Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 795825, 25 pages
doi:10.1155/2012/795825

Research Article

Lyapunov's Type Inequalities for Fourth-Order Differential Equations

Samir H. Saker

College of Science Research Centre, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Samir H. Saker, ssaker@ksu.edu.sa

Received 4 January 2012; Accepted 16 February 2012

Academic Editor: Chaitan Gupta

For a fourth-order differential equation, we will establish some new Lyapunov-type inequalities, which give lower bounds of the distance between zeros of a nontrivial solution and also lower bounds of the distance between zeros of a solution and/or its derivatives. The main results will be proved by making use of Hardy's inequality and some generalizations of Opial-Wirtinger-type inequalities involving higher-order derivatives. Some examples are considered to illustrate the main results.

1. Introduction

In this paper, we are concerned with the lower bounds of the distance between zeros of a nontrivial solution and also lower bounds of the distance between zeros of a solution and/or its derivatives for the fourth-order differential equation

$$\left(r(t)\left(x'''(t)\right)^{\gamma}\right)' + q(t)x^{\gamma}(t) = 0, \quad t \in \mathbb{I}, \tag{1.1}$$

where $\gamma \geq 1$, $r, q : \mathbb{I} \to \mathbb{R}$ are continuous measurable functions and $\mathbb{I}$ is a nontrivial interval of reals. By a solution of (1.1) on the interval $J \subseteq \mathbb{I}$, we mean a nontrivial real-valued function $x \in C^3(J)$, which has the property that $r(t)(x'''(t))^{\gamma} \in C^1(J)$ and satisfies (1.1) on J. We assume that (1.1) possesses such a nontrivial solution on $\mathbb{I}$.

The nontrivial solution $x(t)$ of (1.1) is said to be oscillate or to be oscillatory if it has arbitrarily large zeros. Equation (1.1) is oscillatory if one of its nontrivial solutions is oscillatory. Equation (1.1) is said to be (i, j)-disconjugate if i and j are positive integers such that $i + j = 4$ and no solution of (1.1) has an (i, j)-distribution of zeros, that is, no nontrivial solution has a pair of zeros of multiplicities i and j, respectively. In general, the differential equation of nth-order

$$x^{(n)}(t) + q(t)x(t) = 0 \tag{1.2}$$

is said to be $(k, n-k)$-disconjugate on an interval $\mathbb{I}$ in case no nontrivial solution has a zero of order k followed by a zero of order $n-k$. This means that, for every pair of points $\alpha,\ \beta \in \mathbb{I},\ \alpha < \beta$, a nontrivial solution of (1.1) that satisfies

$$\begin{aligned} x^{(i)}(\alpha) &= 0, \quad i = 0, \ldots, k-1, \\ x^{(j)}(\beta) &= 0, \quad j = 0, \ldots, n-k-1 \end{aligned} \tag{1.3}$$

does not exist.

The least value of β such that there exists a nontrivial solution which satisfies (1.3) is called the $(k, n-k)$-conjugate point of α. The differential equation (1.2) is said to disconjugate on an interval $\mathbb{I}$ if one of its nontrivial solutions has at most $n-1$ zeros. For our case, if no nontrivial solution of (1.1) has more than three zeros, the equation is termed disconjugate. Together with $(k, n-k)$-disconjugacy, we consider the related concept, which is $(k, n-k)$-disfocality. The differential equation (1.2) is said to be disfocal on an interval $\mathbb{I}$ if for every nontrivial solution x at least one of the functions $x, x', \ldots, x^{(n-1)}$ does not vanish on $\mathbb{I}$. If the equation is not disfocal on $\mathbb{I}$, then there exists an integer k $(1 \le k \le n-1)$, a pair of points $\alpha,\ \beta \in \mathbb{I},\ \alpha < \beta$ and a nontrivial solution x such that k of the functions $x, x', \ldots, x^{(n-1)}$ vanishes at α and the remaining $n-k$ functions at β, that is,

$$\begin{aligned} x^{(i)}(\alpha) &= 0, \quad i = 0, \ldots, k-1, \\ x^{(j)}(\beta) &= 0, \quad j = k, \ldots, n-1. \end{aligned} \tag{1.4}$$

The equation (1.1) is said to be $(2,2)$-disconjugate on $[\alpha, \beta]$ if there is no nontrivial solution $x(t)$ and $c, d \in [\alpha, \beta],\ c < d$ such that $x(c) = x'(c) = x(d) = x'(d) = 0$. Equation (1.1) is said to be $(k, 4-k)$-disfocal on an interval $\mathbb{I}$ for some $1 \le k \le 3$ in case there does not exist a solution x with a zero of order k followed by a zero of $x_*^{(j)}$ of order $4-k$, where $x_*^{(j)} = x^{(j)}$ for $j = 0, 1, 2, 3$ and $x_*^{(4)} = (r(x''')^{\gamma})'$.

For nth-order differential equations, $(k, n-k)$-disconjugacy and disfocality are connected by the result of Nehari [1], which states that, if (1.2) is $(k, n-k)$-disfocal on (α, β) it is disconjugate on (α, β). For more details about disconjugacy and disfocality and the relation between them, we refer the reader to the paper [2]. For related results to the present paper, we refer the reader to the papers [3–14] and the references cited therein.

