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Research Article

Numerical Solution of Stochastic Hyperbolic Equations

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A two-step difference scheme for the numerical solution of the initial-boundary value problem for stochastic hyperbolic equations is presented. The convergence estimate for the solution of the difference scheme is established. In applications, the convergence estimates for the solution of the difference scheme are obtained for different initialboundary value problems. The theoretical statements for the solution of this difference scheme are supported by numerical examples.

1. Introduction

Stochastic partial differential equations have been studied extensively by many researchers. For example, the method of operators as a tool for investigation of the solution to stochastic equations in Hilbert and Banach spaces have been used systematically by several authors (see, [1–7] and the references therein). Numerical methods and theory of solutions of initial boundary value problem for stochastic partial differential equations have been studied in [8–16]. Moreover, the authors of [17] presented a two-step difference scheme for the numerical solution of the following initial value problem:

$$\begin{gathered} d\dot{v}(t) = -Av(t)dt + f(t)dw_t, \quad 0 < t < T, \\ v(0) = 0, \qquad \dot{v}(0) = 0, \end{gathered} \tag{1.1}$$

for stochastic hyperbolic differential equations. We have the following.

(i) w_t is a standard Wiener process given on the probability space (Ω, F, P).

(ii) For any $z \in [0,T]$, $f(z)$ is an element of the space $M_w^2([0,T], H_1)$, where H_1 is a subspace of H.

Here, $M_w^2([0,T],H)$ [18] denote the space of H-valued measurable processes which satisfy

(a) $\phi(t)$ is F_t measurable, a.e. in t,

(b) $E\int_0^T \|\phi(t)\|_H dt < \infty$.

The convergence estimates for the solution of the difference scheme are established.

In the present work, we consider the following initial value problem:

$$\begin{gathered} d\dot{v}(t) + Av(t)dt = f(t)dw_t, \quad 0 < t < T, \\ v(0) = \varphi, \quad \dot{v}(0) = \psi, \end{gathered} \tag{1.2}$$

for stochastic hyperbolic equation in a Hilbert space H with a self-adjoint positive definite operator A with $A \geq \delta I$, where $\delta > \delta_0 > 0$. In addition to (i) and (ii), we put the following.

(iii) φ and ψ are elements of the space $M_w^2([0,T],H_2)$ of H_2-valued measurable processes, where H_2 is a subspace of H.

By the solutions provided in [19] (page 423, (0.4)) and in [20] (page 1005, (2.9)), under the assumptions (i), (ii), and (iii), the initial value problem (1.2) has a unique mild solution given by the following formula:

$$v(t) = c(t)\varphi + s(t)\psi + \int_0^t s(t-z)f(z)dw_z. \tag{1.3}$$

For the theory of cosine and sine operator-function we refer to [21, 22].

Our interest in this study is to construct and investigate the difference scheme for the initial value problem (1.2). The convergence estimate for the solution of the difference scheme is proved. In applications, the theorems on convergence estimates for the solution of difference schemes for the numerical solution of initial-boundary value problems for hyperbolic equations are established. The theoretical statements for the solution of this difference scheme are supported by the result of the numerical experiments.

2. The Exact Difference Scheme

We consider the following uniform grid space:

$$[0,T]_\tau = \{t_k = k\tau,\ k = 0,1,\ldots,N,\ N\tau = T\}, \tag{2.1}$$

with step $\tau > 0$. Here, N is a fixed positive integer.

Theorem 2.1. *Let $v(t_k)$ be the solution of the initial value problem (1.2) at the grid points $t = t_k$. Then, $\{v(t_k)\}_0^N$ is the solution of the initial value problem for the following difference equation:*

$$\frac{1}{\tau^2}(v(t_{k+1}) - 2v(t_k) + v(t_{k-1})) + \frac{2}{\tau^2}(I - c(\tau))v(t_k) = \frac{1}{\tau}\left(f_{1,k+1} + s(\tau)f_{2,k} - c(\tau)f_{1,k}\right),$$
$$f_{1,k} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} s(t_k - z)f(z)dw_z, \quad f_{2,k} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} c(t_k - z)f(z)dw_z, \quad 1 \le k \le N-1, \tag{2.2}$$
$$v(0) = \varphi, \quad v(\tau) = c(\tau)\varphi + s(\tau)\psi + \tau f_{1,1}.$$

Proof. Putting $t = t_k$ into the formula (1.3), we can write

$$v(t_k) = c(t_k)\varphi + s(t_k)\psi + \int_0^{t_k} s(t_k - z)f(z)dw_z. \tag{2.3}$$

Using (2.3), the definition of the sine and cosine operator function, we obtain

$$\begin{aligned} v(t_k) &= c(t_k)\varphi + s(t_k)\psi + \sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} s\left(t_k - t_j + t_j - z\right)f(z)dw_z \\ &= c(t_k)\varphi + s(t_k)\psi + \tau\sum_{j=1}^{k}\left(s\left(t_k - t_j\right)f_{2,j} + c\left(t_k - t_j\right)f_{1,j}\right). \end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned} v(t_{k+1}) + v(t_{k-1}) = {} & [c(t_{k+1}) + c(t_{k-1})]\varphi + [s(t_{k+1}) + s(t_{k-1})]\psi \\ & + 2c(\tau)\tau\sum_{j=1}^{k}\left(s\left(t_k - t_j\right)f_{2,j} + c\left(t_k - t_j\right)f_{1,j}\right) \\ & + \tau\left(f_{1,k+1} + s(\tau)f_{2,k} - c(\tau)f_{1,k}\right). \end{aligned} \tag{2.5}$$

Hence, we get the relation between $v(t_k)$ and $v(t_{k\pm1})$ as

$$v(t_{k+1}) + v(t_{k-1}) - 2c(\tau)v(t_k) = \tau\left(f_{1,k+1} + s(\tau)f_{2,k} - c(\tau)f_{1,k}\right). \tag{2.6}$$

