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Research Article

Nonoscillatory Solutions of Second-Order Superlinear Dynamic Equations with Integrable Coefficients

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The asymptotic behavior of nonoscillatory solutions of the superlinear dynamic equation on time scales $(r(t)x^{\Delta}(t))^{\Delta}+p(t)|x(\sigma(t))|^{\gamma}\operatorname{sgn}x(\sigma(t)) = 0$, $\gamma > 1$, is discussed under the condition that $P(t) = \lim_{\tau\to\infty}\int_t^{\tau} p(s)\Delta s$ exists and $P(t) \geq 0$ for large t.

1. Introduction

Consider the second-order superlinear dynamic equation

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)|x(\sigma(t))|^{\gamma} \operatorname{sgn} x(\sigma(t)) = 0, \qquad \gamma > 1, \tag{1.1}$$

where

$$P(t) = \lim_{\tau\to\infty}\int_t^{\tau} p(s)\Delta s \tag{1.2}$$

exists and is finite. $P(t) \geq 0$ for large t.

When $\mathbb{T} = \mathbb{R}$, $r(t) = 1$, (1.1) is the second-order superlinear differential equation

$$x''(t) + p(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \qquad \gamma > 1. \tag{1.3}$$

When $\mathbb{T} = \mathbb{N}$, $r(n) = 1$, (1.1) is the second-order superlinear difference equation

$$\Delta^2 x(n) + p(n)|x(n+1)|^\gamma \operatorname{sgn} x(n+1) = 0, \quad \gamma > 1. \tag{1.4}$$

The following condition is introduced in [1].

Condition (H). *We say that $\mathbb{T}$ satisfies Condition (H) provided one of the following holds.*

(1) *There exists a strictly increasing sequence $\{t_n\}_{n=0}^{\infty} \subset \mathbb{T}$ with $\lim_{n\to\infty} t_n = \infty$ and for each $n \geq 0$ either $\sigma(t_n) = t_{n+1}$ or the real interval $[t_n, t_{n+1}] \subset \mathbb{T}$;*

(2) *$\mathbb{T} \cap \mathbb{R} = [T', \infty)$ for some $T' \in \mathbb{T}$.*

We note that time scales which satisfy Condition (H) include most of the important time scales, such as $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0}$, where $q > 1$ and $\mathbb{N}_0$ is the nonnegative integers and harmonic numbers $\{\sum_{k=1}^{n} 1/k : n \in \mathbb{N}\}$ [2, Example 1.45].

In [3], Naito proved the following result.

Theorem 1.1. *If $P(t) = \int_t^\infty p(s)ds \geq 0$ for large t, then a nonoscillatory solution $x(t)$ of (1.3) satisfies exactly one of the following three asymptotic properties:*

$$\lim_{t\to\infty} x(t) = c \neq 0,$$
$$\lim_{t\to\infty} \frac{x(t)}{t} = 0, \qquad \lim_{t\to\infty} x(t) = \pm\infty, \tag{1.5}$$
$$\lim_{t\to\infty} \frac{x(t)}{t} = c \neq 0.$$

In this paper, we extend Theorem 1.1 to superlinear dynamic equation (1.1) on time scale. As an application, we get the asymptotic behavior of each nonoscillation solution of the difference equation

$$\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{(-1)^n b}{n^c}\right)|x(n+1)|^\gamma \operatorname{sgn} x(n+1) = 0, \quad \gamma > 1, \tag{1.6}$$

where $b > 0$, $c > 1$, and $a/c > b/2$.

2. Main Theorems

Consider the second-order nonlinear dynamic equation

$$\left(r(t)x^\Delta(t)\right)^\Delta + p(t)|x(\sigma(t))|^\gamma \operatorname{sgn} x(\sigma(t)) = 0, \quad \gamma > 0, \tag{2.1}$$

where $r(t), p(t) \in C(\mathbb{T}, R)$, $r(t) > 0$, $t_0 \in \mathbb{T}$, and $\int_{t_0}^\infty [r(t)]^{-1} dt = \infty$. $\lim_{t\to\infty} \int_{t_0}^t p(s)\Delta s$ exists and is finite.

Lemma 2.1. *Suppose that* $\mathbb{T}$ *satisfies Condition* (H). *If* $x(t)$ *is a positive solution of* (1.1) *on* $[T,\infty)$, *then the integral equation*

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} = \alpha + P(t) + \int_{t}^{\infty} \frac{r(s)\gamma\int_0^1 \left[x(s)+h\mu(s)x^{\Delta}(s)\right]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s \tag{2.2}$$

is satisfied for $t \geq T$, *where* α *is a nonnegative constant.*

Proof. Suppose that $x(t)$ is a positive solution of (1.1) on $[T,\infty)$.

In the first place, we will prove

$$\int_{T}^{\infty} \frac{r(s)\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s < \infty, \tag{2.3}$$

where $x_h(t) = x(t) + h\mu(t)x^{\Delta}(t) = (1-h)x(t) + hx(\sigma(t)) > 0$.