In [4, 15], the authors established some new Lyapunov-type inequalities for higher-order differential equations. In the following, we present some of some special cases of their results for fourth-order differential equations that serve and motivate the contents of this paper. In [15], it is proved that if $x(t)$ is a solution of the fourth-order differential equation

$$x^{(4)}(t) + q(t)x(t) = 0, \tag{1.5}$$

which satisfies $x(\alpha) = x'(\alpha) = x(\beta) = x'(\beta) = 0$, then

$$\int_{\alpha}^{\beta} |q(t)| dt \geq \frac{192}{(\beta - \alpha)^3}, \tag{1.6}$$

and if $x(t)$ satisfies $x(\alpha) = x''(\alpha) = x(\beta) = x''(\beta) = 0$, then

$$\int_{\alpha}^{\beta} |q(t)| dt \geq \frac{4}{(\beta - \alpha)^2}. \tag{1.7}$$

In [4], the author proved that if $x(t)$ is a solution of (1.5), which satisfies $x(\alpha) = x(\beta) = x''(\alpha) = x''(\beta) = 0$, then

$$\int_{\alpha}^{\beta} |q(t)| dt \geq \frac{16}{(\beta - \alpha)^3}. \tag{1.8}$$

In this paper, we are concerned with the following problems for the general equation (1.1):

(i) obtain lower bounds for the spacing $\beta-\alpha$, where x is a solution of (1.1) that satisfies $x^{(i)}(\alpha) = 0$ for $i = 0, 1, 2$ and $x''(\beta) = 0$,

(ii) obtain lower bounds for the spacing $\beta-\alpha$, where x is a solution of (1.1) that satisfies $x^{(i)}(\beta) = 0$ for $i = 0, 1, 2$ and $x''(\alpha) = 0$,

(iii) obtain lower bounds for the spacing $\beta-\alpha$, where x is a solution of (1.1) that satisfies $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$ for $i = 0, 1, 2$.

The main results will be proved in Section 2 by making use of Hardy's inequality and some generalizations of Opial-Wirtinger-type inequalities involving higher-order derivatives. The results yield conditions for disfocality and disconjugacy. In Section 3, we will discuss some special cases of our results to derive some new results for (1.5) and give some illustrative examples. To the best of the author knowledge, this technique has not been employed before on (1.1). Of particular interest in this paper is when q is oscillatory and r is a negative function.

2. Main Results

In this section, we will prove the main results by making use of Hardy's inequality and some Opial-Wirtinger-type inequalities. Throughout the paper, all the functions are assumed to be measurable functions and all the integrals that will appear in the inequalities are finite.

The Hardy inequality [16, 17] of the differential form that we will need in this paper states that, if y is absolutely continuous on (α,β), then the following inequality holds

$$\left(\int_{\alpha}^{\beta} q(t)|y(t)|^n dt\right)^{1/n} \leq C\left(\int_{\alpha}^{\beta} r(t)|y'(t)|^m dt\right)^{1/m}, \tag{2.1}$$

where q, r the weighted functions are measurable positive functions in the interval (α, β) and m, n are real parameters that satisfy $0 < n \leq \infty$ and $1 \leq m \leq \infty$. The constant C satisfies

$$C \leq k(m,n)\alpha(\alpha,\beta), \quad \text{for } 1 < m \leq n, \tag{2.2}$$

where

$$\begin{aligned}
\alpha(\alpha,\beta) &:= \sup_{\alpha<t<\beta}\left(\int_t^\beta q(t)dt\right)^{1/n}\left(\int_\alpha^t r^{1-m^*}(s)ds\right)^{1/m^*} \quad \text{if } y(\alpha)=0,\\
\alpha(\alpha,\beta) &:= \sup_{\alpha<t<\beta}\left(\int_\alpha^t q(t)dt\right)^{1/n}\left(\int_t^\beta r^{1-m^*}(s)ds\right)^{1/m^*} \quad \text{if } y(\beta)=0,
\end{aligned} \tag{2.3}$$

and $m^* = m/(m-1)$. Note that the inequality (2.1) has an immediate application to the case when $y(\alpha) = y(\beta) = 0$. In this case, we see that (2.1) is satisfied if and only if

$$\begin{aligned}
\alpha(\alpha,\beta) = &\sup_{(c,d)\subset(\alpha,\beta)}\left(\int_c^d q(t)dt\right)^{1/n}\\
&\times \min\left\{\left(\int_\alpha^c r^{1-m^*}(s)ds\right)^{1/m^*}, \left(\int_d^\beta r^{1-m^*}(s)ds\right)^{1/m^*}\right\}
\end{aligned} \tag{2.4}$$

exists and is finite. The constant $k(m,n)$ in (2.2) appears in various forms. For example,

$$\begin{aligned}
k(m,n) &:= m^{1/m}(m^*)^{1/m^*},\\
k(m,n) &:= \left(1+\frac{n}{m^*}\right)^{1/n}\left(1+\frac{m^*}{n}\right)^{1/m^*}, \quad \left(u^* = \frac{u}{u-1}\right)\\
k(m,n) &:= \left[\frac{\Gamma(n/s)}{\Gamma(1+(1/s))\Gamma((n-1)/s)}\right]^{s/n}, \quad s = \frac{n}{m-1}.
\end{aligned} \tag{2.5}$$

In the following, we present the Opial-Wirtinger-type inequalities that we will need in the proof of the main results.

Theorem 2.1 ([18, Theorem 3.9.1]). *Assume that the functions ϑ and ϕ are nonnegative and measurable on the interval $(\alpha, \beta), m, n$ are real numbers such that $\mu/m > 1$, $x(t) \in C^{(n-1)}[\alpha, \beta]$ is such that $x^{(i)}(\alpha) = 0$, $0 \leq k \leq i \leq n-1$ $(n \geq 1)$, and $x^{(n-1)}(t)$ absolutely continuous on (α, β). Then*

$$\int_\alpha^\beta \phi(t)\left|x^{(k)}(t)\right|^l\left|x^{(n)}(t)\right|^m dt \leq K_1(\alpha,\beta)\left[\int_\alpha^\beta \vartheta(t)\left|x^{(n)}(t)\right|^\mu dt\right]^{(l+m)/\mu}, \tag{2.6}$$

where

$$K_1(\alpha,\beta) = \frac{(m/(l+m))^{m/\mu}}{(K!)^l}\left[\int_\alpha^\beta \left(\frac{\phi^\mu(t)}{\vartheta^m(t)}\right)^{1/(\mu-m)} (P_{1,k}(t))^{l(\mu-1)/(\mu-m)} dt\right]^{(\mu-m)/\mu},$$
$$K = (n-k-1), \qquad P_{1,k}(t) := \int_\alpha^t (t-s)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds. \tag{2.7}$$

