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Research Article

Coupled Coincidence Point Theorem in Partially Ordered Metric Spaces via Implicit Relation

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We prove a coupled coincidence point theorem for mappings $F : X \times X \to X$ and $g : X \to X$, where F has the mixed g-monotone property, in partially ordered metric spaces via implicit relations. Our result extends and improves several results in the literature. Examples are also given to illustrate our work.

1. Introduction and Preliminaries

The notion of coupled fixed point was introduced by Guo and Lakshmikantham [1] in 1987. Later, Bhaskar and Lakshmikantham [2] defined the notions of mixed monotone mapping and proved some coupled fixed point theorems for the mixed monotone mappings. In this pioneer paper [2], they also discussed the existence and uniqueness of solution for a periodic boundary value problem. We start with recalling these basic concepts.

Definition 1.1 (see [2]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,

$$\begin{aligned} &x_1, x_2 \in X, \quad x_1 \preceq x_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y), \\ &y_1, y_2 \in X, \quad y_1 \preceq y_2 \Longrightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned} \tag{1.1}$$

Definition 1.2 (see [2]). An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y), \qquad y = F(y, x). \tag{1.2}$$

The main results of Bhaskar and Lakshmikantham in [2] are the following theorems.

Theorem 1.3 (see [2]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \tag{1.3}$$

for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0), \qquad y_0 \succeq F(y_0, x_0), \tag{1.4}$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \qquad y = F(y, x). \tag{1.5}$$

Theorem 1.4 (see [2]). *Let $(X, \preceq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \tag{1.6}$$

for all $x \succeq u$ and $y \preceq v$. If there exist two elements $x_0, y_0 \in X$ with

$$x_0 \preceq F(x_0, y_0), \qquad y_0 \succeq F(y_0, x_0), \tag{1.7}$$

then there exist $x, y \in X$ such that

$$x = F(x, y), \qquad y = F(y, x). \tag{1.8}$$

Afterwards, a number of coupled coincidence/fixed point theorems and their application to integral equations, matrix equations, and periodic boundary value problem have been established (e.g., see [3–28] and references therein). In particular, Lakshmikantham and Ćirić

[7] established coupled coincidence and coupled fixed point theorems for two mappings $F : X \times X \to X$ and $g : X \to X$, where F has the mixed g-monotone property and the functions F and g commute, as an extension of the fixed point results in [2]. Choudhury and Kundu in [15] introduced the concept of compatibility and proved the result established in [7] under a different set of conditions. Precisely, they established their result by assuming that F and g are compatible mappings. For the sake of completeness, we remind these characterizations.

Definition 1.5 (see [7]). Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$ and $g : X \to X$ are two mappings. We say F has the mixed g-monotone property if $F(x, y)$ is g-nondecreasing in its first argument and is g-nonincreasing in its second argument, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, &\quad gx_1 \preceq gx_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y), \\ y_1, y_2 \in X, &\quad gy_1 \preceq gy_2 \Longrightarrow F(x, y_1) \succeq F(x, y_2). \end{aligned} \tag{1.9}$$

Definition 1.6 (see [7]). An element $(x, y) \in X \times X$ is called a coupled coincident point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y), \qquad gy = F(y, x). \tag{1.10}$$

Definition 1.7 (see [15]). The mappings F and g where $F : X \times X \to X, g : X \to X$ are said to be compatible if

$$\begin{aligned} &\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \\ &\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0, \end{aligned} \tag{1.11}$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for all $x, y \in X$ are satisfied.

Luong and Thuan [11] slightly extended the concept of compatible mappings into the context of partially ordered metric spaces, namely, O-compatible mappings and proved some coupled coincidence point theorems for such mappings in partially ordered generalized metric spaces.

The concept of O-compatible mappings is stated as follows.

Definition 1.8 (cf. [11]). Let $(X, \preceq, d)$ be a partially ordered metric space. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be O-compatible if

$$\begin{aligned} &\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \\ &\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0, \end{aligned} \tag{1.12}$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\{gx_n\}$, $\{gy_n\}$ are monotone and

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x,$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y, \tag{1.13}$$

for all $x, y \in X$ are satisfied.

Let $(X, \preceq, d)$ be a partially metric space. If $F : X \times X \to X$ and $g : X \to X$ are compatible then they are O-compatible. However, the converse is not true. The following example shows that there exist mappings that are O-compatible but not compatible.