Multiplying both sides of (1.1) by $1/x^{\gamma}(\sigma(t))$, we get that

$$\left(\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)}\right)^{\Delta} = -p(t) - \frac{r(t)\gamma\int_0^1 [x_h(t)]^{\gamma-1}dh\left[x^{\Delta}(t)\right]^2}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \tag{2.4}$$

Integrating from T to t,

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} - \frac{r(T)x^{\Delta}(T)}{x^{\gamma}(T)} = -\int_{T}^{t} p(s)\Delta s - \int_{T}^{t} \frac{r(s)\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s. \tag{2.5}$$

If (2.3) fails to hold, that is,

$$\int_{T}^{\infty} \frac{r(s)\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s = \infty, \tag{2.6}$$

from (2.5), we have

$$\lim_{t\to\infty} \frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} = -\infty. \tag{2.7}$$

Without loss of generality, we can assume that for $t \geq T$

$$\frac{r(T)x^{\Delta}(T)}{x^{\gamma}(T)} - \int_{T}^{t} p(s)\Delta s < -1. \tag{2.8}$$

Otherwise, let $L = \max_{t\geq T}|\int_T^t p(s)\Delta s|$. By (2.7), we can take a large $T_1 > T$ such that $r(T_1)x^{\Delta}(T_1)/x^{\gamma}(T_1) < -(2L+1)$. So we have

$$\begin{aligned}\frac{r(T_1)x^{\Delta}(T_1)}{x^{\gamma}(T_1)} - \int_{T_1}^{t} p(s)\Delta s &< -(2L+1) - \left[\int_T^t p(s)\Delta s - \int_T^{T_1} p(s)\Delta s\right] \\ &\leq -(2L+1) - [-2L] = -1.\end{aligned} \tag{2.9}$$

So we can replace T by $T_1 > T$ such that (2.8) still holds.

From (2.5) and (2.8), we get for $t \geq T$

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} + \int_T^t \frac{r(s)\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s < -1. \tag{2.10}$$

In particular, we have

$$x^{\Delta}(t) < 0, \quad \text{for } t \geq T. \tag{2.11}$$

Therefore, $x(t)$ is strictly decreasing.

Assume that $t = t_{i-1} < t_i = \sigma(t)$. Then, $x(\sigma(t)) < x(t)$, and so

$$\begin{aligned}\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh &= \gamma\int_0^1 [(1-h)x(s) + hx(\sigma(s))]^{\gamma-1}dh \\ &= \frac{[(1-h)x(s) + hx(\sigma(s))]^{\gamma}\big|_0^1}{x(\sigma(s)) - x(s)} = \frac{x^{\gamma}(\sigma(s)) - x^{\gamma}(s)}{x(\sigma(s)) - x(s)}.\end{aligned} \tag{2.12}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, then, for $s \in [t_{i-1}, t_i]$, we have

$$\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh = \gamma x^{\gamma-1}(s). \tag{2.13}$$

Let

$$y(t) := 1 + \int_T^t \frac{r(s)\gamma\int_0^1 [x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma-1}(\sigma(s))}\Delta s. \tag{2.14}$$

Hence, from (2.10), we get that

$$-\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} > y(t). \tag{2.15}$$

From (2.14) and (2.15), we get that

$$y^{\Delta}(t)=\frac{r(t)\gamma\int_0^1[x_h(t)]^{\gamma-1}dh\left[x^{\Delta}(t)\right]^2}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}$$
$$>y(t)\frac{\gamma\int_0^1[x_h(t)]^{\gamma-1}dh\left[-x^{\Delta}(t)\right]}{x^{\gamma}(\sigma(t))}. \tag{2.16}$$

Assume that $t=t_{i-1}<t_i=\sigma(t)$. From (2.16) and (2.12), we get that

$$\frac{y(\sigma(t))-y(t)}{y(t)(\sigma(t)-t)}>\frac{x^{\gamma}(\sigma(t))-x^{\gamma}(t)}{x(\sigma(t))-x(t)}\frac{x(t)-x(\sigma(t))}{x^{\gamma}(\sigma(t))[\sigma(t)-t]}. \tag{2.17}$$

So,

$$\frac{y(\sigma(t))}{y(t)}>\frac{x^{\gamma}(t)}{x^{\gamma}(\sigma(t))}, \tag{2.18}$$

that is,

$$\frac{y(t_i)}{y(t_{i-1})}>\frac{x^{\gamma}(t_{i-1})}{x^{\gamma}(t_i)}. \tag{2.19}$$

If the real interval $[t_{i-1},t_i]\subset\mathbb{T}$, then, for $t\in(t_{i-1},t_i]$, it follows from (2.16) and (2.13) that

$$\frac{y'(t)}{y(t)}>\frac{\gamma x^{\gamma-1}(t)[-x'(t)]}{x^{\gamma}(t)}, \tag{2.20}$$

that is,

$$(\ln y(t))'>-(\ln x^{\gamma}(t))'. \tag{2.21}$$

Integrating from t_{i-1} to t, we get that

$$\frac{y(t)}{y(t_{i-1})}>\frac{x^{\gamma}(t_{i-1})}{x^{\gamma}(t)},\qquad t\in(t_{i-1},t_i]. \tag{2.22}$$