If we replace $x^{(i)}(\alpha) = 0$ *by* $x^{(i)}(\beta) = 0$, $0 \le k \le i \le n-1$ $(n \ge 1)$, *then* (2.6) *holds where* K_1 *is replaced by* K_2*, which is given by*

$$K_2(\alpha,\beta) = \frac{(m/(l+m))^{m/\mu}}{(K!)^l}\left[\int_\alpha^\beta \left(\frac{\phi^\mu(t)}{\vartheta^m(t)}\right)^{1/(\mu-m)} (P_{2,k}(t))^{l(\mu-1)/(\mu-m)} dt\right]^{(\mu-m)/\mu}, \tag{2.8}$$

where

$$P_{2,k}(t) := \int_t^\beta (s-t)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds. \tag{2.9}$$

Note that the inequality (2.6) has an immediate application to the case when $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ for $0 \le i \le n-1$. In this case, we will assume that there exists $\tau \in (\alpha,\beta)$ such that

$$\int_\alpha^\tau (\tau-s)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds = \int_\tau^\beta (s-\tau)^{(n-k-1)\mu/(\mu-1)} (\vartheta(s))^{-1/(\mu-1)} ds, \tag{2.10}$$

and we denote by $P(\alpha,\beta)$. This gives us the following theorem.

Theorem 2.2. *Assume that the functions* ϑ *and* ϕ *are nonnegative and measurable on the interval* (α,β), m, n *are real numbers such that* $\mu/m > 1$, $x(t) \in C^{(n-1)}[\alpha,\beta]$ *is such that* $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$, $0 \le k \le i \le n-1$ $(n \ge 1)$, *and* $x^{(n-1)}(t)$ *absolutely continuous on* (α,β)*. Then*

$$\int_\alpha^\beta \phi(t)\left|x^{(k)}(t)\right|^l \left|x^{(n)}(t)\right|^m dt \le K(\alpha,\beta)\left[\int_\alpha^\beta \vartheta(t)\left|x^{(n)}(t)\right|^\mu dt\right]^{(l+m)/\mu}, \tag{2.11}$$

where $K(\alpha,\beta)$ *is defined by*

$$K(\alpha,\beta) = \left(\frac{m}{l+m}\right)^{m/\mu} \frac{[P(\alpha,\beta)]^{l(\mu-1)/\mu}}{(K!)^l}\left[\int_\alpha^\beta \left(\frac{\phi^\mu(t)}{\vartheta^m(t)}\right)^{1/(\mu-m)} dt\right]^{(\mu-m)/\mu}. \tag{2.12}$$

Theorem 2.3 ([18, Theorem 3.9.2]). *Let* r_k, $0 \le k \le n-1$ $(n \ge 1)$ *be nonnegative numbers such that* $\sigma = \sum_{k=0}^{n-1} r_k > 0$ *and* ϑ *and* ϕ *are nonnegative and measurable on the interval* (α,β)*. Further,* *let* $x(t) \in C^{(n-1)}[\alpha,\beta]$ *be such that* $x^{(i)}(\alpha) = 0$, $0 \le i \le n-1$ $(n \ge 1)$ *and* $x^{(n-1)}(t)$ *absolutely continuous on* (α,β). *Then*

$$\int_\alpha^\beta \phi(t)\prod_{k=0}^{n}\left|x^{(k)}(t)\right|^{r_k}dt \le K_1^*(\alpha,\beta)\left[\int_\alpha^\beta \vartheta(t)\left|x^{(n)}(t)\right|^{r}dt\right]^{(\sigma+r_n)/r}, \tag{2.13}$$

where

$$\begin{gathered} K_1^*(\alpha,\beta) = \frac{1}{\Omega}\left(\frac{r_n}{\sigma+r_n}\right)^{r_n/r}\left[\int_\alpha^\beta\left(\frac{\phi^r(t)}{\vartheta^{r_n}(t)}\right)^{1/(r-r_n)}\prod_{k=0}^{n-1}\left(P_{1,k}^*(t)\right)^{r_k(r-1)/(r-r_n)}dt\right]^{(r-r_n)/r}, \\ \Omega = \prod_{k=0}^{n-1}((n-k-1)!)^{r_k}, \qquad P_{1,k}^*(t) := \int_\alpha^t (t-s)^{(n-k-1)r/(r-1)}(\vartheta(s))^{-1/(r-1)}ds. \end{gathered} \tag{2.14}$$

If we replace $x^{(i)}(\alpha) = 0$ *by* $x^{(i)}(\beta) = 0$, $0 \le i \le n-1$ $(n \ge 1)$, *then* (2.13) *holds where* K_1^* *is replaced by* K_2^*, *which is given by*

$$K_2^*(\alpha,\beta) = \frac{1}{\Omega}\left(\frac{r_n}{\sigma+r_n}\right)^{r_n/r}\left[\int_\alpha^\beta\left(\frac{\phi^r(t)}{\vartheta^{r_n}(t)}\right)^{1/(r-r_n)}\prod_{k=0}^{n-1}\left(P_{2,k}^*(t)\right)^{r_k(r-1)/(r-r_n)}dt\right]^{(r-r_n)/r}, \tag{2.15}$$

where

$$P_{2,k}^*(t) := \int_t^\beta (s-t)^{(n-k-1)r/(r-1)}(\vartheta(s))^{-1/(r-1)}ds. \tag{2.16}$$

Note that the inequality (2.13) has an immediate application to the case when $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ for $0 \le i \le n-1$. In this case, we will assume that there exists $\tau \in (\alpha,\beta)$ such that

$$\prod_{k=0}^{n-1}\left(P_{1,k}^*(\tau)\right)^{r_k(r-1)/(r-r_n)} = \prod_{k=0}^{n-1}\left(P_{2,k}^*(\tau)\right)^{r_k(r-1)/(r-r_n)}, \tag{2.17}$$

denoted by $P^*(\alpha,\beta)$. In this case the inequality (2.13) is satisfied but the constant $K_1^*(\alpha,\beta)$ is replaced by $K^*(\alpha,\beta)$, which is defined by

$$K^*(\alpha,\beta) = \left(\frac{r_n}{\sigma+r_n}\right)^{r_n/r}\frac{\left[P^*(\alpha,\beta)\right]^{(r-r_n)/r}}{\Omega}\left[\int_\alpha^\beta\left(\frac{\phi^r(t)}{\vartheta^{r_n}(t)}\right)^{1/(r-r_n)}dt\right]^{(r-r_n)/r}. \tag{2.18}$$

The Wirtinger-type inequality and its general forms have been studied in the literature in various modifications both in the continuous and in the discrete setting. It has an extensive applications on partial differential and difference equations, harmonic analysis, approximations, number theory, optimization, convex geometry, spectral theory of differential and difference operators, and others (see [19]).