Example 1.9 (see [11]). Let $X = \{0\} \cup [1/2, 2]$ with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$. We consider the following order relation on X:

$$x, y \in X \; x \preceq y \Longleftrightarrow x = y \quad \text{or} \quad (x, y) \in \{(0,0), (0,1), (1,1)\}. \tag{1.14}$$

Let $F : X \times X \to X$ be given by

$$F(x, y) = \begin{cases} 0, & \text{if } x, y \in \{0\} \cup \left[\dfrac{1}{2}, 1\right], \\ 1, & \text{otherwise}, \end{cases} \tag{1.15}$$

and $g : X \to X$ be defined by

$$gx = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } \dfrac{1}{2} \le x \le 1, \\ 2 - x, & \text{if } 1 < x \le \dfrac{3}{2}, \\ \dfrac{1}{2}, & \text{if } \dfrac{3}{2} < x \le 2. \end{cases} \tag{1.16}$$

Then F and g are O-compatible but not compatible.

Indeed, let $\{x_n\}, \{y_n\}$ in X such that $\{gx_n\}, \{gy_n\}$ are monotone and

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x,$$
$$\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y, \tag{1.17}$$

for some $x, y \in X$. Since $F(x_n, y_n) = F(y_n, x_n) \in \{0, 1\}$ for all n, $x = y \in \{0, 1\}$. The case $x = y = 1$ is impossible. In fact, if $x = y = 1$. Then since $\{gx_n\}, \{gy_n\}$ are monotone, $gx_n = gy_n = 1$ for all $n \ge n_1$, for some n_1. That is $x_n, y_n \in [1/2, 1]$ for all $n \ge n_1$. This implies $F(x_n, y_n) = F(y_n, x_n) = 0$, for all $n \ge n_1$, which is a contradiction. Thus $x = y = 0$. That implies $gx_n = gy_n = 0$ for all $n \ge n_2$, for some n_2. That is $x_n = y_n = 0$ for all $n \ge n_2$. Thus, for all $n \ge n_2$,

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = d(gF(y_n, x_n), F(gy_n, gx_n)) = 0. \tag{1.18}$$

Hence

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0,$$
$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0 \tag{1.19}$$

hold. Therefore F and g are O-compatible.

Now let $\{x_n\}, \{y_n\}$ in X be defined by

$$x_n = y_n = 1 + \frac{1}{n+1}, \quad n = 1, 2, 3, \ldots. \tag{1.20}$$

We have

$$F(x_n, y_n) = F(y_n, x_n) = F\left(1 + \frac{1}{n+1}, 1 + \frac{1}{n+1}\right) = 1,$$
$$gx_n = gy_n = g\left(1 + \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty, \tag{1.21}$$

but

$$d(gF(x_n, y_n), F(gx_n, gy_n)) = d\left(F\left(1 - \frac{1}{n+1}, 1 - \frac{1}{n+1}\right), g1\right)$$
$$= d(0, 1) = 1 \nrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{1.22}$$

Thus, F and g are not compatible.

Implicit relation on metric spaces has been used in many articles (see, e.g., [29–31] and references therein). In this paper, we use the following implicit relation to prove a coupled coincidence point theorem for mappings $F : X \times X \to X$ and $g : X \to X$, where F has the mixed g-monotone property and F, g are O-compatible.

Let Φ denote all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy

(i) φ is continuous,

(ii) $\varphi(t) < t$ for each $t > 0$.

Obviously, if $\varphi \in \Phi$ then $\varphi(0) = 0$.

Let $\mathbb{H}$ denote all continuous functions $H : (\mathbb{R}^+)^5 \to \mathbb{R}$ which satisfy

(H1) $H(t_1, t_2, t_3, t_4, t_5)$ is nonincreasing in t_2 and t_5,

(H2) there exists a function $\varphi \in \Phi$ such that

$$H(u, u+v, v, w, u+v) \leq 0 \quad \text{implies } u \leq \varphi(\max\{v, w\}). \tag{1.23}$$

It is easy to check that the following functions are in $\mathbb{H}$:

(i) $H_1(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha t_2 - \beta t_3 - \gamma t_4 - \theta t_5$, where $\alpha, \beta, \gamma, \theta$ are nonnegative real numbers satisfying $2\alpha + \beta + \gamma + 2\theta < 1$;

(ii) $H_2(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha \max\{t_2/2, t_3, t_4, t_5/2\}$, where $\alpha \in (0,1)$;

(iii) $H_3(t_1, t_2, t_3, t_4, t_5) = t_1 - \varphi(\max\{t_3, t_4\})$, where $\varphi \in \Phi$.