This relation and equality (2.2) are equivalent. Theorem 2.1 is proved. □

3. Convergence of the Difference Scheme

For the approximate solution of problem (1.2), we need to approximate the following expressions:

$$f_{1,k} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} s(t_k - z)f(z)dw_z, \qquad f_{2,k} = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} c(t_k - z)f(z)dw_z,$$
$$\exp\left(\pm i\tau A^{1/2}\right). \tag{3.1}$$

Using Taylor's formula and Pade approximation of the function $\exp(-z)$ at $z = 0$, we get

$$\exp\left(\pm i\tau A^{1/2}\right) \approx \left(I \pm \frac{i\tau A^{1/2}}{2}\right)\left(I \mp \frac{i\tau A^{1/2}}{2}\right)^{-1} = R\left(\pm i\tau A^{1/2}\right),$$
$$f_{1,k} \approx -\frac{1}{\tau}\int_{t_{k-1}}^{t_k} (z - t_k)f(z)dw_z = \widetilde{f}_{1,k}, \qquad f_{2,k} \approx \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(z)dw_z = \widetilde{f}_{2,k}. \tag{3.2}$$

Applying the difference scheme (2.2) and formula (3.2), we can construct the following difference scheme:

$$\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{2}{\tau^2}(I - c_\tau(\tau))u_k = \frac{1}{\tau}\left(\widetilde{f}_{1,k+1} + s_\tau(\tau)\widetilde{f}_{2,k} - c_\tau(\tau)\widetilde{f}_{1,k}\right), \tag{3.3}$$

$$c_\tau(\tau) = \frac{R(i\tau A^{1/2}) + R(-i\tau A^{1/2})}{2},$$
$$s_\tau(\tau) = A^{-1/2}\frac{R(i\tau A^{1/2}) - R(-i\tau A^{1/2})}{2i}, \quad 1 \le k \le N-1,$$
$$u_0 = \varphi, \quad u_1 = c_\tau(\tau)\varphi + s_\tau(\tau)\psi + \tau\widetilde{f}_{1,1}, \tag{3.4}$$

for the approximate solution of the initial value problem (1.2). Using the definition of $c_\tau(\tau)$ and $s_\tau(\tau)$, we can write (3.3) in the following equivalent form:

$$\frac{1}{\tau^2}(u_{k+1} - 2u_k + u_{k-1}) + \frac{1}{2}Au_k + \frac{1}{4}Au_{k+1} + Au_{k-1}$$
$$= \frac{1}{\tau}\left(\left(I + \frac{1}{4}\tau^2 A\right)\widetilde{f}_{1,k+1} + \tau\widetilde{f}_{2,k} - \left(I - \frac{1}{4}\tau^2 A\right)\widetilde{f}_{1,k}\right),$$
$$t_k = k\tau, \quad 1 \le k \le N-1,$$
$$u_0 = \varphi, \quad u_1 = c_\tau(\tau)\varphi + s_\tau(\tau)\psi + \tau\widetilde{f}_{1,1}. \tag{3.5}$$

Now, let us give the lemma we need in the sequel from papers [23, 24].

Lemma 3.1. *The following estimates hold:*

$$\|c(t)\|_{H\to H} \le 1, \quad \left\|A^{1/2}s(t)\right\|_{H\to H} \le 1 \quad (t \ge 0), \tag{3.6}$$

$$\|c_\tau(k\tau)\|_{H\to H} \le 1, \quad \left\|A^{1/2}s_\tau(k\tau)\right\|_{H\to H} \le 1 \quad (k \ge 0), \tag{3.7}$$

$$\left\|A^{-(1+\alpha)}(c_\tau(k\tau) - c(t_k))\right\|_{H\to H} \le C\tau^{(3/2+\alpha)}, \quad 0 \le \alpha \le \frac{1}{2}, \tag{3.8}$$

$$\left\|A^{-(1/2+\alpha)}(s_\tau(k\tau) - s(t_k))\right\|_{H\to H} \le C\tau^{(3/2+\alpha)}, \quad (k \ge 0), \tag{3.9}$$

where

$$\begin{aligned} c_\tau(k\tau) &= \frac{R^k(i\tau A^{1/2}) + R^k(-i\tau A^{1/2})}{2}, \\ s_\tau(k\tau) &= A^{-1/2}\frac{R^k(i\tau A^{1/2}) - R^k(-i\tau A^{1/2})}{2i}. \end{aligned} \tag{3.10}$$

The following Theorem on convergence of difference scheme (3.5) *is established.*

Theorem 3.2. *Assume that*

$$E\left(\|A\varphi\|_H^2\right) \le C, \qquad E\left(\left\|\left(A^{1/2}\psi\right)\right\|_H^2\right) \le C, \qquad E\int_0^T \|Af(t)\|_H^2 dt \le C, \tag{3.11}$$

then the estimate of convergence

$$\left(\sum_{k=1}^{N} E\|v(t_k) - u_k\|_H^2\right)^{1/2} \le C_1(\delta)\tau \tag{3.12}$$

holds. Here, $C_1(\delta)$ *does not depend on* τ.

Proof. Using the formula for the solution of second order difference equation and the definition of $c_\tau(k\tau)$ and $s_\tau(k\tau)$, we can write

$$u_k = c_\tau(k\tau)\varphi + s_\tau(k\tau)\psi + \tau\sum_{j=1}^{k}\left(s_\tau((k-j)\tau)\widetilde{f}_{2,j} + c_\tau((k-j)\tau)\widetilde{f}_{1,j}\right), \quad 1 \le k \le N. \tag{3.13}$$

