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Research Article

A Note on Fractional Differential Equations with Fractional Separated Boundary Conditions

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We consider a new class of boundary value problems of nonlinear fractional differential equations with fractional separated boundary conditions. A connection between classical separated and fractional separated boundary conditions is developed. Some new existence and uniqueness results are obtained for this class of problems by using standard fixed point theorems. Some illustrative examples are also discussed.

1. Introduction

In this paper, we investigate the existence of solutions for a fractional boundary value problem with fractional separated boundary conditions given by

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,1],\ 1 < q \le 2, \\ \alpha_1 x(0) + \beta_1\left({}^cD^p x(0)\right) = \gamma_1, \quad \alpha_2 x(1) + \beta_2\left({}^cD^p x(1)\right) = \gamma_2, \quad 0 < p < 1, \end{gathered} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order q, f is a given continuous function, and $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2)$ are real constants, with $\alpha_1 \neq 0$.

Fractional calculus has recently gained much momentum as extensive and significant progress on theoretical and practical aspects of the subject can easily be witnessed in the literature. As a matter of fact, the tools of fractional calculus have been effectively applied in the modelling of many physical and engineering problems. The recent development in the theory and methods for fractional calculus indicates its popularity. For some recent work on fractional boundary value problems, See [1–16] and the references therein.

2. Linear Problem

Let us recall some basic definitions of fractional calculus [1, 3].

Definition 2.1. For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)ds, \quad n-1<q<n,\ n=[q]+1, \tag{2.1}$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$ I^{q}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}ds, \quad q>0, \tag{2.2}$$

provided the integral exists.

To define the solution of the boundary value problem (1.1) we need the following lemma, which deals with linear variant of the problem (1.1).

Lemma 2.3. *For a given $\sigma \in C([0,1],\mathbb{R})$, the unique solution of the problem*

$$\begin{gathered} {}^{c}D^{q}x(t) = \sigma(t), \quad t\in[0,1],\ 1<q\le 2, \\ \alpha_1 x(0) + \beta_1\left({}^{c}D^{p}x(0)\right) = \gamma_1, \quad \alpha_2 x(1) + \beta_2\left({}^{c}D^{p}x(1)\right) = \gamma_2, \quad 0<p<1, \end{gathered} \tag{2.3}$$

is given by

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \frac{t}{\nu_1}\left(\alpha_2\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \beta_2\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}\sigma(s)ds\right) \\ & + \frac{\alpha_1\nu_2 t + \gamma_1\nu_1}{\alpha_1\nu_1}, \end{aligned} \tag{2.4}$$

where

$$ \nu_1 = \frac{\alpha_2\Gamma(2-p)+\beta_2}{\Gamma(2-p)}, \qquad \nu_2 = \frac{\gamma_2\alpha_1 - \alpha_2\gamma_1}{\alpha_1}. \tag{2.5}$$

Proof. It is well known [3] that the solution of fractional differential equation in (2.3) can be written as

$$x(t) = I^q\sigma(t) - b_1 - b_2 t = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_1 - b_2 t. \tag{2.6}$$

Using ${}^cD^p b = 0$ (b is a constant), ${}^cD^p t = t^{1-p}/\Gamma(2-p)$, ${}^cD^p I^q\sigma(t) = I^{q-p}\sigma(t)$, (2.6) gives

$${}^cD^p x(t) = \int_0^t \frac{(t-s)^{q-p-1}}{\Gamma(q-p)}\sigma(s)ds - b_2\frac{t^{1-p}}{\Gamma(2-p)}. \tag{2.7}$$

From the boundary condition $\alpha_1 x(0) + \beta_1({}^cD^p x(0)) = \gamma_1$, we have

$$\alpha_1(-b_1) + \beta_1(0) = \gamma_1, \quad \text{which implies that } b_1 = -\frac{\gamma_1}{\alpha_1}. \tag{2.8}$$

By the boundary condition $\alpha_2 x(1) + \beta_2({}^cD^p x(1)) = \gamma_2$, we get

$$\alpha_2\left(\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - b_1 - b_2\right) + \beta_2\left(\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}\sigma(s)ds - \frac{b_2}{\Gamma(2-p)}\right) = \gamma_2, \tag{2.9}$$

which, on inserting the value of b_1, gives

$$-b_2\left(\frac{\alpha_2\Gamma(2-p)+\beta_2}{\Gamma(2-p)}\right) = \frac{\gamma_2\alpha_1 - \alpha_2\gamma_1}{\alpha_1} - \alpha_2\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \beta_2\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}\sigma(s)ds. \tag{2.10}$$