In this paper, we prove a coupled coincidence point theorem for mappings satisfying such implicit relations.

2. Coupled Coincidence Point Theorem

Now we are going to prove our main result.

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Suppose $F : X \times X \to X$ and $g : X \to X$ are mappings such that F has the mixed g-monotone property. Assume that there exists $H \in \mathbb{H}$ such that*

$$H\begin{pmatrix} d(F(x,y),F(u,v)), d(F(x,y),gx) + d(F(u,v),gu), \\ d(gx,gu), d(gy,gv), d(F(x,y),gu) + d(F(u,v),gx) \end{pmatrix} \leq 0, \tag{2.1}$$

for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$. Suppose $F(X \times X) \subseteq g(X)$, g is continuous and g is O-compatible with F. Suppose either

(a) *F is continuous or;*

(b) *X has the following property:*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all n,*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all n.*

If there exist two elements $x_0, y_0 \in X$ with

$$gx_0 \preceq F(x_0, y_0), \qquad gy_0 \succeq F(y_0, x_0), \tag{2.2}$$

then F and g have a coupled coincidence point in X.

Proof. Let $x_0, y_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we construct the sequences $\{x_n\}$ and $\{y_n\}$ in X as follows:

$$gx_{n+1} = F(x_n, y_n), \quad gy_{n+1} = F(y_n, x_n) \quad \forall n \geq 0. \tag{2.3}$$

By using the mathematical induction and the mixed g-monotone property of F, we can show that

$$gx_n \preceq gx_{n+1}, \quad gy_n \succeq gy_{n+1}, \quad \forall n \geq 0. \tag{2.4}$$

If there is some $n_0 \in \mathbb{N}^*$ such that $gx_{n_0} = gx_{n_0+1}$ and $gy_{n_0} = gy_{n_0+1}$ then

$$gx_{n_0} = gx_{n_0+1} = F(x_{n_0}, y_{n_0}), \qquad gy_{n_0} = gy_{n_0+1} = F(y_{n_0}, x_{n_0}), \tag{2.5}$$

that means (x_{n_0}, y_{n_0}) is a coupled coincidence point of F and g. Thus we may assume that $\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} > 0$ for all n.

Since $gx_{n+1} \succeq gx_n$ and $gy_{n+1} \preceq gy_n$, from (2.1), we have

$$H\begin{pmatrix} d(F(x_{n+1}, y_{n+1}), F(x_n, y_n)), d(F(x_{n+1}, y_{n+1}), gx_{n+1}) + d(F(x_n, y_n), gx_n), \\ d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(F(x_{n+1}, y_{n+1}), gx_n) + d(F(x_n, y_n), gx_{n+1}) \end{pmatrix} \le 0 \tag{2.6}$$

or

$$H\begin{pmatrix} d(gx_{n+2}, gx_{n+1}), d(gx_{n+2}, gx_{n+1}) + d(gx_{n+1}, gx_n), \\ d(gx_{n+1}, gx_n),\ d(gy_{n+1}, gy_n),\ d(gx_{n+2}, gx_n) \end{pmatrix} \le 0. \tag{2.7}$$

By the properties of H, we have

$$H\begin{pmatrix} d(gx_{n+2}, gx_{n+1}), d(gx_{n+2}, gx_{n+1}) + d(gx_{n+1}, gx_n), \\ d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n), d(gx_{n+2}, gx_{n+1}) + d(gx_{n+1}, gx_n) \end{pmatrix} \le 0, \tag{2.8}$$

which implies that

$$d(gx_{n+2}, gx_{n+1}) \le \varphi(\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}). \tag{2.9}$$

Similarly, one can show that

$$d(gy_{n+2}, gy_{n+1}) \le \varphi(\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}). \tag{2.10}$$

From (2.9) and (2.10), we have

$$\max\{d(gx_{n+2}, gx_{n+1}), d(gy_{n+2}, gy_{n+1})\} \le \varphi(\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}), \tag{2.11}$$

which implies

$$\max\{d(gx_{n+2}, gx_{n+1}), d(gy_{n+2}, gy_{n+1})\} < \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}. \tag{2.12}$$

This means that $\{d_n := \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}\}$ is a decreasing sequence of positive real numbers. So there is a $d \ge 0$ such that

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} = d. \tag{2.13}$$

We will show that $d = 0$. Assume, to the contrary, that $d > 0$. Taking $n \to \infty$ in (2.11), we have

$$d \leq \lim_{n \to \infty} \varphi(d_n) = \varphi(d) < d, \tag{2.14}$$

which is a contradiction. Thus $d = 0$.