Using (2.4) and (3.13), we obtain

$$\begin{aligned}
v(t_k) - u_k &= [c(k\tau) - c_\tau(k\tau)]\varphi + [s(k\tau) - s_\tau(k\tau)]\psi \\
&\quad + \tau\sum_{j=1}^{k}\big(s(t_k - t_j) f_{2,j} + c(t_k - t_j) f_{1,j}\big) \\
&\quad - \tau\sum_{j=1}^{k}\Big(s_\tau((k-j)\tau)\tilde{f}_{2,j} + c_\tau((k-j)\tau)\tilde{f}_{1,j}\Big) \\
&= [c(k\tau) - c_\tau(k\tau)]\varphi + [s(k\tau) - s_\tau(k\tau)]\psi \\
&\quad + \tau\sum_{j=1}^{k-1} s_\tau((k-j)\tau)\Big(f_{2,j} - \tilde{f}_{2,j}\Big) \\
&\quad + \tau\sum_{j=1}^{k-1}\big(s(t_k - t_j) - s_\tau((k-j)\tau)\big) f_{2,j} \\
&\quad + \tau\sum_{j=1}^{k} c_\tau((k-j)\tau)\Big(f_{1,j} - \tilde{f}_{1,j}\Big) \\
&\quad + \tau\sum_{j=1}^{k}\big(c(t_k - t_j) - c_\tau((k-j)\tau)\big) f_{1,j} \\
&= J_{1,k} + J_{2,k} + J_{3,k} + J_{4,k} + J_{5,k} + J_{6,k}, \quad 1 \le k \le N,
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
J_{1,k} &= [c(k\tau) - c_\tau(k\tau)]\varphi, \qquad J_{2,k} = [s(k\tau) - s_\tau(k\tau)]\psi, \\
J_{3,k} &= \tau\sum_{j=1}^{k-1} s_\tau((k-j)\tau)\Big(f_{2,j} - \tilde{f}_{2,j}\Big), \\
J_{4,k} &= \tau\sum_{j=1}^{k-1}\big(s(t_k - t_j) - s_\tau((k-j)\tau)\big) f_{2,j}, \\
J_{5,k} &= \tau\sum_{j=1}^{k} c_\tau((k-j)\tau)\Big(f_{1,j} - \tilde{f}_{1,j}\Big), \\
J_{6,k} &= \tau\sum_{j=1}^{k}\big(c(t_k - t_j) - c_\tau((k-j)\tau)\big) f_{1,j}.
\end{aligned} \tag{3.15}$$

Let us estimate the expected value of $J_{m,k}$ for all $m = 1,\dots,6$, separately. We start with $J_{1,k}$ and $J_{2,k}$. Using (3.6), (3.7), and (3.8), we obtain

$$\begin{aligned}
\left(\sum_{k=1}^{N} E\|J_{1,k}\|_H^2\right)^{1/2} &= \left(\sum_{k=1}^{N} E\left\|A^{-1}[c(k\tau)-c_\tau(k\tau)]A\varphi\right\|_H^2\right)^{1/2} \\
&\leq C\left(\sum_{k=1}^{N}\tau^3 E\|A\varphi\|_H^2\right)^{1/2} \leq \tau C\left(E\|A\varphi\|_H^2\right)^{1/2}, \\
\left(\sum_{k=1}^{N} E\|J_{2,k}\|_H^2\right)^{1/2} &= \left(\sum_{k=1}^{N} E\left\|A^{-1/2}[s(k\tau)-s_\tau(k\tau)]A^{1/2}\psi\right\|_H^2\right)^{1/2} \\
&\leq C\left(\sum_{k=1}^{N}\tau^3 E\left\|A^{1/2}\psi\right\|_H^2\right)^{1/2} \leq \tau C\left(E\left\|A^{1/2}\psi\right\|_H^2\right)^{1/2}.
\end{aligned} \tag{3.16}$$

Estimates for the expected value of $J_{m,k}$ for all $m = 3,\dots,6$, separately, were also used in paper [17]. Combining these estimates, we obtain (3.12). Theorem 3.2 is proved. □

4. Applications

First, let Λ be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n = \{x = (x_1,\dots,x_n) : 0 < x_i < 1,\ i = 1,\dots,n\}$ with boundary S, $\overline{\Lambda} = \Lambda \cup S$. In $[0,T]\times\Lambda$, the initial-boundary value problem for the following multidimensional hyperbolic equation:

$$\begin{gathered}
d\dot{u}(t,x) - \sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}dt = f(t,x)dw_t, \quad 0<t<T,\ x=(x_1,\dots,x_n)\in\Lambda, \\
u(0,x)=\varphi(x), \quad \dot{u}(0,x)=\psi(x), \quad x\in\overline{\Lambda}; \quad u(t,x)=0, \quad x\in S,\ 0\leq t\leq T
\end{gathered} \tag{4.1}$$

with the Dirichlet condition is considered. Here, $a_r(x)$, $(x\in\Lambda)$, $\delta\geq 0$ and $f(t,x)$ $(t\in(0,1),\ x\in\Lambda)$ are given smooth functions with respect to x and $a_r(x)\geq a>0$.

The discretization of (4.1) is carried out in two steps. In the first step, define the grid space $\widetilde{\Lambda}_h = \{x = x_m = (h_1m_1,\dots,h_nm_n); m=(m_1,\dots,m_n),\ 0\leq m_r\leq N_r,\ h_rN_r=1, r=1,\dots,n\}$, $\Lambda_h=\widetilde{\Lambda}_h\cap\Lambda$, $S_h=\widetilde{\Lambda}_h\cap S$.

Let L_{2h} denote the Hilbert space as

$$L_{2h}=L_2\left(\widetilde{\Lambda}_h\right)=\left\{\varphi^h(x):\left(\sum_{x\in\widetilde{\Lambda}_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n\right)^{1/2}<\infty\right\}. \tag{4.2}$$

The differential operator A in (4.1) is replaced with

$$A_h^x u^h(x) = -\sum_{r=1}^{n}\left(a_r(x)u_{\overline{x}_r}^h\right)_{x_r,j_r}+\delta u^h(x), \tag{4.3}$$

where the difference operator A_h^x is defined on these grid functions $u^h(x) = 0$, for all $x \in S_h$. As it is proved in [25], A_h^x is a self-adjoint positive definite operator in L_{2h}. Using (4.1) and (4.3), we get

$$\begin{gathered} d\dot{u}^h(t,x) + A_h^x u^h(t,x)dt = f^h(t,x)dw_t, \quad 0 < t < T, \ x \in \Lambda_h, \\ u^h(0,x) = \varphi^h(x), \quad \dot{u}^h(0,x) = \psi^h(x), \quad x \in \widetilde{\Lambda}_h. \end{gathered} \tag{4.4}$$