Let $T=t_{n_0}$, and let $t\in(T,\infty)_{\mathbb{T}}$. Then, there is an $n>n_0$ such that $t\in(t_{n-1},t_n]_{\mathbb{T}}$. From (2.22) and (2.19), we get that

$$\frac{y(t)}{y(t_{n-1})}>\frac{x^{\gamma}(t_{n-1})}{x^{\gamma}(t)},\ \frac{y(t_{n-1})}{y(t_{n-2})}>\frac{x^{\gamma}(t_{n-2})}{x^{\gamma}(t_{n-1})},\dots,\ \frac{y(t_{n_0+1})}{y(t_{n_0})}>\frac{x^{\gamma}(t_{n_0})}{x^{\gamma}(t_{n_0+1})}. \tag{2.23}$$

Multiplying, we get that

$$\frac{y(t)}{y(t_{n_0})} > \frac{x^{\gamma}(t_{n_0})}{x^{\gamma}(t)}. \tag{2.24}$$

Using (2.15) again, we get

$$-\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} > y(t) > \frac{y(t_{n_0})x^{\gamma}(t_{n_0})}{x^{\gamma}(t)}. \tag{2.25}$$

If we set $L := y(t_{n_0})x^{\gamma}(t_{n_0})$, we get

$$x^{\Delta}(t) < -\frac{L}{r(t)}. \tag{2.26}$$

Integrating from T to t, we get that

$$x(t) - x(T) < -\int_T^t \frac{L}{r(s)}\Delta s \longrightarrow -\infty, \qquad \text{as } t \longrightarrow \infty, \tag{2.27}$$

which contradicts $x(t) > 0$.

In (2.5), letting $t \to \infty$, replacing T by τ, and denoting $\alpha = \lim_{t\to\infty} r(t)x^{\Delta}(t)/x^{\gamma}(t)$, we get that

$$\alpha + \int_{\tau}^{\infty} p(s)\Delta s + \int_{\tau}^{\infty} \frac{r(s)\gamma \int_0^1 (x_h(s))^{\gamma-1} dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s = \frac{r(\tau)x^{\Delta}(\tau)}{x^{\gamma}(\tau)}. \tag{2.28}$$

We need to show that $\alpha \geq 0$.

Suppose that $\alpha < 0$. Then, there exists a large T_1 such that, for $t > T_1$, we have

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)} \leq \frac{\alpha}{2}. \tag{2.29}$$

So,

$$x^{\Delta}(t) \leq \frac{\alpha x^{\gamma}(t)}{2r(t)}. \tag{2.30}$$

Thus,

$$\begin{aligned} M(T_1) =: &\int_{T_1}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s \\ &\geq -\frac{\alpha}{2}\int_{T_1}^{\infty} \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh\left[-x^{\Delta}(s)\right]}{x^{\gamma}(\sigma(s))}\Delta s. \end{aligned} \tag{2.31}$$

Assume that $t = t_{i-1} < t_i = \sigma(t)$. From (2.12) and $x^{\Delta}(t) < 0$, we have

$$\begin{aligned}
\int_t^{\sigma(t)} \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh\left[-x^{\Delta}(s)\right]}{x^{\gamma}(\sigma(s))} \Delta s
&= \frac{\gamma \int_0^1 [x_h(t)]^{\gamma-1} dh\left[-x^{\Delta}(t)\right](\sigma(t)-t)}{x^{\gamma}(\sigma(t))} \\
&= \frac{x^{\gamma}(t) - x^{\gamma}(\sigma(t))}{x^{\gamma}(\sigma(t))} \\
&\geq \int_{x^{\gamma}(\sigma(t))}^{x^{\gamma}(t)} \frac{1}{v} dv \\
&= \ln \frac{x^{\gamma}(t)}{x^{\gamma}(\sigma(t))} \\
&= \ln \frac{x^{\gamma}(t_{i-1})}{x^{\gamma}(t_i)}.
\end{aligned} \tag{2.32}$$

If the real interval $[t_{i-1}, t_i] \subset \mathbb{T}$, from (2.13) we have, for $t \in (t_{i-1}, t_i]$,

$$\begin{aligned}
\int_{t_{i-1}}^{t} \frac{\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh\left[-x^{\Delta}(s)\right]}{x^{\gamma}(\sigma(s))} \Delta s
&= \int_{t_{i-1}}^{t} \frac{\gamma x^{\gamma-1}(s)\left[-x'(s)\right]}{x^{\gamma}(s)} ds \\
&= \ln \frac{x^{\gamma}(t_{i-1})}{x^{\gamma}(t)}.
\end{aligned} \tag{2.33}$$

From (2.31), (2.32), (2.33) and the additivity of the integral, it is easy to get

$$M(T_1) \geq -\frac{\alpha}{2} \lim_{u \to \infty} \ln \frac{x^{\gamma}(T_1)}{x^{\gamma}(u)}. \tag{2.34}$$

So, for large u, we have

$$\ln \frac{x^{\gamma}(T_1)}{x^{\gamma}(u)} \leq -\frac{2M(T_1)}{\alpha} + 1. \tag{2.35}$$

Thus,

$$x^{\gamma}(u) \geq x^{\gamma}(T_1) \exp\left(\frac{2M(T_1)}{\alpha} - 1\right). \tag{2.36}$$

By (2.30) and noticing that $\alpha < 0$, we get that

$$x^{\Delta}(u) \le \frac{\alpha x^{\gamma}(T_1)}{2r(u)} \exp\left(\frac{2M(T_1)}{\alpha} - 1\right). \tag{2.37}$$

Integrating (2.37), we get that $x(u) \to -\infty$, which is a contradiction.