In what follows, we will show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. This means that there exists an $\varepsilon > 0$ for wich we can find subsequences $\{gx_{n(k)}\}, \{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}, \{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \geq \varepsilon. \tag{2.15}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.15). Then

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)})\} < \varepsilon. \tag{2.16}$$

Using the triangle inequality and (2.16), we have

$$\begin{aligned} d(gx_{n(k)}, gx_{m(k)}) &\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) \\ &< d(gx_{n(k)}, gx_{n(k)-1}) + \varepsilon, \\ d(gy_{n(k)}, gy_{m(k)}) &\leq d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) \\ &< d(gy_{n(k)}, gy_{n(k)-1}) + \varepsilon. \end{aligned} \tag{2.17}$$

From (2.15) and (2.17), we have

$$\begin{aligned} \varepsilon &\leq \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &< \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} + \varepsilon. \end{aligned} \tag{2.18}$$

Letting $k \to \infty$ in the inequalities above and using (2.13) we get

$$\lim_{k \to \infty} \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} = \varepsilon. \tag{2.19}$$

By the triangle inequality

$$\begin{aligned} d(gx_{n(k)}, gx_{m(k)}) &\leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{m(k)}), \\ d(gy_{n(k)}, gy_{m(k)}) &\leq d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}) + d(gy_{m(k-1}, gy_{m(k)}). \end{aligned} \tag{2.20}$$

From the last two inequalities and (2.15), we have

$$\begin{aligned}\varepsilon &\le \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}\\ &\le \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\}\\ &\quad + \max\{d(gx_{m(k)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\}\\ &\quad + \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}.\end{aligned} \tag{2.21}$$

Again, by the triangle inequality,

$$\begin{aligned}d(gx_{n(k)-1}, gx_{m(k)-1}) &\le d(gx_{n(k)-1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)-1})\\ &< d(gx_{m(k)}, gx_{m(k)-1}) + \varepsilon,\\ d(gy_{n(k)-1}, gy_{m(k)-1}) &\le d(gy_{n(k)-1}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)-1})\\ &< d(gy_{m(k)}, gy_{m(k)-1}) + \varepsilon.\end{aligned} \tag{2.22}$$

Therefore,

$$\begin{aligned}&\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}\\ &\quad < \max\{d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\} + \varepsilon.\end{aligned} \tag{2.23}$$

From (2.21) and (2.23), we have

$$\begin{aligned}\varepsilon &- \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\}\\ &\quad - \max\{d(gx_{m(k)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\}\\ &\le \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}\\ &< \max\{d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\} + \varepsilon.\end{aligned} \tag{2.24}$$

Taking $k \to \infty$ in the inequalities above and using (2.13), we get

$$\lim_{k\to\infty} \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} = \varepsilon. \tag{2.25}$$

From (2.19) and (2.25), the sequences $\{d(gx_{n(k)}, gx_{m(k)})\}$, $\{d(gy_{n(k)}, gy_{m(k)})\}$, $\{d(gx_{n(k)-1}, gx_{m(k)-1})\}$, and $\{d(gy_{n(k)-1}, gy_{m(k)-1})\}$ have subsequences converging to ε_1, ε_2, ε_3 and ε_4, respectively, and $\max\{\varepsilon_1, \varepsilon_2\} = \max\{\varepsilon_3, \varepsilon_4\} = \varepsilon > 0$. We may assume that

$$\begin{aligned}\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) &= \varepsilon_1, \qquad \lim_{k\to\infty} d(gy_{n(k)}, gy_{m(k)}) = \varepsilon_2,\\ \lim_{k\to\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) &= \varepsilon_3, \qquad \lim_{k\to\infty} d(gy_{n(k)-1}, gy_{m(k)-1}) = \varepsilon_4.\end{aligned} \tag{2.26}$$