Using (2.5) in the above equation, we obtain

$$b_2 = -\frac{\nu_2}{\nu_1} + \frac{1}{\nu_1}\left(\alpha_2\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \beta_2\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}\sigma(s)ds\right). \tag{2.11}$$

Substituting the values of b_1 and b_2 in (2.6), we get (2.4). □

Remark 2.4. In the limit $p \to 1^-$, it has been observed that the solution (2.4) of problem (2.3) is not reduced to the solution of the resulting problem given by

$$\begin{gathered} {}^cD^q x(t) = f(t, x(t)), \quad t \in [0,1],\ 1 < q \le 2, \\ \alpha_1 x(0) + \beta_1 x'(0) = \gamma_1, \qquad \alpha_2 x(1) + \beta_2 x'(1) = \gamma_2. \end{gathered} \tag{2.12}$$

The solution of (2.12) is

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \frac{(\beta_1 - \alpha_1 t)}{\Delta}\left(\alpha_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds + \beta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds\right)$$
$$+ \frac{1}{\Delta}(\gamma_1(\alpha_2+\beta_2) - \gamma_2\beta_1 + (\gamma_2\alpha_1 - \gamma_1\alpha_2)t), \tag{2.13}$$

where $\Delta = \alpha_1(\alpha_2+\beta_2) - \alpha_2\beta_1 \neq 0$. However, we notice that the solution (2.4) of problem (2.3) does not depend on the parameter β_1 (appearing in the boundary conditions of (2.3)). Thus we conclude that the parameter β_1 is of arbitrary nature. Furthermore, it has been found that the solutions (2.4) and (2.13) coincide by taking $\beta_1 = 0$ in (2.13). Hence, for a particular choice of $\beta_1 = 0$ in problems (2.3) and (2.12), the two problems have the same solution.

3. Main Results

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$.

In view of Lemma 2.3, we define an operator $\mathbf{F} : \mathcal{C} \to \mathcal{C}$ by

$$(\mathbf{F}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds$$
$$-\frac{t}{\nu_1}\left(\alpha_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \beta_2 \int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\right) + \frac{\alpha_1\nu_2 t + \gamma_1\nu_1}{\alpha_1\nu_1}. \tag{3.1}$$

Observe that the problem (1.1) has solutions if and only if the operator equation $\mathbf{F}x = x$ has fixed points.

Now we are in a position to present our main results. The methods used to prove the existence results are standard; however, their exposition in the framework of problem (1.1) is new.

Theorem 3.1. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies the following assumption:*

(A_1) $|f(t,x) - f(t,y)| \leq L|x-y|$, *for all* $t \in [0,1]$, $L > 0$, $x, y \in \mathbb{R}$.

Then the boundary value problem (1.1) *has a unique solution provided*

$$\frac{L}{\Gamma(q+1)}\left(1 + \frac{|\alpha_2|}{|\nu_1|}\right) + \frac{|\beta_2|}{|\nu_1|}\frac{L}{\Gamma(q-p+1)} < 1. \tag{3.2}$$

Proof. Setting $\sup_{t\in[0,1]}|f(t,0)| = M < \infty$ and choosing $r \geq (\Lambda M + N)/(1 - L\Lambda)$, where

$$\Lambda = \frac{1}{\Gamma(q+1)}\left(1 + \frac{|\alpha_2|}{|\nu_1|}\right) + \frac{|\beta_2|}{|\nu_1|}\frac{1}{\Gamma(q-p+1)}, \qquad N = \frac{|\alpha_1\nu_2| + |\gamma_1\nu_1|}{|\alpha_1\nu_1|}, \tag{3.3}$$