This completes the proof of the lemma. □

Consider the second-order superlinear dynamic equation

$$\left[r(t)x^{\Delta}(t)\right]^{\Delta} + p(t)|x(\sigma(t))|^{\gamma} \operatorname{sgn} x(\sigma(t)) = 0, \qquad \gamma > 1, \tag{2.38}$$

where $r(t) > 0$, $\int_T^{\infty}(1/r(s))\Delta s = \infty$,

$$P(t) = \lim_{\tau \to \infty} \int_t^{\tau} p(s)\Delta s \tag{2.39}$$

exists and is finite, and $P(t) \ge 0$ for $t \ge T$.

Theorem 2.2. *Suppose that $\mathbb{T}$ satisfies Condition (H) and $P(t) \ge 0$ for $t \ge T$. Then each nonoscillatory solution $x(t)$ of* (2.38) *satisfies exactly one of the following three asymptotic properties:*

$$\lim_{t \to \infty} x(t) = c \ne 0, \tag{2.40}$$

$$\lim_{t \to \infty} \frac{x(t)}{\int_T^t \Delta s / r(s)} = 0, \qquad \lim_{t \to \infty} x(t) = \pm\infty, \tag{2.41}$$

$$\lim_{t \to \infty} \frac{x(t)}{\int_T^t \Delta s / r(s)} = c \ne 0. \tag{2.42}$$

Proof. Let $x(t)$ be a nonoscillatory solution of (2.38), say, $x(t) > 0$ for $t \ge T > 0$. From Lemma 2.1, it is known that $x(t)$ satisfies the equality

$$\begin{aligned} r(t)x^{\Delta}(t) &= \alpha x^{\gamma}(t) + P(t)x^{\gamma}(t) \\ &\quad + \gamma x^{\gamma}(t) \int_t^{\infty} \frac{r(s)\int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh \left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))} \Delta s \end{aligned} \tag{2.43}$$

for $t \ge T$. Therefore, we have

$$r(t)x^{\Delta}(t) \ge P(t)x^{\gamma}(t), \tag{2.44}$$

for $t \geq T$. Since $P(t) \geq 0$ for $t \geq T$, it follows that $x^{\Delta}(t) \geq 0$ for $t \geq T$. An integration by parts of (2.38) gives

$$\begin{aligned} r(u)x^{\Delta}(u) - P(u)x^{\gamma}(u) + \gamma \int_t^u P(s)x^{\Delta}(s) \int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh\, \Delta s \\ = r(t)x^{\Delta}(t) - P(t)x^{\gamma}(t), \end{aligned} \tag{2.45}$$

where $u \geq t \geq T$. Let t be fixed. Since $P(s)x^{\Delta}(s)\int_0^1 [x(s)+h\mu(s)x^{\Delta}(s)]^{\gamma-1}dh$ is nonnegative, the integral term in (2.45) has a finite limit or diverges to ∞ as $t \to \infty$. If the latter case occurs, then $r(u)x^{\Delta}(u) - P(u)x^{\gamma}(u) \to -\infty$ as $u \to \infty$, which is a contradiction to (2.44). Thus, the former case occurs, that is,

$$\int_t^{\infty} P(s)x^{\Delta}(s) \int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh\, \Delta s < \infty. \tag{2.46}$$

Define the function $k_1(t)$ as

$$k_1(t) = \int_t^{\infty} P(s)x^{\Delta}(s) \int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh\, \Delta s < \infty \tag{2.47}$$

and the finite constant $\beta = \lim_{u\to\infty}[r(u)x^{\Delta}(u) - P(u)x^{\gamma}(u)]$. Then, equality (2.45) yields

$$r(t)x^{\Delta}(t) = \beta + P(t)x^{\gamma}(t) + \gamma k_1(t), \tag{2.48}$$

for $t \geq T$. Observe by (2.44) that $\beta \geq 0$. From (2.44) and (2.46), it follows that

$$\int_t^{\infty} \frac{P^2(s)x^{\gamma}(s)}{r(s)} \int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh\, \Delta s \leq k_1(t) < \infty. \tag{2.49}$$