In the second step, we replace (4.4) with the difference scheme (3.5) as

$$\begin{gathered} \frac{1}{\tau^2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)\right) + \frac{1}{2}A_h^x u_k^h(x) \\ + \frac{1}{4}\left(A_h^x u_{k+1}^h(x) + A_h^x u_{k-1}^h(x)\right) = \varphi_k^h(x), \\ \varphi_k^h(x) = \frac{1}{\tau}\left[\left(I + \frac{1}{4}\tau^2 A_h^x\right)\varphi_{1,k+1}^h(x) + \tau\varphi_{2,k}^h(x) - \left(I - \frac{1}{4}\tau^2 A_h^x\right)\varphi_{1,k}^h(x)\right], \\ \varphi_{1,k}^h(x) = -\frac{1}{\tau}\int_{t_{k-1}}^{t_k}(z - t_k)f^h(z,x)dw_z, \qquad \varphi_{2,k}^h(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f^h(z,x)dw_z, \\ t_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = T, \quad x \in \Lambda_h, \\ u_1^h(x) = c_\tau(\tau)\varphi^h(x) + s_\tau(\tau)\psi^h(x) - \int_0^\tau (z-\tau)f^h(z,x)dw_z, \quad x \in \Lambda_h, \\ u_0^h(x) = \varphi^h(x). \end{gathered} \tag{4.5}$$

Theorem 4.1. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solution of difference scheme* (4.5) *satisfies the convergence estimate as*

$$\left(\sum_{k=1}^{N} E\left\|u^h(t_k) - u_k^h\right\|_{L_{2h}}^2\right)^{1/2} \le C(\delta)\left(\tau + |h|^2\right), \tag{4.6}$$

where $C(\delta)$ does not depend on τ and $|h|$.

The proof of Theorem 4.1 is based on the abstract Theorem 3.2 and the symmetry properties of the difference operator A_h^x defined by (4.3).

Second, in $[0,T] \times \Lambda$, the initial-boundary value problem for the following multidimensional hyperbolic equation:

$$\begin{gathered} d\dot{u}(t,x) - \sum_{r=1}^{n}\left(a_r(x)u_{x_r}\right)_{x_r}dt + \delta u(t,x)dt = f(t,x)dw_t, \\ 0 < t < T, \ x = (x_1,\dots,x_n) \in \Lambda, \\ u(0,x) = \varphi(x), \quad \dot{u}(0,x) = \psi(x), \quad x \in \overline{\Lambda}, \quad \frac{\partial u(t,x)}{\partial \vec{n}} = 0, \quad x \in S, \ 0 \le t \le T \end{gathered} \tag{4.7}$$

with the Neumann condition is considered. Here, $\vec{n}$ is the normal vector to Λ, $\delta > 0$, $a_r(x)$, $(x \in \Lambda)$, and $f(t,x)$ $(t \in (0,1),\ x \in \Lambda)$ are given smooth functions with respect to x and $a_r(x) \geq a > 0$.

The discretization of (4.7) is carried out in two steps. In the first step, the differential operator A in (4.7) is replaced with

$$A_h^x u^h(x) = -\sum_{r=1}^{n} \left(a_r(x) u_{\bar{x}_r}^h \right)_{x_r, j_r} + \delta u^h(x), \tag{4.8}$$

where the difference operator A_h^x is defined on those grid functions $D^h u^h(x) = 0$, for all $x \in S_h$, where $D^h u^h(x) = 0$ is the second order of approximation of $\partial u(t,x)/\partial \vec{n}$. As it is proved in [25], A_h^x is a self-adjoint positive definite operator in L_{2h}. Using (4.7) and (4.8), we get

$$\begin{gathered} d\dot{u}^h(t,x) + A_h^x u^h(t,x)dt = f^h(t,x)dw_t, \quad 0 < t < T,\ x \in \Lambda_h, \\ u^h(0,x) = \varphi^h(x), \quad \dot{u}^h(0,x) = \psi^h(x), \quad x \in \widetilde{\Lambda}_h. \end{gathered} \tag{4.9}$$

In the second step, we replace (4.9) with the difference scheme (3.5) as

$$\begin{gathered} \frac{1}{\tau^2}\left(u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x) \right) + \frac{1}{2} A_h^x u_k^h(x) + \frac{1}{4}\left(A_h^x u_{k+1}^h(x) + A_h^x u_{k-1}^h(x) \right) = \varphi_k^h(x), \\ \varphi_k^h(x) = \frac{1}{\tau}\left(\left(I + \frac{1}{4}\tau^2 A_h^x \right) \varphi_{1,k+1}^h(x) + \tau \varphi_{2,k}^h(x) - \left(I - \frac{1}{4}\tau^2 A_h^x \right) \varphi_{1,k}^h(x) \right), \\ \varphi_{1,k}^h(x) = -\frac{1}{\tau}\int_{t_{k-1}}^{t_k} (z - t_k) f^h(z,x) dw_z, \qquad \varphi_{2,k}^h(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f^h(z,x) dw_z, \\ t_k = k\tau, \quad 1 \leq k \leq N-1, \qquad N\tau = T, \quad x \in \Lambda_h, \\ u_1^h(x) = c_\tau(\tau)\varphi^h(x) + s_\tau(\tau)\psi^h(x) - \int_0^\tau (z-\tau) f^h(z,x) dw_z, \\ u_0^h(x) = \varphi^h(x), \quad x \in \Lambda_h. \end{gathered} \tag{4.10}$$

Theorem 4.2. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small numbers. Then, the solution of difference scheme* (4.10) *satisfies the convergence estimate as*

$$\left(\sum_{k=1}^{N} E \left\| u^h(t_k) - u_k^h \right\|_{L_{2h}}^2 \right)^{1/2} \leq C(\delta)\left(\tau + |h|^2 \right), \tag{4.11}$$

where $C(\delta)$ does not depend on τ and $|h|$.