In the following, we present a special case of the Wirtinger-type inequality that has been proved by Agarwal et al. in [20] and will be need in the proof the main results.

Theorem 2.4. *For* $\mathbb{I} = [\alpha, \beta], \gamma \geq 1$ *is a positive integer and a positive function* $\lambda \in C^1(\mathbb{I})$ *with either* $\lambda'(t) > 0$ *or* $\lambda'(t) < 0$ *on* $\mathbb{I}$*; we have*

$$\int_\alpha^\beta \frac{\lambda^{\gamma+1}(t)}{|\lambda'(t)|^\gamma} \left|y'(t)\right|^{\gamma+1} dt \geq \frac{1}{(\gamma+1)^{\gamma+1}} \int_\alpha^\beta \left|\lambda'(t)\right| \left|y(t)\right|^{\gamma+1} dt, \tag{2.19}$$

for any $y \in C^1(\mathbb{I})$ *with* $y(\alpha) = 0 = y(\beta)$.

Remark 2.5. It is clear that Theorem 2.4 is satisfied for any function y that satisfies the assumptions of theorem. So if $y(t) = x''(t)$ with $x''(\alpha) = 0 = \lambda(\beta)$, $x''(\beta) = 0 = \lambda(\alpha)$, or $x''(\alpha) = 0 = x''(\beta)$ and $p(t) = \lambda'(t)$, we have the following inequality, which gives a relation between $x''(t)$ and $x'''(t)$ on the interval $[\alpha, \beta]$.

Corollary 2.6. *For* $\mathbb{I} = [\alpha, \beta]$ *and* $\gamma \geq 1$ *being a positive integer, then we have*

$$\int_\alpha^\beta |r(t)| \left|x'''(t)\right|^{\gamma+1} dt \geq \frac{1}{(\gamma+1)^{\gamma+1}} \int_\alpha^\beta |p(t)| \left|x''(t)\right|^{\gamma+1} dt, \tag{2.20}$$

for any $x \in C^3(\mathbb{I})$ *with* $x''(\alpha) = 0 = r(\beta)$, $x''(\beta) = 0 = r(\alpha)$, *or* $x''(\alpha) = 0 = x''(\beta)$, *where* $r(t)$ *and* $p(t)$ *satisfy the equation*

$$\left(r(t)\left(\lambda'(t)\right)^\gamma\right)' - (\gamma+1)p(t)\lambda^\gamma(t) = 0, \tag{2.21}$$

for any function $\lambda(t)$ *satisfing* $\lambda'(t) \neq 0$.

For illustration, we apply the inequality (2.20) with $x''(t) = \sin t$ in the interval $[0, \pi]$. If $p(t) = 1$ and $\gamma = 1$ and by choosing $r(t) = t^2$, we see that (2.21) is satisfied when $\lambda(t) = t$. So one can see that

$$\int_0^\pi t^2 \cos^2 t \, dt \simeq 5.9531 > 0.39270 \simeq \frac{1}{4} \int_0^\pi \sin^2 t \, dt. \tag{2.22}$$

Note also that (2.21) holds if one chooses $r(t) = p(t) = 1$, where in this case

$$\lambda(t) = \exp\left(\frac{\gamma+1}{\gamma}\right)^{1/(\gamma+1)} t. \tag{2.23}$$

Now, we are ready to state and prove the main results when $r(t) > 0$. For simplicity, we introduce the following notations:

$$\Phi_1(Q, r, P_{1,0}) := 2^{-\gamma} \Lambda \left[\int_\alpha^\beta \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} P_{1,0}^\gamma(t) dt\right]^{\gamma/(\gamma+1)}, \tag{2.24}$$

where $\Lambda = (1/(\gamma+1))^{1/(\gamma+1)}$, $P_{1,0}(t) = \int_\alpha^t (t-s)^{2(\gamma+1)/\gamma} r^{-1/\gamma}(s)ds$,

$$\Phi_2(Q, r, P_{2,0}) := 2^{-\gamma}\Lambda\left[\int_\alpha^\beta \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} P_{2,0}^{\gamma}(t)dt\right]^{\gamma/(\gamma+1)}, \tag{2.25}$$

where $P_{2,0}(t) = \int_t^\beta (s-t)^{2(\gamma+1)/\gamma} r^{-1/\gamma}(s)ds$,

$$\Psi_1\left(Q, r, P_{1,0}^* P_{1,1}^*\right) := \Lambda\left[\int_\alpha^\beta \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} \left(P_{1,0}^*(t)\right)^{\gamma-1} P_{1,1}^*(t)dt\right]^{\gamma/(\gamma+1)}, \tag{2.26}$$

where $P_{1,0}^*(t) = \int_\alpha^t (t-s)^{(\gamma+1)/\gamma} r^{-1/\gamma}(s)ds$, $P_{1,1}^*(t) = \int_\alpha^t r^{-1/\gamma}(s)ds$, and

$$\Psi_2\left(Q, r, P_{2,0}^* P_{2,1}^*\right) := \Lambda\left[\int_\alpha^\beta \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} \left(P_{2,0}^*(t)\right)^{\gamma-1} P_{2,1}^*(t)dt\right]^{\gamma/(\gamma+1)}, \tag{2.27}$$

where $P_{2,0}^*(t) = \int_t^\beta (s-t)^{(\gamma+1)/\gamma} r^{-1/\gamma}(s)ds$, $P_{2,1}^*(t) = \int_t^\beta r^{-1/\gamma}(s)ds$.

Remark 2.7. Note that when $\gamma = 1$, then $\Psi_1(Q, r, P_{1,0}^* P_{1,1}^*)$ and $\Psi_2(Q, r, P_{2,0}^* P_{2,1}^*)$ become $\Psi_1(Q, r, P_{1,1}^*)$ and $\Psi_2(Q, r, P_{2,1}^*)$.