Next, we define the function $k_2(t)$ by (noticing that $\gamma > 1$)

$$\begin{aligned} k_2(t) &= \int_t^{\infty} \frac{P^2(s)x^{\gamma}(s)}{r(s)} \int_0^1 \left[x(s) + h\mu(s)x^{\Delta}(s)\right]^{\gamma-1} dh\, \Delta s \\ &= \int_t^{\infty} \frac{P^2(s)x^{\gamma}(s)}{r(s)} \int_0^1 [(1-h)x(s) + hx(\sigma(s))]^{\gamma-1} dh\, \Delta s \\ &\geq \int_t^{\infty} \frac{P^2(s)x^{\gamma}(s)}{r(s)} \int_0^1 [(1-h)x(s) + hx(s)]^{\gamma-1} dh\, \Delta s \\ &= \int_t^{\infty} \frac{P^2(s)x^{2\gamma-1}(s)}{r(s)} \Delta s \end{aligned} \tag{2.50}$$

for $t \geq T$. Dividing (2.48) by $r(t)$ and integrating from T to t, we get

$$x(t) = x(T) + \int_T^t \frac{\beta}{r(s)}\Delta s + \int_T^t \frac{P(s)x^\gamma(s)}{r(s)}\Delta s + \gamma \int_T^t \frac{k_1(s)}{r(s)}\Delta s. \tag{2.51}$$

By Schwartz's inequality and the fact that $x^\Delta(t) \geq 0$ for $t \geq T$, the second integral term in (2.51) can be estimated as follows:

$$\begin{aligned}\int_T^t \frac{P(s)x^\gamma(s)}{r(s)}\Delta s &\leq \left(\int_T^t \frac{P^2(s)x^{2\gamma-1}(s)}{r(s)}\Delta s\right)^{1/2}\left(\int_T^t \frac{x(s)}{r(s)}\Delta s\right)^{1/2} \\ &\leq k_2^{1/2}(T)\left(\int_T^t \frac{\Delta s}{r(s)}\right)^{1/2} x^{1/2}(t),\end{aligned} \tag{2.52}$$

for $t \geq T$. From (2.51) and (2.52) and noticing that $k_1(t)$ is decreasing, we get that

$$x(t) \leq x(T) + \int_T^t \frac{\beta}{r(s)}\Delta s + k_2^{1/2}(T)\left(\int_T^t \frac{\Delta s}{r(s)}\right)^{1/2} x^{1/2}(t) + \gamma k_1(T)\int_T^t \frac{\Delta s}{r(s)}. \tag{2.53}$$

The above inequality may be regarded as a quadratic inequality in $x^{1/2}(t)$. Then, we have

$$x^{1/2}(t) \leq \frac{1}{2}k_2^{1/2}(T)\left(\int_T^t \frac{\Delta s}{r(s)}\right)^{1/2} + \frac{1}{2}D^{1/2}(t) \tag{2.54}$$

for $t \geq T$, where

$$D(t) = k_2(T)\int_T^t \frac{\Delta s}{r(s)} + 4\left[x(T) + (\beta + \gamma k_1(T))\int_T^t \frac{\Delta s}{r(s)}\right]. \tag{2.55}$$

It is obvious that $D(t) = O(\int_T^t \Delta s/r(s))$ as $t \to \infty$, and, consequently, there exists a positive constant m such that

$$x(t) \leq m\int_T^t \frac{\Delta s}{r(s)} \tag{2.56}$$

for $t \geq T$. Let $T_1(\geq T)$ be an arbitrary number. It is clear that

$$0 \leq \int_T^t \frac{P(s)x^\gamma(s)}{r(s)}\Delta s = \int_T^{T_1} \frac{P(s)x^\gamma(s)}{r(s)}\Delta s + \int_{T_1}^t \frac{P(s)x^\gamma(s)}{r(s)}\Delta s \tag{2.57}$$

for $t \geq T_1$. Arguing as in (2.52), we find

$$\int_{T_1}^{t} \frac{P(s)x^{\gamma}(s)}{r(s)}\Delta s \leq \left[k_2(T_1)x(t)\int_{T_1}^{t}\frac{\Delta s}{r(s)}\right]^{1/2} \tag{2.58}$$

for $t \geq T_1$, which when combined with (2.56) yields

$$\begin{aligned}\int_{T_1}^{t} \frac{P(s)x^{\gamma}(s)}{r(s)}\Delta s &\leq \left[mk_2(T_1)\int_{T}^{t}\frac{\Delta s}{r(s)}\int_{T_1}^{t}\frac{\Delta s}{r(s)}\right]^{1/2} \\ &\leq [mk_2(T_1)]^{1/2}\int_{T}^{t}\frac{\Delta s}{r(s)},\end{aligned} \tag{2.59}$$

for $t \geq T_1 \geq T$. Using (2.57) and (2.59) and noticing that $\int_T^{\infty}\Delta s/r(s) = \infty$, we obtain

$$0 \leq \limsup_{t\to\infty}\frac{1}{\int_T^t \Delta s/r(s)}\int_{T}^{t}\frac{P(s)x^{\gamma}(s)}{r(s)}\Delta s \leq [mk_2(T_1)]^{1/2}. \tag{2.60}$$