The proof of Theorem 4.2 is based on the abstract Theorem 3.2 and the symmetry properties of the difference operator A_h^x defined by (4.8).

Third, in $[0,T]\times\Lambda$, the mixed boundary value problem for the following multidimensional hyperbolic equation:

$$\begin{gathered} d\dot{u}(t,x)-\sum_{r=1}^{n}(a_r(x)u_{x_r})_{x_r}dt+\delta u(t,x)dt=f(t,x)dw_t, \\ 0<t<T,\ x=(x_1,\dots,x_n)\in\Lambda, \\ u(0,x)=\varphi(x),\quad \dot{u}(0,x)=\psi(x),\quad x\in\overline{\Lambda}, \\ \frac{\partial u(t,x)}{\partial\vec{n}}=0,\quad x\in S_2,\ \ 0\le t\le T,\quad S_1\cup S_2=S, \\ u(t,x)=0,\quad x\in S_1 \end{gathered} \tag{4.12}$$

with the Dirichlet-Neumann condition is considered. Here, $\vec{n}$ is the normal vector to Λ, $\delta>0$, $a_r(x)$, $(x\in\Lambda)$, and $f(t,x)$ $(t\in(0,1),\ x\in\Lambda)$ are given smooth functions with respect to x and $a_r(x)\ge a>0$.

The discretization of (4.12) is carried out in two steps. In the first step, the differential operator A in (4.12) is replaced with

$$A_h^x u^h(x)=-\sum_{r=1}^{n}\left(a_r(x)u_{\overline{x}_r}^h\right)_{x_r,j_r}+\delta u^h(x), \tag{4.13}$$

where the difference operator A_h^x is defined on those grid functions $u^h(x)=0$, for all $x\in S_h^1$ and $D^h u^h(x)=0$, for all $x\in S_h^2$, $S_h^1\cup S_h^2=S_h$, where $D^h u^h(x)=0$ is the second order of approximation of $\partial u(t,x)/\partial\vec{n}$. By [25], we can conclude that A_h^x is a self-adjoint positive definite operator in L_{2h}. Using (4.12) and (4.13), we get

$$\begin{gathered} d\dot{u}^h(t,x)+A_h^x u^h(t,x)dt=f^h(t,x)dw_t,\quad 0<t<T,\ x\in\Lambda_h, \\ u^h(0,x)=\varphi(x),\quad \dot{u}^h(0,x)=\psi(x),\quad x\in\widetilde{\Lambda}_h. \end{gathered} \tag{4.14}$$

In the second step, we replace (4.14) with the difference scheme (3.5) as

$$\begin{gathered} \frac{1}{\tau^2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{1}{2}A_h^x u_k^h(x)+\frac{1}{4}\left(A_h^x u_{k+1}^h(x)+A_h^x u_{k-1}^h(x)\right)=\varphi_k^h(x), \\ \varphi_k^h(x)=\frac{1}{\tau}\left(\left(I+\frac{1}{4}\tau^2A_h^x\right)\varphi_{1,k+1}^h(x)+\tau\varphi_{2,k}^h(x)-\left(I-\frac{1}{4}\tau^2A_h^x\right)\varphi_{1,k}^h(x)\right), \end{gathered}$$

$$\varphi_{1,k}^h(x) = -\frac{1}{\tau}\int_{t_{k-1}}^{t_k}(z-t_k)f^h(z,x)dw_z, \varphi_{2,k}^h(x) = \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f^h(z,x)dw_z,$$
$$t_k = k\tau, \quad 1 \le k \le N-1, \qquad N\tau = T, \quad x \in \Lambda_h,$$
$$u_1^h(x) = c_\tau(\tau)\varphi^h(x) + s_\tau(\tau)\psi^h(x) - \int_0^\tau (z-\tau)f^h(z,x)dw_z,$$
$$u_0^h(x) = \varphi^h(x), \quad x \in \Lambda_h. \tag{4.15}$$

Theorem 4.3. *Let τ and $|h| = \sqrt{h_1^2 + \cdots + h_n^2}$ be sufficiently small positive numbers. Then, the solution of difference scheme* (4.15) *satisfies the convergence estimate as*

$$\left(\sum_{k=1}^N E\left\|u^h(t_k) - u_k^h\right\|_{L_{2h}}^2\right)^{1/2} \le C(\delta)\left(\tau + |h|^2\right), \tag{4.16}$$

where $C(\delta)$ does not depend on τ and $|h|$.

The proof of Theorem 4.3 is based on the abstract Theorem 3.2 and the symmetry properties of the difference operator A_h^x defined by (4.13).

5. Numerical Examples

In this section, we apply finite difference scheme (2.2) to four examples which are stochastic hyperbolic equation with Neumann, Dirichlet, Dirichlet-Neumann, and Neumann-Dirichlet conditions.

Example 5.1. The following initial-boundary value problem:

$$d\dot{u}(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2}dt + u(t,x)dt = f(t,x)dw_t,$$
$$f(t,x) = \sqrt{2}\cos x, \quad w_t = \sqrt{t}\xi, \quad 0 < t < 1, \quad 0 < x < \pi,$$
$$u(0,x) = \cos x, \quad \dot{u}(0,x) = 0, \quad 0 \le x \le \pi,$$
$$u_x(t,0) = u_x(t,\pi) = 0, \quad 0 \le t \le 1 \tag{5.1}$$

for a stochastic hyperbolic equation is considered. The exact solution of this problem is

$$u(t,x) = \int_0^t \sin\left(\sqrt{2}(t-s)\right)\cos x\, dw_s + \cos\left(\sqrt{2}t\right)\cos x. \tag{5.2}$$