Theorem 2.8. *Suppose that x is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0$, for $i = 0, 1, 2$ and $x''(\beta) = 0$, then*

$$\Phi_1(Q, r, P_{1,0}) + \gamma(\gamma+1)^{\gamma+1}\Psi_1\left(Q, r, P_{1,0}^* P_{1,1}^*\right) \geq 1, \tag{2.28}$$

where $Q(t) = \int_t^\beta q(s)ds$. If $x^{(i)}(\beta) = 0$, for $i = 0, 1, 2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q, r, P_{2,0}) + \gamma(\gamma+1)^{\gamma+1}\Psi_2\left(Q, r, P_{2,0}^* P_{2,1}^*\right) \geq 1, \tag{2.29}$$

where $Q(t) = \int_\alpha^t q(s)ds$.

Proof. We prove (2.28). Multiplying (1.1) by $x''(t)$ and integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta \left(r(t)\left(x'''(t)\right)^\gamma\right)' x''(t)dt &= r(t)\left(x'''(t)\right)^\gamma x''(t)\Big|_\alpha^\beta - \int_\alpha^\beta r(t)\left(x'''(t)\right)^{\gamma+1}dt \\ &= -\int_\alpha^\beta q(t)x''(t)x^\gamma(t)dt.\end{aligned} \tag{2.30}$$

Using the boundary conditions $x''(\alpha) = x''(\beta) = 0$ and the assumption $Q(t) = \int_t^\beta q(s)ds$, we have

$$\int_\alpha^\beta r(t)\left(x'''(t)\right)^{\gamma+1}dt = \int_\alpha^\beta q(t)x''(t)x^\gamma(t)dt = -\int_\alpha^\beta Q'(t)x''(t)x^\gamma(t)dt. \tag{2.31}$$

Integrating by parts the right-hand side, we see that

$$\begin{aligned}\int_\alpha^\beta Q'(t)x''(t)x^\gamma(t)dt = {} & Q(t)x''(t)x^\gamma(t)\big|_\alpha^\beta - \gamma\int_\alpha^\beta Q(t)x^{\gamma-1}(t)x'(t)x''(t)dt \\ & -\int_\alpha^\beta Q(t)x^\gamma(t)x'''(t)dt.\end{aligned} \tag{2.32}$$

Using the boundary conditions $x''(\beta) = x''(\alpha) = 0$, we have

$$\int_\alpha^\beta Q'(t)x''(t)x^\gamma(t)dt = -\gamma\int_\alpha^\beta Q(t)x^{\gamma-1}(t)x'(t)x''(t)dt - \int_\alpha^\beta Q(t)x^\gamma(t)x'''(t)dt. \tag{2.33}$$

Substituting (2.33) into (2.31), we obtain

$$\begin{aligned}\int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1}dt \le {} & \gamma\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right|dt \\ & + \int_\alpha^\beta |Q(t)||x(t)|^\gamma\left|x'''(t)\right|dt.\end{aligned} \tag{2.34}$$

Applying the inequality (2.6) on the integral

$$\int_\alpha^\beta |Q(t)||x(t)|^\gamma\left|x'''(t)\right|dt, \tag{2.35}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $m = 1$, $k = 0$, $l = \gamma$, $n = 3$, and $\mu = \gamma + 1$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0, 1, 2$) that

$$\int_\alpha^\beta |Q(t)||x(t)|^\gamma\left|x'''(t)\right|dt \le \Phi_1(Q(t), r, P_{1,0})\int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1}dt, \tag{2.36}$$

where $\Phi_1(Q, r, P_{1,0})$ is defined as in (2.24). Applying the inequality (2.13) on the integral

$$\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right|dt, \tag{2.37}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $n = 2$, $r_0 = \gamma - 1$, $r_1 = 1$, $r_2 = 1$, $\sigma + r_2 = \gamma + 1$, and $r = \gamma + 1$, we see that

$$\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right| dt \le \Psi_1^*\left(Q, r, P_{1,0}^* P_{1,1}^*\right) \int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1} dt, \tag{2.38}$$

where $\Psi_1(Q, r, P_{1,0}^* P_{1,1}^*)$ is defined as in (2.26). Applying the Wirtinger inequality (2.20) on the integral

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1} dt, \tag{2.39}$$

where $x''(\alpha) = 0 = x''(\beta)$, we see that

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1} dt \le (\gamma+1)^{\gamma+1} \int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1} dt, \tag{2.40}$$

where $r(t)$ satisfies (2.21) for any positive function $\lambda(t)$ and $p(t)$ is replaced by $r(t)$. Substituting (2.40) into (2.38), we have

$$\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right| dt \le \frac{\Psi_1\left(Q, r, P_{1,0}^* P_{1,1}^*\right)}{(\gamma+1)^{-(\gamma+1)}} \int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1} dt. \tag{2.41}$$

Substituting (2.36) and (2.41) into (2.34) and cancelling the term $\int_\alpha^\beta r(t)|x'''(t)|^{\gamma+1} dt$, we have

$$\Phi_1(Q, r, P_{1,0}) + \gamma(\gamma+1)^{\gamma+1}\Psi_1\left(Q, r, P_{1,0}^* P_{1,1}^*\right) \ge 1, \tag{2.42}$$

which is the desired inequality (2.28). The proof of (2.29) is similar by using the integration by parts and $\Phi_1(Q, r, P_{1,0})$ is replaced by $\Phi_2(Q, r, P_{2,0})$, which is defined as in (2.25), and $\Psi_1(Q, r, P_{1,0}^* P_{1,1}^*)$ is replaced by $\Psi_2(Q, r, P_{2,0}^* P_{2,1}^*)$, which is defined as in (2.27). The proof is complete. □