Since T_1 is arbitary and $k_2(T_1) \to 0$ as $T_1 \to \infty$, letting $T_1 \to \infty$ in (2.60), we get

$$0 \leq \lim_{t\to\infty}\frac{1}{\int_T^t \Delta s/r(s)}\int_{T}^{t}\frac{P(s)x^{\gamma}(s)}{r(s)}\Delta s = 0. \tag{2.61}$$

Using L'Hospital's rule of time scale (see Theorem 1.119 of [2]), we have

$$\lim_{t\to\infty}\frac{\int_T^t (k_1(s)/r(s))\Delta s}{\int_T^t \Delta s/r(s)} = \lim_{t\to\infty} k_1(t) = 0. \tag{2.62}$$

In view of (2.51), (2.61), and (2.62), we find $\lim_{t\to\infty}x(t)/\int_T^t \Delta s/r(s) = \beta$.

Recall that $x(t)$ is nondecreasing for $t \geq T$. Now, there are three cases to consider:

(i) $\beta = 0$ and $x(t)$ is bounded above,

(ii) $\beta = 0$ and $x(t)$ is unbounded,

(iii) $\beta > 0$ (and hence $x(t)$ is unbounded).

Case (i) implies (2.40) with $c = \lim_{t\to\infty}x(t) > 0$, while case (iii) implies (2.42) with $c = \beta > 0$. It is also clear that case (ii) implies (2.41). This completes the proof. □

The following lemma is from [1].

Lemma 2.3. *Suppose that $\mathbb{T}$ satisfies Condition (H). $x(t) > 0$ is a solution of* (1.1). *Then, one has*

$$\int_{T}^{t}\frac{x^{\Delta}(s)}{x^{\gamma}(\sigma(s))}\Delta s \leq \frac{x^{-\gamma+1}(T) - x^{-\gamma+1}(t)}{\gamma - 1} \leq \frac{x^{-\gamma+1}(T)}{\gamma-1}. \tag{2.63}$$

Using Lemma 2.1, we can prove the following corollary.

Corollary 2.4. *Under the assumptions of Lemma 2.1, if $x(t)$ is a positive solution of* (1.1) *on* $[T,\infty)$, *then the integral equation*

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))}=\alpha+P(t)+\int_{\sigma(t)}^{\infty}\frac{r(s)\left[\int_0^1\gamma(x_h(s))^{\gamma-1}dh\right]\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s \tag{2.64}$$

is satisfied for $t\geq T$, where $P(t)=\int_t^{\infty}p(s)\Delta s$, $x_h(s)=x(s)+h\mu(s)x^{\Delta}(s)$.

Proof. In the left side of (2.2), using

$$\begin{aligned}\frac{1}{x^{\gamma}(t)}&=\frac{1}{x^{\gamma}(\sigma(t))}-\left(\frac{1}{x^{\gamma}(t)}\right)^{\Delta}\mu(t)\\&=\frac{1}{x^{\gamma}(\sigma(t))}+\frac{\int_0^1\gamma(x_h(t))^{\gamma-1}dhx^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\mu(t)\end{aligned} \tag{2.65}$$

and using (2.2), (2.65), the additivity of the integral, and [2, Theorem 1.75], we have that

$$\begin{aligned}\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(t)}&=\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))}+\frac{r(t)\int_0^1\gamma(x_h(t))^{\gamma-1}dh\left[x^{\Delta}(t)\right]^2}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\mu(t)\\&=\alpha+P(t)+\int_{\sigma(t)}^{\infty}\frac{r(s)\left[\int_0^1\gamma(x_h(s))^{\gamma-1}dh\right]\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s\\&\quad+\int_t^{\sigma(t)}\frac{r(s)\left[\int_0^1\gamma(x_h(s))^{\gamma-1}dh\right]\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s\\&=\alpha+P(t)+\int_{\sigma(t)}^{\infty}\frac{r(s)\left[\int_0^1\gamma(x_h(s))^{\gamma-1}dh\right]\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s\\&\quad+\frac{r(t)\left[\int_0^1\gamma(x_h(t))^{\gamma-1}dh\right]\left[x^{\Delta}(t)\right]^2}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}\mu(t).\end{aligned} \tag{2.66}$$

From (2.66), we get (2.64). □

The following theorem can be regarded as a time scale version of [4, Theorem 1].

Theorem 2.5. *Suppose that $\mathbb{T}$ satisfies Condition (H), $r(t)>0$ with $\int_T^{\infty}[r(t)]^{-1}\Delta t=\infty$, and suppose that $\lim_{t\to\infty}\int_T^t p(s)\Delta s$ exists and is finite. Let $P(t)=\int_t^{\infty}p(s)\Delta s$. Then, the superlinear dynamic equation* (1.1) *is oscillatory if*

$$\limsup_{t\to\infty}\int_T^t\frac{P(s)}{r(s)}\Delta s=\infty. \tag{2.67}$$

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1) on $[T,\infty)$. Without loss of generality, assume that $x(t)$ is positive for $t \in [T,\infty)$. From Corollary 2.4, $x(t)$ satisfies the integral equation (2.64). Dropping the last integral term in (2.64), we have the inequality