we show that $\mathbf{F}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
\|(\mathbf{F}x)(t)\| &\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds + \frac{|\alpha_2 t|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s))|ds \right.\\
&\qquad\left. + \frac{|\beta_2 t|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}|f(s,x(s))|ds + \frac{|\alpha_1\nu_2 t + \gamma_1\nu_1|}{|\alpha_1\nu_1|}\right\}\\
&\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds \right.\\
&\qquad + \frac{|\alpha_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds\\
&\qquad\left. + \frac{|\beta_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)ds + \frac{|\alpha_1\nu_2 + \gamma_1\nu_1|}{|\alpha_1\nu_1|}\right\}\\
&\leq (Lr + M)\sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}ds + \frac{|\alpha_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}ds \right.\\
&\qquad\left. + \frac{|\beta_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}ds\right\} + \frac{|\alpha_1\nu_2 t + \gamma_1\nu_1|}{|\alpha_1\nu_1|}\\
&\leq (Lr + M)\left\{\frac{1}{\Gamma(q+1)} + \frac{|\alpha_2|}{|\nu_1|}\frac{1}{\Gamma(q+1)} + \frac{|\beta_2|}{|\nu_1|}\frac{1}{\Gamma(q-p+1)}\right\} + \frac{|\alpha_1\nu_2| + |\gamma_1\nu_1|}{|\alpha_1\nu_1|}\\
&= (Lr + M)\Lambda + N \leq r.
\end{aligned} \tag{3.4}$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$\begin{aligned}
\|(\mathbf{F}x)(t) - (\mathbf{F}y)(t)\| &\leq \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds \right.\\
&\qquad + \frac{|\alpha_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|f(s,x(s)) - f(s,y(s))|ds\\
&\qquad\left. + \frac{|\beta_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)}|f(s,x(s)) - f(s,y(s))|ds\right\}\\
&\leq L\left\{\frac{1}{\Gamma(q+1)} + \frac{|\alpha_2|}{|\nu_1|}\frac{1}{\Gamma(q+1)} + \frac{|\beta_2|}{|\nu_1|}\frac{1}{\Gamma(q-p+1)}\right\}\|x - y\|.
\end{aligned} \tag{3.5}$$

As $(L/\Gamma(q+1))(1+|\alpha_2|/|\nu_1|)+(|\beta_2|/|\nu_1|)(L/\Gamma(q-p+1)) < 1$, therefore $\mathbf{F}$ is a contraction. Thus, the conclusion of the theorem followed by the contraction mapping principle (Banach fixed point theorem). □

Example 3.2. Consider the following fractional boundary value problem

$$
\begin{gathered}
{}^cD^{3/2}x(t) = \frac{1}{(t+2)^2}\frac{|x|}{1+|x|}, \quad t \in [0,1], \\
x(0) + \beta_1\left({}^cD^{1/2}x(0)\right) = \frac{1}{2}, \qquad \frac{1}{2}x(1) + \frac{1}{3}\left({}^cD^{1/2}\,x(1)\right) = 2.
\end{gathered} \tag{3.6}
$$

Here, $q = 3/2$, $p = 1/2$, $\alpha_1 = 1$, $\alpha_2 = 1/2$, $\beta_2 = 1/3$, $\gamma_1 = 1/2$, $\gamma_2 = 2$, β_1 is arbitrary, and $f(t,x) = (1/(t+2)^2)(|x|/1+|x|)$. As $|f(t,x) - f(t,y)| \le (1/4)|x-y|$, therefore, (A_1) is satisfied with $L = 1/4$. Further, $\nu_1 = 1/2 + 2/3\sqrt{\pi}$, $\nu_2 = 7/4$ and

$$
\frac{L}{\Gamma(q+1)}\left(1+\frac{|\alpha_2|}{|\nu_1|}\right) + \frac{|\beta_2|}{|\nu_1|}\frac{L}{\Gamma(q-p+1)} = \frac{1}{3\sqrt{\pi}} + \frac{2+\sqrt{\pi}}{2(3\sqrt{\pi}+4)} \simeq 0.390505 < 1. \tag{3.7}
$$

Thus, by the conclusion of Theorem 3.1, the boundary value problem (3.6) has a unique solution on $[0,1]$.

Now, we prove the existence of solutions of (1.1) by applying Krasnoselskii's fixed point theorem [17].

Theorem 3.3 (Krasnoselskii's fixed point theorem). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.4. *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a jointly continuous function satisfying the assumption* (A_1). *In addition one assumes that*

(A_2) $|f(t,x)| \le \mu(t)$, *for all* $(t,x) \in [0,1] \times \mathbb{R}$, *and* $\mu \in C([0,1], \mathbb{R}^+)$.