In the following, we apply the Hardy inequality (2.1) on the term

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1} dt, \tag{2.43}$$

by replacing $y(t)$ by $x''(t)$ and use the assumption $x''(\alpha) = 0 = x''(\beta)$. In this case, we see that

$$\int_\alpha^\beta r(t)\left|x''(t)\right|^{\gamma+1} dt \le C_\Gamma \alpha(\alpha, \beta, r) \int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1} dt, \tag{2.44}$$

where

$$C_\Gamma := \Gamma^{1/\gamma}(\gamma)\Gamma^{-1/\gamma}\left(1+\frac{\gamma}{\gamma+1}\right)\Gamma^{-1/\gamma}\left(\frac{\gamma^2}{\gamma+1}\right), \tag{2.45}$$

$$\begin{aligned}\alpha(\alpha,\beta,r) &:= \sup_{(c,d)\subset(\alpha,\beta)}\left(\int_c^d r(t)dt\right)^{1/(\gamma+1)} \\ &\quad\times \min\left\{\left(\int_\alpha^c \frac{ds}{r^{1/\gamma}(s)}\right)^{\gamma/(\gamma+1)},\left(\int_d^\beta \frac{ds}{r^{1/\gamma}(s)}\right)^{\gamma/(\gamma+1)}\right\}.\end{aligned} \tag{2.46}$$

This implies that

$$\begin{aligned}\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right|dt &\le C_\Gamma\alpha(\alpha,\beta,r)\Psi_1\left(Q,r,P_{1,0}^*P_{1,1}^*\right) \\ &\quad\times\int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1}dt.\end{aligned} \tag{2.47}$$

Proceeding as in the proof of Theorem 2.8 and using (2.47) instead of (2.41), we have the following result.

Theorem 2.9. *Assume that $Q'(t) = q(t)$ and x is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then*

$$\Phi_1(Q,r,P_{1,0}) + \gamma C_\Gamma\alpha(\alpha,\beta,r)\Psi_1\left(Q,r,P_{1,0}^*P_{1,1}^*\right) \ge 1, \tag{2.48}$$

where $Q(t) = \int_t^\beta q(s)ds$. If $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then

$$\Phi_2(Q,r,P_{2,0}) + \gamma C_\Gamma\alpha(\alpha,\beta,r)\Psi_2\left(Q,r,P_{2,0}^*(t)P_{2,1}^*\right) \ge 1, \tag{2.49}$$

where $Q(t) = \int_\alpha^t q(s)ds$.

In the following, we will apply a new inequality to establish a new result but on the interval $[0,\beta]$. The inequality that we will apply is given in the following theorem.

Theorem 2.10 ([18, Theorem 3.7.4]). *Let $r_k, 0 \le k \le n-1$ $(n \ge 1)$ be nonnegative numbers and ϑ and ϕ nonnegative and measurable on the interval $(0,\beta)$. Let $x(t) \in C^{(n-1)}[0,\beta]$ be such that $x^{(i)}(0) = 0$, $0 \le i \le n-1$ $(n \ge 1)$, $x^{(n-1)}(t)$, is absolutely continuous on $(0,\beta)$, and let s_1, s_2 be constants greater than 1, $1/s_1 + 1/s_1^* = 1$, $1/s_2 + 1/s_2^* = 1$ and μ a constant such that $\mu > s_2$. Further assume that*

$$\sigma = \sum_{k=0}^{n-1} r_k > 0, \qquad F(\phi,\vartheta) := \left(\int_0^\beta \left(\frac{1}{\vartheta(t)}\right)^{s_2/\mu} dt\right)^{\sigma/s_2^*}\left(\int_0^\beta \phi^{s_1^*}(t)dt\right)^{1/s_1^*}. \tag{2.50}$$

Then the following inequality holds

$$\int_0^\beta \phi(t)\prod_{k=0}^{n-1}\left|x^{(k)}(t)\right|^{r_k}dt \le C\beta^\lambda\left[\int_0^\beta \vartheta(t)\left|x^{(n)}(t)\right|^{\mu}dt\right]^{\sigma/\mu}, \tag{2.51}$$

where $\lambda = \sum_{k=0}^{n-1}(n-k-1)r_k + \sigma\delta + 1/s_1$, $\delta = (\mu - s_2)/\mu s_2$ *and*

$$C := F(\phi,\vartheta)\prod_{k=0}^{n-1}\frac{[K!]^{-r_k}[(n-k-1/\delta)+1]^{-r_k\delta}}{\left[\sum_{k=0}^{n-1} Kr_k s_1 + \sigma s_1\delta + 1\right]^{1/s_1}}. \tag{2.52}$$

Now, by applying the inequality (2.51) on the term

$$\gamma\int_0^\beta |Q(t)|\left|x^{\gamma-1}(t)\right|\left|x'(t)\right|\left|x''(t)\right|dt, \tag{2.53}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $n = 3$, $r_0 = \gamma-1$, $r_1 = 1$, $r_2 = 1, \sigma = \gamma+1$, $\mu = \gamma+1$, $s_1 = s_2 = \gamma$, and $\alpha = 0$, we obtain

$$\gamma\int_0^\beta |Q(t)|\left|x^{\gamma-1}(t)\right|\left|x'(t)\right|\left|x''(t)\right|dt \le \gamma L_\gamma F(r,Q)\beta^{2\gamma-1+2/\gamma}\int_0^\beta r(t)\left|x'''(t)\right|^{\gamma+1}dt, \tag{2.54}$$

where

$$\begin{aligned} L_\gamma &:= \frac{1}{2^{\gamma-1}}\left[\frac{\gamma}{4\gamma+1}\right]^{1/\gamma}\frac{\left[2\gamma(\gamma+1)+1\right]^{-(\gamma-1)/\gamma(\gamma+1)}}{\left[\gamma(\gamma+1)+1\right]^{1/\gamma(\gamma+1)}}, \\ F(r,Q) &:= \left(\int_0^\beta\left(\frac{1}{r(t)}\right)^{\gamma/(\gamma+1)}dt\right)^{(\gamma+1)(\gamma-1)/\gamma}\left(\int_0^\beta |Q(t)|^{\gamma/\gamma-1}dt\right)^{(\gamma-1)/\gamma}. \end{aligned} \tag{2.55}$$

Proceeding as in the proof of Theorem 2.8 by using (2.54) instead of (2.38), we get the following result.

