contradicts to the assumption in Condition (F1). Hence A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$i(A, K \cap \Omega_1, K) = 1. \tag{3.8}$$

On the other hand, choose $R > \max\{(M/m\sigma)H, \delta\}$, where H is the positive constant in condition (F2), and let $e(t) \equiv 1$. Clearly, $e \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$, namely, $u - Au \neq \mu v$ for every $u \in K \cap \partial\Omega_2$ and $\mu \geq 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1 e$, since $u_1 - \mu_1 e = Au_1$, by definition of A and Lemma 2.2, $u_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$-u_1''(t) + a(t)\big(u_1(t) - \mu_1\big) = f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.9}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t-\tau_k) \geq \frac{m\sigma}{M}\|u_1\|_C = \frac{m\sigma}{M}R > H, \quad t \in I, \ k = 1, \ldots, n. \tag{3.10}$$

From this and Condition (F2), it follows that

$$f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)) \geq d_1 u_1(t-\tau_1) + \cdots + d_n u_n(t-\tau_n), \quad t \in I. \tag{3.11}$$

By this inequality and (3.9), we have

$$-u_1''(t) + a(t)\big(u_1(t) - \mu_1\big) \geq d_1 u_1(t-\tau_1) + \cdots + d_n u_1(t-\tau_n), \quad t \in I. \tag{3.12}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of u_1, we obtain that

$$\begin{aligned} \int_0^\omega a(t)\big(u_1(t) - \mu_1\big)dt &\geq d_1 \int_0^\omega u_1(t-\tau_1)dt + \cdots + d_n \int_0^\omega u_1(t-\tau_n)dt \\ &= (d_1 + \cdots + d_n)\int_0^\omega u_1(t)ds. \end{aligned} \tag{3.13}$$

Consequently, we have that

$$\begin{aligned} M\int_0^\omega u_1(t)dt \geq \int_0^\omega a(t)u_1(t)dt &\geq \int_0^\omega a(t)\big(u_1(t) - \mu_1\big)dt \\ &\geq (d_1 + \cdots + d_n)\int_0^\omega u_1(t)ds. \end{aligned} \tag{3.14}$$

Since $\int_0^\omega u_1(t)dt \geq (m\sigma/M)\|u_1\|_C \cdot \omega > 0$, form this inequality it follows that $M \geq d_1 + \cdots + d_n$, which contradicts to the assumption in Condition (F2). This means that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5,

$$i(A, K \cap \Omega_2, K) = 0. \tag{3.15}$$

Now by the additivity of fixed point index, (3.8), and (3.15) we have

$$i\Big(A, K \cap \Big(\Omega_2 \setminus \overline{\Omega}_1\Big), K\Big) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1. \tag{3.16}$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

Proof of Theorem 1.2. Let $\Omega_1, \Omega_2 \subset C_\omega(\mathbb{R})$ be defined by (3.1). We prove that the operator A defined by (2.24) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R large enough.

Let $r \in (0, \delta)$, where δ is the positive constant in Condition (F2), and choose $e(t) \equiv 1$. We prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$, namely, $u - Au \neq \mu e$ for every

$u \in K \cap \partial\Omega_1$ and $\mu \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\mu_0 \geq 0$ such that $u_0 - Au_0 = \mu_0 e$, since $u_0 - \mu_0 e = Au_0$, by definition of A and Lemma 2.2, $u_0 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$-u_0''(t) + a(t)\left(u_0(t) - \mu_0\right) = f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.17}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (3.3). From (3.3) and Condition (F3), it follows that

$$f(t, u_0(t-\tau_1), \ldots, u_0(t-\tau_n)) \geq d_1 u_0(t-\tau_1) + \cdots + d_n u_0(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.18}$$

From this and (3.17), we see that

$$-u_0''(t) + a(t)\left(u_0(t) - \mu_0\right) \geq d_1 u_0(t-\tau_1) + \cdots + d_n u_0(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.19}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of $u_0(t)$, we have

$$\begin{aligned}\int_0^\omega a(t)\left(u_0(t) - \mu_0\right)dt &\geq d_1 \int_0^\omega u_0(t-\tau_1)dt + \cdots + d_n \int_0^\omega u_0(t-\tau_n)dt \\ &= (d_1 + \cdots + d_n) \int_0^\omega u_0(t)ds.\end{aligned} \tag{3.20}$$

From this we obtain that

$$\begin{aligned}M \int_0^\omega u_0(t)dt \geq \int_0^\omega a(t)u_0(t)dt &\geq \int_0^\omega a(t)\left(u_0(t) - \mu_0\right)dt \\ &\geq (d_1 + \cdots + d_n) \int_0^\omega u_0(t)ds.\end{aligned} \tag{3.21}$$