Then the problem (1.1) *has at least one solution on* $[0,1]$ *if*

$$
\frac{|\alpha_2|}{|\nu_1|}\frac{L}{\Gamma(q+1)} + \frac{|\beta_2|}{|\nu_1|}\frac{L}{\Gamma(q-p+1)} < 1. \tag{3.8}
$$

Proof. Letting $\sup_{t\in[0,1]}|\mu(t)| = \|\mu\|$, we choose a real number $\overline{r}$ satisfying the inequality

$$
\overline{r} \ge \|\mu\|\left\{\frac{1}{\Gamma(q+1)} + \frac{|\alpha_2|}{|\nu_1|}\frac{1}{\Gamma(q+1)} + \frac{|\beta_2|}{|\nu_1|}\frac{1}{\Gamma(q-p+1)} + \frac{|\alpha_1\nu_2| + |\gamma_1\nu_1|}{|\alpha_1\nu_1|}\right\}, \tag{3.9}
$$

and consider $B_{\overline{r}} = \{x \in C : \|x\| \leq \overline{r}\}$. We define the operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\overline{r}}$ as

$$(\mathcal{P}x)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))ds,$$
$$(\mathcal{Q}x)(t) = -\frac{t}{\nu_1}\left(\alpha_2 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,x(s))ds + \beta_2 \int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)} f(s,x(s))ds\right) \quad (3.10)$$
$$+ \frac{\alpha_1\nu_2 t + \gamma_1\nu_1}{\alpha_1\nu_1}.$$

For $x, y \in B_{\overline{r}}$, we find that

$$\|\mathcal{P}x + \mathcal{Q}y\| \leq \|\mu\|\left\{\frac{1}{\Gamma(q+1)} + \frac{|\alpha_2|}{|\nu_1|}\frac{1}{\Gamma(q+1)} + \frac{|\beta_2|}{|\nu_1|}\frac{1}{\Gamma(q-p+1)} + \frac{|\alpha_1\nu_2| + |\gamma_1\nu_1|}{|\alpha_1\nu_1|}\right\} \leq \overline{r}. \quad (3.11)$$

Thus, $\mathcal{P}x + \mathcal{Q}y \in B_{\overline{r}}$. It follows from the assumption (A_1) together with (3.8) that $\mathcal{Q}$ is a contraction mapping. Continuity of f implies that the operator $\mathcal{P}$ is continuous. Also, $\mathcal{P}$ is uniformly bounded on $B_{\overline{r}}$ as

$$\|\mathcal{P}x\| \leq \frac{\|\mu\|}{\Gamma(q+1)}. \quad (3.12)$$

Now we prove the compactness of the operator $\mathcal{P}$.

In view of (A_1), we define $\sup_{(t,x)\in[0,1]\times B_{\overline{r}}}|f(t,x)| = \overline{f}$, and consequently we have

$$|(\mathcal{P}x)(t_1) - (\mathcal{P}x)(t_2)| = \left|\frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_2-s)^{q-1} - (t_1-s)^{q-1}\right] f(s,x(s))ds + \int_{t_1}^{t_2}(t_2-s)^{q-1} f(s,x(s))ds\right|$$
$$\leq \frac{\overline{f}}{\Gamma(q+1)}\left|2(t_2-t_1)^q + t_1^q - t_2^q\right|, \quad (3.13)$$

which is independent of x. Thus, $\mathcal{P}$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}$ is compact on $B_{\overline{r}}$. Thus, all the assumptions of Theorem 3.3 are satisfied. So the conclusion of Theorem 3.3 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$. □

Our next existence result is based on Leray-Schauder nonlinear alternative [18].

Lemma 3.5 (nonlinear alternative for single-valued maps). *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (i.e., $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in* $\overline{U}$, *or*

(ii) *there is a* $u \in \partial U$ *(the boundary of* U *in* C*) and* $\lambda \in (0,1)$ *with* $u = \lambda F(u)$.

Theorem 3.6. *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be a jointly continuous function. Assume that:*

(A_3) *there exist a function* $p \in C([0,1], \mathbb{R}^+)$ *and a nondecreasing function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $|f(t,x)| \le p(t)\psi(\|x\|)$, *for all* $(t,x) \in [0,1] \times \mathbb{R}$;

(A_4) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\{(1/\Gamma(q+1))(1+|\alpha_2|/|\nu_1|)+(|\beta_2|/|\nu_1|\Gamma(q-p+1))\}\|p\|+(|\alpha_1\nu_2|+|\gamma_1\nu_1|)/|\alpha_1\nu_1|} > 1. \tag{3.14}$$

Then the boundary value problem (1.1) *has at least one solution on* $[0,1]$.