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))} \geq P(t). \tag{2.68}$$

Dividing (2.68) by $r(t)$, integrating from T to t, and using Lemma 2.3, we find

$$\frac{x^{-\gamma+1}(T)}{\gamma-1} \geq \int_T^t \frac{x^{\Delta}(s)}{x^{\gamma}(\sigma(s))}\Delta s \geq \int_T^t \frac{P(s)}{r(s)}\Delta s. \tag{2.69}$$

This contradicts (2.67), and so (1.1) is oscillatory. □

Consider the second-order superlinear dynamic equation with forced term

$$\left[r(t)x^{\Delta}(t)\right]^{\Delta} + p(t)|x(\sigma(t))|^{\gamma}\operatorname{sgn} x(\sigma(t)) = h(t), \qquad \gamma > 1, \tag{2.70}$$

where $r(t) > 0$, $\int_T^{\infty}(1/r(s))\Delta s = \infty$, and

$$P(t) = \lim_{\tau\to\infty}\int_t^{\tau} p(s)\Delta s \tag{2.71}$$

exists and is finite.

Lemma 2.6. *Suppose that*

$$\int_T^{\infty}|h(s)|\Delta s < +\infty. \tag{2.72}$$

If $x(t)$ is a positive solution of (2.70) *and* $\liminf_{t\to\infty}x(t) > 0$, *then*

$$\int_T^{\infty}\frac{r(s)\gamma\int_0^1[x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s < \infty, \tag{2.73}$$

$$\begin{aligned}\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))} = \alpha &+ \int_t^{\infty}\left[p(s) - \frac{h(s)}{x^{\gamma}(\sigma(s))}\right]\Delta s \\ &+ \int_{\sigma(t)}^{\infty}\frac{r(s)\gamma\int_0^1[x_h(s)]^{\gamma-1}dh\left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))}\Delta s\end{aligned} \tag{2.74}$$

are satisfied for sufficiently large t, where $x_h(s) = x(s) + h\mu(s)x^{\Delta}(s)$, α is a nonnegative constant.

Proof. The fact that $\liminf_{t\to\infty} x(t) > 0$ implies the existence of $t_1 \geq T$ and $m > 0$ such that $x(t) \geq m$ for $t \geq t_1$. Then, using (2.72), we find

$$\left|\int_{t_1}^{t} \frac{h(s)}{x^{\gamma}(\sigma(s))}\Delta s\right| \leq \int_{t_1}^{t}\left|\frac{h(s)}{x^{\gamma}(\sigma(s))}\right|\Delta s \leq \frac{1}{m^{\gamma}}\int_{t_1}^{t}|h(s)|\Delta s \leq M, \qquad t \geq t_1, \tag{2.75}$$

where M is some finite positive constant.

So, $\lim_{\tau\to\infty}\int_t^{\tau}[p(s) - h(s)/x^{\gamma}(\sigma(s))]\Delta s$ exists and is finite.

Similar to the proof of Lemma 2.1 and Corollary 2.4, it is easy to know that (2.73) and (2.74) hold. □

For subsequent results, we define

$$\Phi_0(t) = \int_t^{\infty}\left[p(s) - k|h(s)|\right]\Delta s, \qquad t \geq T, \tag{2.76}$$

where k is a positive constant. It is noted that, if (2.71) and (2.72) hold, then $\Phi_0(t)$ is finite for any k. Assume that $\Phi_0(t) > 0$ for sufficiently large t. Define, for a positive integer n and a positive constant ρ, the following functions:

$$\Phi_1(t) = \int_{\sigma(t)}^{\infty}\frac{[\Phi_0(s)]^2}{r(s)}\Delta s,$$
$$\Phi_{n+1}(t) = \int_{\sigma(t)}^{\infty}\frac{\left[\Phi_0(s) + \rho\Phi_n(s)\right]^2}{r(s)}\Delta s. \tag{2.77}$$

We introduce the following condition.

Condition (A)**.** *For every $\rho > 0$, there exists a positive integer N such that $\Phi_n(t)$ is finite for $n = 1, 2, \ldots, N-1$ and $\Phi_N(t)$ is infinite.*

Theorem 2.7. *Suppose that* (2.71), (2.72), *and Condition (A) hold. Then, every solution $x(t)$ of* (2.70) *is either oscillatory or satisfies*

$$\liminf_{t\to\infty} x(t) = 0. \tag{2.78}$$

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (2.70) and $\liminf_{t\to\infty}|x(t)| > 0$. Without loss of generality, let $x(t)$ be eventually positive. By Lemma 2.6, $x(t)$ satisfies (2.73) and (2.74). Further, there exist $t_1 \geq T$ and $m > 0$ such that $x(t) \geq m$ for $t \geq t_1$. Let

$$\Phi_0(t) = \int_t^{\infty}\left[p(s) - \frac{|h(s)|}{m^{\gamma}}\right]\Delta s. \tag{2.79}$$