For the approximate solution of the (5.1), we apply the finite difference scheme (2.2) and we get

$$\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau}+\frac{\tau}{2}\left[-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k\right]$$
$$+\frac{\tau}{4}\left[-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}+u_n^{k+1}-\frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2}+u_n^{k-1}\right]=f_n^k,$$
$$f_n^k=\sqrt{2}\xi\cos x_n\left[\sqrt{t_{k+1}}-\sqrt{t_{k-1}}+\frac{\tau^2-2}{2\tau}\left[\tau\left(\sqrt{t_{k+1}}-\sqrt{t_{k-1}}\right)-\frac{2}{3}\left(\sqrt{t_{k+1}^3}-\sqrt{t_k^3}-\sqrt{t_{k-1}^3}\right)\right]\right],$$
$$N\tau=1,\quad x_n=nh,\quad 1\le n\le M-1,\quad Mh=\pi,\quad 1\le k\le N-1,\quad t_k=k\tau,$$
$$u_n^0=\cos x_n,\quad u_n^1-u_n^0=\frac{4+\tau^2}{6}\sqrt{2\tau^3}\xi\cos x_n,\quad 1\le n\le M-1,$$
$$u_0^k=u_1^k,\quad u_M^k=u_{M-1}^k,\quad 1\le k\le N. \tag{5.3}$$

The system can be written in the following matrix form:

$$Au_{n+1}+Bu_n+Cu_{n-1}=D\varphi_n,\quad 1\le n\le M-1,$$
$$u_0=u_1,\qquad u_M=u_{M-1}. \tag{5.4}$$

Here,

$$A=\begin{bmatrix}0&0&0&0&\cdots&0&0&0&0\\ a&2a&a&0&\cdots&0&0&0&0\\ 0&a&2a&a&\cdots&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&\cdots&a&2a&a&0\\ 0&0&0&0&\cdots&0&a&2a&a\\ s_a&-s_a&0&0&\cdots&0&0&0&0\end{bmatrix}_{(N+1)\times(N+1)},$$
$$B=\begin{bmatrix}1&0&0&0&\cdots&0&0&0&0\\ b&c&b&0&\cdots&0&0&0&0\\ 0&b&c&b&\cdots&0&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots\\ 0&0&0&0&\cdots&b&c&b&0\\ 0&0&0&0&\cdots&0&b&c&b\\ s_b&-s_b&0&0&\cdots&0&0&0&0\end{bmatrix}_{(N+1)\times(N+1)}, \tag{5.5}$$

the matrix is $C = A$,

$$a = \frac{-\tau}{4h^2}, \qquad b = \frac{1}{\tau} + \frac{\tau}{2h^2} + \frac{\tau}{4}, \qquad c = \frac{-2}{\tau} + \frac{\tau}{h^2} + \frac{\tau}{2},$$
$$s_a = \frac{\tau^2}{4h^2}, \qquad s_b = 1 + \frac{\tau^2}{2h^2} + \frac{\tau^2}{4}, \tag{5.6}$$

$$f_n = \begin{bmatrix} f_n^0 \\ f_n^1 \\ f_n^2 \\ \vdots \\ f_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$f_n^k = \sqrt{2}\xi \cos x_n \left[\sqrt{t_{k+1}} - \sqrt{t_{k-1}} + \frac{\tau^2 - 2}{2\tau} \times \left[\tau\left(\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right) - \frac{2}{3}\left(\sqrt{t_{k+1}^3} - \sqrt{t_k^3} - \sqrt{t_{k-1}^3}\right)\right]\right], \quad 1 \le k \le N-1,$$
$$f_n^0 = \cos x_n, \quad 0 \le n \le M,$$
$$f_n^N = \frac{4+\tau^2}{6}\sqrt{2\tau^3}\xi \cos x_n, \quad 0 \le n \le M, \tag{5.7}$$

and $D = I_{N+1}$ is the identity matrix,

$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ u_s^2 \\ \vdots \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \quad s = n-1, n, n+1. \tag{5.8}$$

This type of system was used by [26] for difference equations. For the solution of matrix equation (5.4), we will use modified Gauss elimination method. We seek a solution of the matrix equation by the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1, \tag{5.9}$$

where $u_M = (I - \alpha_M)^{-1}\beta_M$, α_j $(j = 1, \ldots, M-1)$ are $(N+1) \times (N+1)$ square matrices, β_j $(j = 1, \ldots, M-1)$ are $(N+1) \times 1$ column matrices α_1 is an identity and β_1 is a zero matrices, and

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1} A,$$
$$\beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, 2, 3, \ldots, M-1. \tag{5.10}$$

Example 5.2. The following initial-boundary value problem:

$$\begin{gathered}
d\dot{u}(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2}dt + u(t,x)dt = f(t,x)dw_t, \\
f(t,x) = \sqrt{2}\sin x, \quad w_t = \sqrt{t}\xi, \quad 0 < t < 1,\ 0 < x < \pi, \\
u(0,x) = \sin x, \quad \dot{u}(0,x) = 0, \quad 0 \le x \le \pi, \\
u(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1
\end{gathered} \tag{5.11}$$

for a stochastic hyperbolic equation is considered. We use the same procedure as in the first example. The exact solution of this problem is

$$u(t,x) = \int_0^t \sin\left(\sqrt{2}(t-s)\right)\sin x\,dw_s + \cos\left(\sqrt{2}t\right)\sin x. \tag{5.12}$$

For the approximate solution of the (5.11), we can construct the following difference scheme:

$$\begin{gathered}
\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau} + \frac{\tau}{2}\left[-\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k\right] \\
\times \frac{\tau}{4}\left[-\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2} + u_n^{k+1} - \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} + u_n^{k-1}\right] = f_n^k, \\
f_n^k = \sqrt{2}\xi\sin x_n\left[\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right. \\
\left. + \frac{\tau^2 - 2}{2\tau}\left[\tau\left(\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right) - \frac{2}{3}\left(\sqrt{t_{k+1}^3} - \sqrt{t_k^3} - \sqrt{t_{k-1}^3}\right)\right]\right], \\
N\tau = 1, \quad x_n = nh, \quad 1 \le n \le M-1, \quad Mh = \pi, \quad 1 \le k \le N-1, \quad t_k = k\tau, \\
u_n^0 = \sin x_n, \quad u_n^1 - u_n^0 = \frac{4+\tau^2}{6}\sqrt{2\tau^3}\xi\sin x_n, \quad 1 \le n \le M-1, \\
u_0^k = u_M^k = 0, \quad 1 \le k \le N,
\end{gathered} \tag{5.13}$$

and it can be written in the following matrix form:

$$\begin{gathered}
Au_{n+1} + Bu_n + Cu_{n-1} = Df_n, \quad 1 \le n \le M-1, \\
u_0 = u_M = \vec{0}.
\end{gathered} \tag{5.14}$$