Proof. Consider the operator $F : \mathcal{C} \to \mathcal{C}$ defined by (3.1). We show that F *maps bounded sets into bounded sets in* $C([0,1], \mathbb{R})$. For a positive number r, let $B_r = \{x \in C([0,1], \mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1], \mathbb{R})$. Then

$$\begin{aligned}
|(Fx)(t)| &\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds + \frac{|\alpha_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds \\
&\quad + \frac{|\beta_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)} p(s)\psi(\|x\|)ds + \frac{|\alpha_1\nu_2 + \gamma_1\nu_1|}{|\alpha_1\nu_1|} \\
&\le \psi(\|x\|)\left\{\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)ds + \frac{|\alpha_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)ds \right. \\
&\quad \left. + \frac{|\beta_2|}{|\nu_1|}\int_0^1 \frac{(1-s)^{q-p-1}}{\Gamma(q-p)} p(s)ds\right\} + \frac{|\alpha_1\nu_2| + |\gamma_1\nu_1|}{|\alpha_1\nu_1|} \\
&\le \psi(\|x\|)\left\{\frac{1}{\Gamma(q+1)}\left(1 + \frac{|\alpha_2|}{|\nu_1|}\right) + \frac{|\beta_2|}{|\nu_1|\Gamma(q-p+1)}\right\}\|p\| + \frac{|\alpha_1|\nu_2 + |\gamma_1\nu_1|}{|\alpha_1\nu_1|}.
\end{aligned} \tag{3.15}$$

Thus

$$\|Fx\| \le \psi(r)\left\{\frac{1}{\Gamma(q+1)}\left(1 + \frac{|\alpha_2|}{|\nu_1|}\right) + \frac{|\beta_2|}{|\nu_1|\Gamma(q-p+1)}\right\}\|p\| + \frac{|\alpha_1\nu_2 + \gamma_1\nu_1|}{|\alpha_1\nu_1|}. \tag{3.16}$$

Next we show that F *maps bounded sets into equicontinuous sets of* $C([0,1],\mathbb{R})$. Let $t', t'' \in [0,1]$ with $t' < t''$ and $x \in B_r$, where B_r is a bounded set of $C([0,1],\mathbb{R})$. Then we obtain

$$\begin{aligned}
\left|(Fx)(t'') - (Fx)(t')\right| &= \left|\frac{1}{\Gamma(q)}\int_0^{t''}(t''-s)^{q-1}f(s,x(s))ds\right.\\
&\quad\left.-\frac{1}{\Gamma(q)}\int_0^{t'}(t'-s)^{q-1}f(s,x(s))ds\right| + \frac{|\nu_2|}{|\nu_1|}\left|t''-t'\right|\\
&\leq \left|\frac{1}{\Gamma(q)}\int_0^{t'}\left[(t''-s)^{q-1}-(t'-s)^{q-1}\right]\psi(r)p(s)ds\right|\\
&\quad+\left|\frac{1}{\Gamma(q)}\int_{t'}^{t''}(t''-s)^{q-1}\psi(r)p(s)ds\right| + \frac{|\nu_2|}{|\nu_1|}\left|t''-t'\right|.
\end{aligned} \tag{3.17}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $t'' - t' \to 0$. As F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $F : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

Let x be a solution. Then, for $t \in [0,1]$ and using the computations in proving that F is bounded, we have

$$|x(t)| = |\lambda(Fx)(t)| \leq \psi(\|x\|)\left\{\frac{1}{\Gamma(q+1)}\left(1+\frac{|\alpha_2|}{|\nu_1|}\right)+\frac{|\beta_2|}{|\nu_1|\Gamma(q-p+1)}\right\}\|p\| + \frac{|\alpha_1\nu_2|+|\gamma_1\nu_1|}{|\alpha_1\nu_1|}. \tag{3.18}$$

Consequently, we have

$$\frac{\|x\|}{\psi(\|x\|)\{(1/\Gamma(q+1))(1+|\alpha_2|/|\nu_1|)+(|\beta_2|/|\nu_1|\Gamma(q-p+1))\}\|p\|+(|\alpha_1\nu_2|+|\gamma_1\nu_1|)/|\alpha_1\nu_1|} \leq 1. \tag{3.19}$$

In view of (A_4), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M+1\}. \tag{3.20}$$

Note that the operator $F : \overline{U} \to C([0,1],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda F(x)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.5), we deduce that F has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

In the special case when $p(t) = 1$ and $\psi(|x|) = \kappa|x| + N$ we have the following corollary.