Then, from (2.74) we find

$$\begin{aligned}\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))} &\geq \Phi_0(t) + \int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh \left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))} \Delta s \\ &\geq \Phi_0(t) > 0,\end{aligned} \tag{2.80}$$

for $t \geq t_1$. From (2.80), we get

$$x^{\Delta}(t) \geq \frac{\Phi_0(t)x^{\gamma}(\sigma(t))}{r(t)} > 0, \qquad t \geq t_1. \tag{2.81}$$

Applying (2.81) and noticing that $\gamma > 1$, we find for $t \geq t_1$

$$\begin{aligned}&\int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh \left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))} \Delta s \\ &\quad\geq \int_{\sigma(t)}^{\infty} \frac{\gamma[x(s)]^{\gamma-1}[\Phi_0(s)x^{\gamma}(\sigma(s))]^2}{r(s)x^{\gamma}(s)x^{\gamma}(\sigma(s))} \Delta s \\ &\quad\geq \gamma m^{\gamma-1} \int_{\sigma(t)}^{\infty} \frac{[\Phi_0(s)]^2}{r(s)} \Delta s = \gamma m^{\gamma-1} \Phi_1(t).\end{aligned} \tag{2.82}$$

If $N = 1$ in Condition (A), then the right side of (2.82) is infinite. This is a contradiction to (2.73).

Next, it follows from (2.80) and (2.82) that

$$\frac{r(t)x^{\Delta}(t)}{x^{\gamma}(\sigma(t))} \geq \Phi_0(t) + \gamma m^{\gamma-1} \Phi_1(t). \tag{2.83}$$

Using a similar technique and relations (2.83), we get

$$\begin{aligned}&\int_{\sigma(t)}^{\infty} \frac{r(s)\gamma \int_0^1 [x_h(s)]^{\gamma-1} dh \left[x^{\Delta}(s)\right]^2}{x^{\gamma}(s)x^{\gamma}(\sigma(s))} \Delta s \\ &\quad\geq \gamma m^{\gamma-1} \int_{\sigma(t)}^{\infty} \frac{\left[\Phi_0(s) + \gamma m^{\gamma-1}\Phi_1(s)\right]^2}{r(s)} \Delta s = \gamma m^{\gamma-1} \Phi_2(t), \qquad t \geq t_1.\end{aligned} \tag{2.84}$$

If $N = 2$ in Condition (A), then the right side of (2.84) is infinite. This again contradicts (2.73).

A similar argument yields a contradiction for any integer $N > 2$. This completes the proof of the theorem. □

Example 2.8. We have

$$\Delta^2 x(n) + \left(\frac{a}{n^{1+c}} + \frac{b(-1)^n}{n^c}\right)|x(n+1)|^{\gamma} \operatorname{sgn} x(n+1) = 0, \qquad \gamma > 1, \tag{2.85}$$

where $b > 0$, $c > 1$, and $a/c > b/2$. It is easy to see that

$$\sum_{k=n}^{\infty} \frac{1}{k^{1+c}} \geq \int_{n}^{\infty} \frac{1}{t^{1+c}} dt = \frac{1}{cn^{c}}. \tag{2.86}$$

By [5], we have $\sum_{k=n}^{\infty}(-1)^n/n^c \sim (-1)^n/2n^c$. So,

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{k^c} = [1 + o(1)] \frac{(-1)^n}{2n^c}. \tag{2.87}$$

Using (2.86) and (2.87), we get that, for large n,

$$P(n) = \sum_{k=n}^{\infty} \left(\frac{a}{k^{1+c}} + \frac{b(-1)^k}{k^c} \right) \geq \frac{(a/c) + (b/2)[1 + o(1)](-1)^n}{n^c} > 0. \tag{2.88}$$

By Theorem 2.2, each nonoscillatory solution $x(t)$ of (2.85) satisfies exactly one of the following three asymptotic properties:

$$\begin{gathered} \lim_{n \to \infty} x(n) = c \neq 0, \\ \lim_{n \to \infty} \frac{x(n)}{n} = 0, \qquad \lim_{n \to \infty} x(n) = \pm\infty, \\ \lim_{n \to \infty} \frac{x(n)}{n} = c \neq 0. \end{gathered} \tag{2.89}$$

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References

[1] B. Baoguo, L. Erbe, and A. Peterson, "Kiguradze-type oscillation theorems for second order superlinear dynamic equations on time scales," *Canadian Mathematical Bulletin*, vol. 54, no. 4, pp. 580–592, 2011.
[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[3] M. Naito, "Nonoscillatory solutions of second order differential equations with integrable coefficients," *Proceedings of the American Mathematical Society*, vol. 109, no. 3, pp. 769–774, 1990.
[4] J. S. W. Wong, "Oscillation criteria for second order nonlinear differential equations with integrable coefficients," *Proceedings of the American Mathematical Society*, vol. 115, no. 2, pp. 389–395, 1992.
[5] L. Erbe, J. Baoguo, and A. Peterson, "Nonoscillation for second order sublinear dynamic equations on time scales," *Journal of Computational and Applied Mathematics*, vol. 232, no. 2, pp. 594–599, 2009.