We first assume that $\varepsilon_1 = \max\{\varepsilon_1, \varepsilon_2\} = \varepsilon$. Since $n(k) > m(k)$, $gx_{n(k)-1} \succeq gx_{m(k)-1}$ and $gy_{n(k)-1} \preceq gy_{m(k)-1}$. From (2.1), we have

$$H\begin{pmatrix} d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})), d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{n(k)-1}) \\ + d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{m(k)-1}), d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1}), \\ d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{m(k)-1}) + d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{n(k)-1}) \end{pmatrix} \leq 0 \tag{2.27}$$

or

$$H\begin{pmatrix} d(gx_{n(k)}, gx_{m(k)}),\ d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{m(k)}, gx_{m(k)-1}), \\ d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1}), d(gx_{n(k)}, gx_{m(k)-1}) + d(gx_{m(k)}, gx_{n(k)-1}) \end{pmatrix} \leq 0 \tag{2.28}$$

or

$$H\begin{pmatrix} d(gx_{n(k)}, gx_{m(k)}),\ d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{m(k)}, gx_{m(k)-1}), \\ d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1}), d(gx_{n(k)}, gx_{m(k)}) \\ + d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{n(k)-1}) \end{pmatrix} \leq 0. \tag{2.29}$$

Letting $k \to \infty$, we have

$$H(\varepsilon_1, 0, \varepsilon_3, \varepsilon_4, \varepsilon_1 + \varepsilon_3) \leq 0. \tag{2.30}$$

Thus,

$$H(\varepsilon_1, \varepsilon_1 + \varepsilon_3, \varepsilon_3, \varepsilon_4, \varepsilon_1 + \varepsilon_3) \leq 0, \tag{2.31}$$

which implies $\varepsilon = \varepsilon_1 \leq \varphi(\max\{\varepsilon_3, \varepsilon_4\}) = \varphi(\varepsilon) < \varepsilon$. That is a contradiction.

Using the same argument as above for the case $\varepsilon_2 = \max\{\varepsilon_1, \varepsilon_2\} = \varepsilon$, we also get a contradiction. Thus $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \qquad \lim_{n\to\infty} gy_n = y. \tag{2.32}$$

Thus

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x, \qquad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{2.33}$$

Since F and g are O-compatible, from (2.33), we have

$$\lim_{n\to\infty} d(gF(x_n, y_n), F(gx_n, gy_n)) = 0, \tag{2.34}$$

$$\lim_{n\to\infty} d(gF(y_n, x_n), F(gy_n, gx_n)) = 0. \tag{2.35}$$

Now, suppose that assumption (a) holds. We have

$$d(gx, F(g(x_n), g(y_n))) \leq d(gx, gF(x_n, y_n)) + d(gF(x_n, y_n), F(gx_n, gy_n)). \tag{2.36}$$

Taking the limit as $n \to \infty$ in (2.36) and by (2.32), (2.34) and the continuity of F and g we get $d(gx, F(x, y)) = 0$.

Similarly, we can show that $d(gy, F(y, x)) = 0$. Therefore, $gx = F(x, y)$ and $gy = F(y, x)$.

Finally, suppose that assumption (b) holds. Since $\{gx_n\}$ is nondecreasing sequence and $gx_n \to x$ and $\{gy_n\}$ is nonincreasing sequence and $gy_n \to y$, by the assumption, we have $ggx_n \preceq gx$ and $ggy_n \succeq gy$ for all n.

Since g is continuous, from (2.32), (2.34), and (2.35) we have

$$\begin{aligned} \lim_{n\to\infty} ggx_n = gx = \lim_{n\to\infty} gF(x_n, y_n) = \lim_{n\to\infty} F(gx_n, gy_n), \\ \lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} gF(y_n, x_n) = \lim_{n\to\infty} F(gy_n, gx_n). \end{aligned} \tag{2.37}$$

We have

$$H\begin{pmatrix} d(F(gx_n, gy_n), F(x, y)), d(F(gx_n, gy_n), ggx_n) + d(F(x, y), gx) \\ d(ggx_n, gx), d(ggy_n, gy), d(F(gx_n, gy_n), gx) + d(F(x, y), ggx_n) \end{pmatrix} \leq 0. \tag{2.38}$$

Letting $n \to \infty$ and using (2.37), we have

$$H(d(gx, F(x, y)), d(gx, F(x, y)), 0, 0, d(gx, F(x, y))) \leq 0, \tag{2.39}$$

which implies that $d(gx, F(x, y)) \leq \varphi(\max\{0, 0\}) = 0$. Hence $gx = F(x, y)$. Similarly, one can show that $gy = F(y, x)$.