Corollary 3.7. *Let* $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *be a continuous function. Assume that there exist constants* $0 \leq \kappa < 1/\Lambda_1$, *where* $\Lambda_1 = (1/\Gamma(q+1)(1+|\alpha_2|/|\nu_1|)) + (|\beta_2|/|\nu_1|\Gamma(q-p+1))$ *and* $N_1 > 0$ *such that* $|f(t,x)| \leq \kappa|x| + N_1$ *for all* $t \in [0,1]$, $x \in C[0,1]$. *Then the boundary value problem* (1.1) *has at least one solution.*

Example 3.8. *Consider the following boundary value problem:*

$$
\begin{gathered}
{}^{c}D^{3/2}x(t) = \frac{1}{(6\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|}, \quad t \in [0,1], \\
x(0) + \beta_1\left({}^{c}D^{1/2}x(0)\right) = \frac{1}{2}, \qquad \frac{1}{2}x(1) + \frac{1}{3}\left({}^{c}D^{1/2}x(1)\right) = 2.
\end{gathered} \tag{3.21}
$$

Here, $q = 3/2$, $p = 1/2$, $\alpha_1 = 1$, $\alpha_2 = 1/2$, $\beta_2 = 1/3$, $\gamma_1 = 1/2$, $\gamma_2 = 2$, β_1 is arbitrary, and

$$
\left|f(t,x)\right| = \left|\frac{1}{(6\pi)}\sin(2\pi x) + \frac{|x|}{1+|x|}\right| \leq \frac{1}{3}|x| + 1. \tag{3.22}
$$

Clearly $N_1 = 1$ and

$$
\kappa = \frac{1}{3} < \frac{1}{\Lambda_1} = 0.640196. \tag{3.23}
$$

Thus, all the conditions of Corollary 3.7 are satisfied, and consequently the problem (3.21) has at least one solution.

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References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2008.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[4] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publisher, West Redding, Conn, USA, 2006.

[5] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Eds., *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, The Netherlands, 2007.

[6] R. P. Agarwal, M. Belmekki, and M. Benchohra, "A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative," *Advances in Difference Equations*, Article ID 981728, p. 47, 2009.

[7] B. Ahmad and S. Sivasundaram, "Existence and uniqueness results for nonlinear boundary value problems of fractional differential equations with separated boundary conditions," *Communications in Applied Analysis*, vol. 13, no. 1, pp. 121–127, 2009.

[8] B. Ahmad and J. J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1838–1843, 2009.

[9] B. Ahmad, "Existence of solutions for irregular boundary value problems of nonlinear fractional differential equations," *Applied Mathematics Letters*, vol. 23, no. 4, pp. 390–394, 2010.

[10] B. Ahmad and J. J. Nieto, "Existence of solutions for anti-periodic boundary value problems involving fractional differential equations via Leray-Schauder degree theory," *Topological Methods in Nonlinear Analysis*, vol. 35, no. 2, pp. 295–304, 2010.

[11] D. Baleanu, O. G. Mustafa, and R. P. Agarwal, "An existence result for a superlinear fractional differential equation," *Applied Mathematics Letters*, vol. 23, no. 9, pp. 1129–1132, 2010.

[12] E. Hernandez, D. O'Regan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis*, vol. 73, no. 10, pp. 3462–3471, 2010.

[13] B. Ahmad and S. Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 480–487, 2010.

[14] B. Ahmad, S. K. Ntouyas, and A. Alsaedi, "New existence results for nonlinear fractional differential equations with three-point integral boundary conditions," *Advances in Difference Equations*, Article ID 107384, p. 11, 2011.

[15] B. Ahmad and S. K. Ntouyas, "A four-point nonlocal integral boundary value problem for fractional differential equations of arbitrary order," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 22, p. 15, 2011.

[16] B. Ahmad and R. P. Agarwal, "On nonlocal fractional boundary value problems," *Dynamics of Continuous, Discrete & Impulsive Systems Series A*, vol. 18, no. 4, pp. 535–544, 2011.

[17] M. A. Krasnoselskii, "Two remarks on the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, pp. 123–127, 1955.

[18] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2005.