Here, the matrices A, B, C, D are given in the previous example, and

$$f_n = \begin{bmatrix} f_n^0 \\ f_n^1 \\ f_n^2 \\ \vdots \\ f_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\begin{gathered} f_n^k = \sqrt{2}\xi \sin x_n \left[\sqrt{t_{k+1}} - \sqrt{t_{k-1}} + \frac{\tau^2 - 2}{2\tau} \right. \\ \left. \times \left[\tau\left(\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right) - \frac{2}{3}\left(\sqrt{t_{k+1}^3} - \sqrt{t_k^3} - \sqrt{t_{k-1}^3}\right)\right]\right], \quad 1 \le k \le N-1, \\ f_n^0 = \sin x_n, \quad 0 \le n \le M, \\ f_n^N = \frac{4+\tau^2}{6}\sqrt{2\tau^3}\xi \sin x_n, \quad 0 \le n \le M. \end{gathered} \tag{5.15}$$

For the solution of matrix equation (5.14), we will use modified Gauss elimination method.

We seek a solution of the matrix equation in the following form:

$$u_n = \alpha_{n+1} u_{n+1} + \beta_{n+1}, \quad n = M-1, \ldots, 2, 1, \tag{5.16}$$

where $u_M = 0$, α_j $(j = 1, \ldots, M-1)$ are $(N+1)\times(N+1)$ square matrices, β_j $(j = 1, \ldots, M-1)$ are $(N+1)\times 1$ column matrices. α_1 and β_1 are zero matrices, and

$$\begin{gathered} \alpha_{n+1} = -(B + C\alpha_n)^{-1} A, \\ \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1, 2, 3, \ldots, M-1. \end{gathered} \tag{5.17}$$

Example 5.3. The following initial-boundary value problem:

$$\begin{gathered} d\dot{u}(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2} dt + u(t,x)dt = f(t,x)dw_t, \\ f(t,x) = \frac{\sqrt{5}}{2}\sin\left(\frac{x}{2}\right), \quad w_t = \sqrt{t}\xi, \quad 0 < t < 1, \ 0 < x < \pi, \\ u(0,x) = \sin\left(\frac{x}{2}\right), \quad \dot{u}(0,x) = 0, \quad 0 \le x \le \pi, \\ u(t,0) = u_x(t,\pi) = 0, \quad 0 \le t \le 1 \end{gathered} \tag{5.18}$$

for a stochastic hyperbolic equation is considered. The exact solution of this problem is

$$u(t,x) = \left[\int_0^t \sin\left(\frac{\sqrt{5}}{2}(t-s)\right)dw_s + \cos\left(\frac{\sqrt{5}}{2}t\right)\right]\sin\left(\frac{x}{2}\right). \tag{5.19}$$

We get the following difference scheme:

$$\begin{gathered}
\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau} + \frac{\tau}{2}\left[-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k\right]\frac{\tau}{4}\left[-\frac{u_{n+1}^{k+1}-2u_n^{k+1}+u_{n-1}^{k+1}}{h^2}\right. \\
\left. + u_n^{k+1} - \frac{u_{n+1}^{k-1}-2u_n^{k-1}+u_{n-1}^{k-1}}{h^2} + u_n^{k-1}\right] = f_n^k, \\
f_n^k = \frac{\sqrt{5}}{2}\xi\sin\frac{x_n}{2}\left[\sqrt{t_{k+1}}-\sqrt{t_{k-1}}+\frac{\tau^2-2}{2\tau}\right. \\
\left.\times\left[\tau\left(\sqrt{t_{k+1}}-\sqrt{t_{k-1}}\right)-\frac{2}{3}\left(\sqrt{t_{k+1}^3}-\sqrt{t_k^3}-\sqrt{t_{k-1}^3}\right)\right]\right], \\
N\tau = 1, \quad x_n = nh, \quad 1 \le n \le M-1, \quad Mh = \pi, \quad 1 \le k \le N-1, \quad t_k = k\tau, \\
u_n^0 = \sin\frac{x_n}{2}, \quad u_n^1 - u_n^0 = \frac{4+\tau^2}{12}\sqrt{5\tau^3}\xi\sin\frac{x_n}{2}, \quad 1 \le n \le M-1, \\
u_0^k = 0, \quad u_M^k = u_{M-1}^k, \quad 1 \le k \le N,
\end{gathered} \tag{5.20}$$

for the approximate solutions of (5.18), and we obtain the following matrix equation:

$$\begin{gathered}
Au_{n+1} + Bu_n + Cu_{n-1} = Df_n, \quad 1 \le n \le M-1, \\
u_0^k = 0, \quad u_M^k = u_{M-1}^k, \quad 1 \le k \le N.
\end{gathered} \tag{5.21}$$

Here, the matrices A, B, C, D are same as in the first example, and

$$f_n = \begin{bmatrix} f_n^0 \\ f_n^1 \\ f_n^2 \\ \vdots \\ f_n^N \end{bmatrix}_{(N+1)\times 1},$$

$$\begin{gathered}
f_n^k = \frac{\sqrt{5}}{2}\xi\sin\frac{x_n}{2}\left[\sqrt{t_{k+1}}-\sqrt{t_{k-1}}+\frac{\tau^2-2}{2\tau}\right. \\
\left.\times\left[\tau\left(\sqrt{t_{k+1}}-\sqrt{t_{k-1}}\right)-\frac{2}{3}\left(\sqrt{t_{k+1}^3}-\sqrt{t_k^3}-\sqrt{t_{k-1}^3}\right)\right]\right], \quad 1 \le k \le N-1,
\end{gathered}$$

$$f_n^0 = \sin\frac{x_n}{2}, \quad 0 \le n \le M,$$
$$f_n^N = \frac{4+\tau^2}{12}\sqrt{5\tau^3}\xi \sin\frac{x_n}{2}, \quad 0 \le n \le M. \tag{5.22}$$