Since $\int_0^\omega u_0(t)dt \geq (m\sigma/M)\|u_0\|_C \cdot \omega > 0$, from the inequality above, it follows that $M \geq d_1 + \cdots + d_n$, which contradicts to the assumption in (F3). Hence A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$. By Lemma 2.5, we have

$$i(A, K \cap \Omega_1, K) = 0. \tag{3.22}$$

Then, choosing $R > \max\{(M/m\sigma)H, \delta\}$, we show that A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_2$, namely, $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 Au_1 = u_1$, then by the definition of A and Lemma 2.2, $u_1 \in C^2_\omega(\mathbb{R})$ satisfies the differential equation

$$-u_1''(t) + a(t)u_1(t) = \lambda_1 f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)), \quad t \in \mathbb{R}. \tag{3.23}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (3.10). From (3.10) and condition (F4), it follows that

$$f(t, u_1(t-\tau_1), \ldots, u_1(t-\tau_n)) \le c_1 u_1(t-\tau_1) + \cdots + c_n u_1(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.24}$$

By this and (3.23), we have

$$-u_1''(t) + a(t)u_1(t) \le c_1 u_1(t-\tau_1) + \cdots + c_n u_1(t-\tau_n), \quad t \in \mathbb{R}. \tag{3.25}$$

Integrating this inequality on $[0, \omega]$ and using the periodicity of $u_1(t)$, we have

$$\begin{aligned} \int_0^\omega a(t)u_1(t)dt &\le c_1 \int_0^\omega u_1(t-\tau_1)dt + \cdots + c_n \int_0^\omega u_1(t-\tau_n)dt \\ &= (c_1 + \cdots + c_n) \int_0^\omega u_1(t)ds. \end{aligned} \tag{3.26}$$

From this we obtain that

$$m\int_0^\omega u_1(t)dt \le \int_0^\omega a(t)u_1(t)dt \le (c_1 + \cdots + c_n)\int_0^\omega u_1(t)ds. \tag{3.27}$$

Since $\int_0^\omega u_1(t)dt \ge (m\sigma/M)\|u_0\|_C \cdot \omega > 0$, from the inequality (3.27), it follows that $m \le c_1 + \cdots + c_n$, which contradicts to the assumption in Condition (F4). Hence A satisfies the condition of Lemma 2.4 in $K \cap \partial\Omega_1$. By Lemma 2.4, we have

$$i(A, K \cap \Omega_2, K) = 1. \tag{3.28}$$

Now, from (3.22) and (3.28), it follows that

$$i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_1\right), K\right) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1. \tag{3.29}$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive ω-periodic solution of (1.1). □

4. Remarks

In Theorems 1.1 and 1.2, the conditions (F1) and (F4) can be replaced by the following condition:

(F5) there exist positive constants $c_1, \ldots, c_n$ satisfying $c_1 + \cdots + c_n < m$ and $H > 0$ such that

$$f(t, x_1, \ldots, x_n) \le c_1 x_1 + \cdots + c_n x_n, \tag{4.1}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \in [(m\sigma/M)H, H]$;

and (F2) and (F3) can be replaced by the

(F6) there exist positive constants $d_1, \ldots, d_n$ satisfying $d_1 + \cdots + d_n > M$ and $H > 0$ such that

$$f(t, x_1, \ldots, x_n) \geq d_1 x_1 + \cdots + d_n x_n, \tag{4.2}$$

for $t \in \mathbb{R}$ and $x_1, \ldots, x_n \in [(m\sigma/M)H, H]$.

In fact, if condition (F5) holds, setting

$$\Omega_3 = \{u \in C_\omega(\mathbb{R}) \mid \|u\|_C < H\}, \tag{4.3}$$

similar to the proof of (3.28), we can prove that

$$i(A, K \cap \Omega_3, K) = 1, \tag{4.4}$$

and if condition (F6) holds, similar to the proof of (3.15), we can prove that

$$i(A, K \cap \Omega_3, K) = 0. \tag{4.5}$$

Therefore, by the proofs of Theorems 1.1 and 1.2, we have the following theorem.