Theorem 2.11. *Assume that* $Q'(t) = q(t)$ *and* x *is a nontrivial solution of* (1.1). *If* $x^{(i)}(0) = 0$, *for* $i = 0, 1, 2$ *and* $x''(\beta) = 0$, *then*

$$\Phi_1(Q,r,P_{1,0}) + \gamma L_\gamma F(r,Q)\beta^{2\gamma-1+2/\gamma} \ge 1, \tag{2.56}$$

where $Q(t) = \int_t^\beta q(s)ds$ *and* L_γ, $F(r,Q)$ *are defined as in* (2.55).

Remark 2.12. Note that in the proof of Theorem 2.11, we do not require additional inequalities like the Hardy inequality or the Wirtinger inequality. So it will be interesting to extend the proof of Theorem 2.10 to cover the boundary conditions $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$ and replace the interval $[0,\beta]$ by $[\alpha,\beta]$.

In the following, we will assume that (2.10) and (2.17) hold. First, we assume that (2.10) holds and there exists $\tau \in (\alpha, \beta)$ such that

$$\int_{\alpha}^{\tau} \frac{(\tau - s)^{2(\gamma+1)/\gamma}}{r^{1/\gamma}(s)} ds = \int_{\tau}^{\beta} \frac{(s - \tau)^{2(\gamma+1)/\gamma}}{r^{1/\gamma}(s)} ds, \tag{2.57}$$

denoted by $P(\alpha, \beta)$. In this case, we see that

$$\Phi(Q, r) := \Phi_1(Q, r, P(t)) = \Phi_2(Q, r, P(t)), \tag{2.58}$$

where in this case $\Phi(Q, r)$ is given by

$$\Phi(Q, r) := \frac{\Lambda}{2^{\gamma}} \left[\int_{\alpha}^{\beta} \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} P^{\gamma}(\alpha, \beta) dt \right]^{\gamma/(\gamma+1)}. \tag{2.59}$$

Second, we assume that (2.17) holds and there exists $\tau \in (\alpha, \beta)$ such that

$$\left(P_{1,0}^{*}(\alpha, \tau)\right)^{\gamma-1} P_{1,1}^{*}(\alpha, \tau) = \left(P_{2,0}^{*}(\tau, \beta)\right)^{\gamma-1} P_{2,1}^{*}(\tau, \beta), \tag{2.60}$$

denoted by $P^{*}(\alpha, \beta)$. In this case, we get that

$$\Psi(r, Q) = \Psi_1^{*}(r, Q) = \Psi_2^{*}(r, Q), \tag{2.61}$$

where $\Psi(r, Q)$ is given by

$$\Psi(r, Q) := \Lambda \left[\int_{\alpha}^{\beta} \frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)} P^{*}(\alpha, \beta) dt \right]^{\gamma/(\gamma+1)}. \tag{2.62}$$

Note that when $r(t) = 1$, we have that the condition (2.57) is satisfied when $(\tau - \alpha)^{(3\gamma+2)/\gamma} = (\beta - \tau)^{(3\gamma+2)/\gamma}$. This in fact is satisfied when $\tau = (\alpha + \beta)/2$. In this case, we see that

$$P(\alpha, \beta) := \frac{2^{-(3\gamma+2)/\gamma} \gamma}{(3\gamma + 2)} (\beta - \alpha)^{(3\gamma+2)/\gamma}. \tag{2.63}$$

Also, when $r(t) = 1$, then $P^{*}(\alpha, \beta)$ becomes

$$P^{*}(\alpha, \beta) := \left(\frac{\gamma}{2\gamma + 1}\right)^{\gamma-1} \left(\frac{\beta - \alpha}{2}\right)^{((2\gamma+1)(\gamma-1)+\gamma)/\gamma}. \tag{2.64}$$

Theorem 2.13. *Assume that $Q'(t) = q(t)$ and $x(t)$ is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$, for $i = 0, 1, 2$, then*

$$\Phi(Q, r) + \gamma C_\Gamma \alpha(\alpha, \beta, r)\Psi(r, Q) \geq 1, \tag{2.65}$$

where C_Γ and $\alpha(\alpha, \beta, r)$ are defined as in (2.45) *and* (2.46) *and $\Phi(Q, r)$ and $\Psi(Q, r)$ are defined as in* (2.59) *and* (2.62).

Proof. Multiplying (1.1) by $x''(t)$ and integrating by parts, we have

$$\begin{aligned}\int_\alpha^\beta \left(r(t)\left(x'''(t)\right)^\gamma\right)' x''(t)dt &= r(t)\left(x'''(t)\right)^\gamma x''(t)\Big|_\alpha^\beta - \int_\alpha^\beta r(t)\left(x'''(t)\right)^{\gamma+1} dt \\ &= -\int_\alpha^\beta q(t)x^\gamma(t)x''(t)dt.\end{aligned} \tag{2.66}$$

Using the boundary conditions $x''(\alpha) = x''(\beta) = 0$, we get that

$$\int_\alpha^\beta r(t)\left(x'''(t)\right)^{\gamma+1} dt = \int_\alpha^\beta q(t)x^\gamma(t)x''(t)dt = \int_\alpha^\beta Q'(t)x^\gamma(t)x''(t)dt. \tag{2.67}$$

Integrating by parts the right-hand side, we see that

$$\begin{aligned}\int_\alpha^\beta Q'(t)x''(t)x^\gamma(t)dt = Q(t)x''(t)x^\gamma(t)\Big|_\alpha^\beta &- \gamma\int_\alpha^\beta Q(t)x^{\gamma-1}(t)x'(t)x''(t)dt \\ &- \int_\alpha^\beta Q(t)x^\gamma(t)x'''(t)dt.\end{aligned} \tag{2.68}$$

Using the boundary conditions $x''(\beta) = x''(\alpha) = 0$, we see that

$$\begin{aligned}\int_\alpha^\beta Q'(t)x''(t)x^\gamma(t)dt = -\gamma &\int_\alpha^\beta Q(t)x^{\gamma-1}(t)x'(t)x''(t)dt \\ &- \int_\alpha^\beta Q(t)x^\gamma(t)x'''(t)dt.\end{aligned} \tag{2.69}$$