For the solution of matrix equation (5.21), we use the same procedure as in the previous examples. Moreover, $u_M = \vec{0}$, α_1 is an identity and β_1 is a zero matrices, and

$$\alpha_{n+1} = -(B + C\alpha_n)^{-1}A, \qquad \beta_{n+1} = (B + C\alpha_n)^{-1}(D\varphi_n - C\beta_n), \quad n = 1,2,3,\ldots,M-1. \tag{5.23}$$

Example 5.4. The following initial boundary value problem:

$$\begin{gathered} d\dot{u}(t,x) - \frac{\partial^2 u(t,x)}{\partial x^2}dt + u(t,x)dt = f(t,x)dw_t, \\ f(t,x) = \frac{\sqrt{5}}{2}\cos\left(\frac{x}{2}\right), \quad w_t = \sqrt{t}\xi, \quad 0 < t < 1, \quad 0 < x < \pi, \\ u(0,x) = \cos\left(\frac{x}{2}\right), \quad \dot{u}(0,x) = 0, \quad 0 \le x \le \pi, \\ u_x(t,0) = u(t,\pi) = 0, \quad 0 \le t \le 1 \end{gathered} \tag{5.24}$$

for a stochastic hyperbolic equation is considered. The exact solution of this problem is

$$u(t,x) = \left[\int_0^t \sin\left(\frac{\sqrt{5}}{2}(t-s)\right)dw_s + \cos\left(\frac{\sqrt{5}}{2}t\right)\right]\cos\left(\frac{x}{2}\right). \tag{5.25}$$

The following difference scheme:

$$\begin{aligned} &\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau} + \frac{\tau}{2}\left[-\frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{h^2} + u_n^k\right]\frac{\tau}{4}\left[-\frac{u_{n+1}^{k+1} - 2u_n^{k+1} + u_{n-1}^{k+1}}{h^2}\right. \\ &\qquad\qquad \left. + u_n^{k+1} - \frac{u_{n+1}^{k-1} - 2u_n^{k-1} + u_{n-1}^{k-1}}{h^2} + u_n^{k-1}\right] = f_n^k, \\ &f_n^k = \frac{\sqrt{5}}{2}\xi\cos\frac{x_n}{2}\left[\sqrt{t_{k+1}} - \sqrt{t_{k-1}} + \frac{\tau^2 - 2}{2\tau}\right. \\ &\qquad\qquad \left.\times\left[\tau\left(\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right) - \frac{2}{3}\left(\sqrt{t_{k+1}^3} - \sqrt{t_k^3} - \sqrt{t_{k-1}^3}\right)\right]\right], \end{aligned}$$

$$N\tau = 1, \quad x_n = nh, \quad 1 \le n \le M-1, \quad Mh = \pi, \quad 1 \le k \le N-1, \quad t_k = k\tau,$$
$$u_n^0 = \cos\frac{x_n}{2}, \quad u_n^1 - u_n^0 = \frac{4+\tau^2}{12}\sqrt{5\tau^3}\xi\cos\frac{x_n}{2}, \quad 1 \le n \le M-1,$$
$$u_0^k = u_1^k, \quad u_M^k = 0, \quad 1 \le k \le N \tag{5.26}$$

is obtained for the approximate solutions of (5.24), and we obtain the following matrix equation:

$$\begin{gathered} Au_{n+1} + Bu_n + Cu_{n-1} = Df_n, \quad 1 \le n \le M-1, \\ u_1 = u_2, \qquad u_M = \vec{0}. \end{gathered} \tag{5.27}$$

Here, the matrices A, B, C, D are same as in the first example, and

$$f_n = \begin{bmatrix} f_n^0 \\ f_n^1 \\ f_n^2 \\ \vdots \\ f_n^N \end{bmatrix}_{(N+1)\times 1},$$
$$f_n^k = \frac{\sqrt{5}}{2}\xi\cos\frac{x_n}{2}\Bigg[\sqrt{t_{k+1}} - \sqrt{t_{k-1}} + \frac{\tau^2-2}{2\tau}$$
$$\times\left[\tau\left(\sqrt{t_{k+1}} - \sqrt{t_{k-1}}\right) - \frac{2}{3}\left(\sqrt{t_{k+1}^3} - \sqrt{t_k^3} - \sqrt{t_{k-1}^3}\right)\right]\Bigg], \quad 1 \le k \le N-1,$$
$$f_n^0 = \cos\frac{x_n}{2}, \quad 0 \le n \le M,$$
$$f_n^N = \frac{4+\tau^2}{12}\sqrt{5\tau^3}\xi\cos\frac{x_n}{2}, \quad 0 \le n \le M. \tag{5.28}$$

Using (5.27) that we get α_1 is an identity and β_1 is a zero matrices and $u_M = (I - \alpha_M)^{-1}\beta_M$. The rest are the same as in Example 5.3.

For these examples, the errors of the numerical solution derived by difference scheme (2.2) computed by

$$E_M^N = \max_{\substack{1\le k\le N-1, \\ 1\le n\le M-1}} \left(\sum_{k=1}^{N}\left|u(t_k, x_n) - u_n^k\right|^2\right)^{1/2} \tag{5.29}$$

and the results are given in Table 1.

The numerical solutions are recorded for different values of $N = M$, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). To obtain

Table 1

	$N = M = 10$	$N = M = 20$	$N = M = 40$
Example 5.1 (Neumann)	0.3028	0.1219	0.0554
Example 5.2 (Dirichlet)	0.4004	0.2137	0.1342
Example 5.3 (Dirichlet-Neumann)	0.3040	0.1145	0.0494
Example 5.4 (Neumann-Dirichlet)	0.3439	0.1844	0.0957

the results, we simulated the 1000 sample paths of Brownian motion for each level of discretization.

Thus, results show that the error is stable and decreases in an exponential manner.

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