Thus proved that F and g have a coupled coincidence point in X. □

Example 2.2 (see, e.g., [11]). Let $(X, d, \preceq)$, F and g be defined as in Example 1.9. Then

(i) X is complete and X has the property

  (a) if a nondecreasing sequence $\{x_n\} \to x$, then $gx_n \preceq gx$ for all n,
  (b) if a nonincreasing sequence $\{y_n\} \to y$, then $gy \preceq gy_n$ for all n;

(ii) $F(X \times X) = \{0, 1\} \subset \{0\} \cup [1/2, 1] = g(X)$;

(iii) g is continuous and g and F are O-compatible;

(iv) there exist $x_0 = 0, y_0 = 1$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$;

(v) F has the mixed g-monotone property. Indeed, for every $y \in X$, let $x_1, x_2 \in X$ such that $gx_1 \preceq gx_2$

(a) if $gx_1 = gx_2$ then $x_1, x_2 = 0$ or $x_1, x_2 \in [1/2, 1]$ or $x_1, x_2 \in (1, 3/2]$ or $x_1, x_2 \in (3/2, 2]$. Thus,

$$F(x_1, y) = 0 = F(x_2, y) \quad \text{if } y \in \{0\} \cup \left[\frac{1}{2}, 1\right],\ x_1, x_2 = 0 \text{ or } x_1, x_2 \in \left[\frac{1}{2}, 1\right], \tag{2.40}$$

otherwise $F(x_1, y) = 1 = F(x_2, y)$,

(b) if $gx_1 \prec gx_2$, then $gx_1 = 0$ and $gx_2 = 1$, that is, $x_1 = 0$ and $x_2 \in [1/2, 1]$. Thus

$$F(x_1, y) = 0 = F(x_2, y) \quad \text{if } y \in \{0\} \cup \left[\frac{1}{2}, 1\right], \qquad F(x_1, y) = 1 = F(x_2, y) \quad \text{if } y \in (1, 2], \tag{2.41}$$

therefore, F is the g-nondecreasing in its first argument. Similarly, F is the g-nonincreasing in its second argument;

(vi) for $x, y, u, v \in X$, if $gx \succeq gu$ and $gy \preceq gv$ then $d(F(x, y), F(u, v)) = 0$. Indeed,

(a) if $gx \succ gu$ and $gy \prec gv$ then $y = u = 0$ and $x, v \in [1/2, 1]$. Thus $d(F(x, y), F(u, v)) = d(0, 0) = 0$,

(b) if $gx = gu$ and $gy \prec gv$ then $y = 0$ and $v \in [1/2, 1]$. Thus if $x = u = 0$ or $x, u \in [1/2, 1]$ then $d(F(x, y), F(u, v)) = d(0, 0) = 0$, otherwise $d(F(x, y), F(u, v)) = d(1, 1) = 0$. Similarly, if $gx \succ gu$ and $gy = gv$ then $d(F(x, y), F(u, v)) = 0$,

(c) if $gx = gu$ and $gy = gv$ then both x, u are in one of the sets $\{0\}$, $[1/2, 1]$, $(1, 3/2]$ or $(3/2, 2]$ and both y, v are also in one of the sets $\{0\}$, $[1/2, 1]$, $(1, 3/2]$ or $(3/2, 2]$. Thus $d(F(x, y), F(u, v)) = d(0, 0) = 0$ if $x = u = 0$ or $x, u \in [1/2, 1]$ and $y = v = 0$ or $y, v \in [1/2, 1]$, otherwise, $d(F(x, y), F(u, v)) = d(1, 1) = 0$.

Therefore, all the conditions of Theorem 2.1 are satisfied with $H(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_3, t_4\}/2$. Applying Theorem 2.1, we conclude that F and g have a coupled coincidence point.

Note that, we cannot apply the result of Choudhury and Kundu [15], the result of Choudhury et al. [32] as well as the result of Lakshmikantham and Ćirić [7] to this example.

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