Theorem 4.1. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$ and $f(t, x_1, \ldots, x_n)\omega$-periodic in t. Then in each case of the following:*

(1) *(F1) and (F6) hold,*

(2) *(F2) and (F5) hold,*

(3) *(F3) and (F5) hold,*

(4) *(F4) and (F6) hold.*

Equation (1.1) *has at least one positive ω-periodic solution.*

Now we consider the existence of two positive periodic solutions of (1.1). If the conditions (F2), (F3), and (F5) hold, by the proof of Theorem 1.1, condition (F2) implies that (3.15) holds when R is large enough and $R > H$, and by the proof of Theorem 1.2, condition (F3) implies that (3.22) holds when r is small enough and $r < H$. Since $\overline{\Omega}_1 \subset \Omega_3$ and $\overline{\Omega}_3 \subset \Omega_2$, by (3.15), (3.22), and (4.4), we have

$$\begin{aligned} i\left(A, K \cap \left(\Omega_3 \setminus \overline{\Omega}_1\right), K\right) &= i(A, K \cap \Omega_3, K) - i(A, K \cap \Omega_1, K) = 1, \\ i\left(A, K \cap \left(\Omega_2 \setminus \overline{\Omega}_3\right), K\right) &= i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_3, K) = -1. \end{aligned} \tag{4.6}$$

This means that A has fixed-points $u_1 \in K \cap (\Omega_3 \setminus \overline{\Omega}_1)$ and $u_2 \in K \cap (\Omega_2 \setminus \overline{\Omega}_3)$, and u_1 and u_2 are two positive ω-periodic solution of (1.1). Consequently, we have the following theorem.

Theorem 4.2. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function and $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$ and $f(t, x_1, \ldots, x_n)$ be ω-periodic in t. If (F2), (F3), and (F5) hold, then* (1.1) *has two positive ω-periodic solutions.*

Similar to Theorem 4.2, we have the following theorem.

Theorem 4.3. *Let $a \in C(\mathbb{R}, (0, \infty))$ be a ω-periodic function, $f \in C(\mathbb{R} \times [0, \infty)^n, [0, \infty))$, and $f(t, x_1, \ldots, x_n)\omega$-periodic in t. If (F1), (F4), and (F6) hold, then* (1.1) *has two positive ω-periodic solutions.*

References

[1] B. Liu, "Periodic solutions of a nonlinear second-order differential equation with deviating argument," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 1, pp. 313–321, 2005.
[2] J. W. Li and S. S. Cheng, "Periodic solutions of a second order forced sublinear differential equation with delay," *Applied Mathematics Letters*, vol. 18, no. 12, pp. 1373–1380, 2005.
[3] Y. Wang, H. Lian, and W. Ge, "Periodic solutions for a second order nonlinear functional differential equation," *Applied Mathematics Letters*, vol. 20, no. 1, pp. 110–115, 2007.
[4] J. Wu and Z. Wang, "Two periodic solutions of second-order neutral functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 329, no. 1, pp. 677–689, 2007.
[5] Y. X. Wu, "Existence nonexistence and multiplicity of periodic solutions for a kind of functional differential equation with parameter," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 1, pp. 433–443, 2009.
[6] C. J. Guo and Z. M. Guo, "Existence of multiple periodic solutions for a class of second-order delay differential equations," *Nonlinear Analysis*, vol. 10, no. 5, pp. 3285–3297, 2009.
[7] W. S. Cheung, J. Ren, and W. Han, "Positive periodic solution of second-order neutral functional differential equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 9, pp. 3948–3955, 2009.
[8] X. Lv, P. Yan, and D. Liu, "Anti-periodic solutions for a class of nonlinear second-order Rayleigh equations with delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 11, pp. 3593–3598, 2010.
[9] F. M. Atici and G. Sh. Guseinov, "On the existence of positive solutions for nonlinear differential equations with periodic boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 132, no. 2, pp. 341–356, 2001.
[10] P. J. Torres, "Existence of one-signed periodic solutions of some second-order differential equations via a Krasnoselskii fixed point theorem," *Journal of Differential Equations*, vol. 190, no. 2, pp. 643–662, 2003.
[11] Y. X. Li, "Positive periodic solutions of nonlinear second order ordinary differential equations," *Acta Mathematica Sinica*, vol. 45, no. 3, pp. 481–488, 2002 (Chinese).
[12] Y. Li, "Positive periodic solutions of first and second order ordinary differential equations," *Chinese Annals of Mathematics B*, vol. 25, no. 3, pp. 413–420, 2004.
[13] F. Li and Z. Liang, "Existence of positive periodic solutions to nonlinear second order differential equations," *Applied Mathematics Letters*, vol. 18, no. 11, pp. 1256–1264, 2005.
[14] J. R. Graef, L. Kong, and H. Wang, "Existence, multiplicity, and dependence on a parameter for a periodic boundary value problem," *Journal of Differential Equations*, vol. 245, no. 5, pp. 1185–1197, 2008.
[15] K. Deimling, *Nonlinear Functional Analysis*, Springer, NewYork, NY, USA, 1985.
[16] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, NewYork, NY, USA, 1988.