Substituting (2.69) into (2.67), we have

$$\begin{aligned}\int_\alpha^\beta r(t)\left|x'''(t)\right|^{\gamma+1} dt \leq \gamma &\int_\alpha^\beta |Q(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right|dt \\ &+ \int_\alpha^\beta |Q(t)||x(t)|^\gamma\left|x'''(t)\right|dt.\end{aligned} \tag{2.70}$$

Applying the inequality (2.6) on the integral

$$\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma} \left|x'''(t)\right| dt, \tag{2.71}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $m = 1$, $k = 0$, $l = \gamma$, $n = 3$, and $\mu = \gamma + 1$, we get (noting that $x^{(i)}(\alpha) = x^{(i)}(\beta) = 0$, for $i = 0, 1, 2$) that

$$\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma} \left|x'''(t)\right| dt \leq \Phi(Q, r) \int_{\alpha}^{\beta} r(t) \left|x'''(t)\right|^{\gamma+1} dt, \tag{2.72}$$

where $\Phi(Q, r)$ is defined as in (2.59). Applying the inequality (2.13) on the integral

$$\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt, \tag{2.73}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $n = 2$, $r_0 = \gamma - 1$, $r_1 = 1$, $r_2 = 1$, $\sigma + r_2 = \gamma + 1$, and $r = \gamma + 1$, we see that

$$\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt \leq \Psi(Q, r) \int_{\alpha}^{\beta} r(t) \left|x''(t)\right|^{\gamma+1} dt, \tag{2.74}$$

where $\Psi(Q, r)$ is defined as in (2.62). Applying the Hardy inequality (2.1) on the term

$$\int_{\alpha}^{\beta} r(t) \left|x''(t)\right|^{\gamma+1} dt \tag{2.75}$$

with $y(t) = x''(t)$ where $x''(\alpha) = 0 = x''(\beta)$, we see that

$$\int_{\alpha}^{\beta} r(t) \left|x''(t)\right|^{\gamma+1} dt \leq C_{\Gamma} \alpha(\alpha, \beta, r) \int_{\alpha}^{\beta} r(t) \left|x'''(t)\right|^{\gamma+1} dt, \tag{2.76}$$

where C_{Γ} and $\alpha(\alpha, \beta, r)$ are defined as in (2.45) and (2.46). Substituting (2.76) into (2.74), we have

$$\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt \leq \gamma C_{\Gamma} \alpha(\alpha, \beta, r) \Psi(Q, r) \int_{\alpha}^{\beta} r(t) \left|x'''(t)\right|^{\gamma+1} dt. \tag{2.77}$$

Substituting (2.72) and (2.77) into (2.70) and cancelling the term $\int_{\alpha}^{\beta} r(t) |x'''(t)|^{\gamma+1} dt$, we have

$$\Phi(Q, r) + \gamma C_{\Gamma} \alpha(\alpha, \beta, r) \Psi(Q, r) \Psi(Q, r) \geq 1, \tag{2.78}$$

which is the desired inequality (2.28). The proof is complete. □

In the proof of Theorem 2.13 if we apply the Wirtinger inequality (2.20) instead of the Hardy inequality (2.1), then we have the following result.

Theorem 2.14. *Assume that* $rQ'(t) = q(t)$ *and* $x(t)$ *is a nontrivial solution of* (1.1). *If* $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$, *for* $i = 0, 1, 2$, *then*

$$\Phi(Q, r) + \gamma(\gamma+1)^{\gamma+1}\Psi(Q, r) \geq 1, \tag{2.79}$$

where $\Phi(Q, r)$ *and* $\Psi(Q, r)$ *are defined as in* (2.59) *and* (2.62).

Next, in the following, we establish some results which allow us to consider the case when $r(t) < 0$. For simplicity, we denote

$$\begin{aligned} \alpha_1\left(\alpha, \beta, r^2\right) &:= \sup_{(c,d)\subset(\alpha,\beta)} \left(\int_c^d r^2(t)dt\right)^{1/(\gamma+1)} \\ &\quad \times \min\left\{\left(\int_\alpha^c \frac{ds}{r^{2/\gamma}(s)}\right)^{\gamma/(\gamma+1)}, \left(\int_d^\beta \frac{ds}{r^{2/\gamma}(s)}\right)^{\gamma/(\gamma+1)}\right\}, \end{aligned} \tag{2.80}$$

$$\begin{aligned} K_1^* &:= \left(\frac{\gamma}{\gamma+1}\right)^{\gamma/(\gamma+1)} \left[\int_\alpha^\beta \frac{|r(t)r'(t)|^{\gamma+1}}{r^{2\gamma}(t)} P_{1,2}^\gamma(t)dt\right]^{1/(\gamma+1)}, \\ K_2^* &:= \left(\frac{\gamma}{\gamma+1}\right)^{\gamma/(\gamma+1)} \left[\int_\alpha^\beta \frac{|r(t)r'(t)|^{\gamma+1}}{r^{2\gamma}(t)} P_{2,2}^\gamma(t)dt\right]^{1/(\gamma+1)}, \end{aligned} \tag{2.81}$$

where

$$\begin{aligned} P_{1,1}(t) &:= \int_\alpha^t (t-s)^{\gamma+1/\gamma}\left(\frac{1}{r^2(s)}\right)^{1/\gamma} ds, \qquad P_{1,2}(t) := \int_\alpha^t \frac{ds}{r^{2/\gamma}(s)}, \\ P_{2,1}(t) &:= \int_t^\beta (t-s)^{\gamma+1/\gamma}\left(\frac{1}{r^2(s)}\right)^{1/\gamma} ds, \qquad P_{2,2}(t) := \int_t^\beta \frac{ds}{r^{2/\gamma}(s)}. \end{aligned} \tag{2.82}$$

Theorem 2.15. *Suppose that* x *is a nontrivial solution of* (1.1). *If* $x^{(i)}(\alpha) = 0$, *for* $i = 0, 1, 2$ *and* $x''(\beta) = 0$, *then*

$$\Phi_1\left(Q_1, r^2, P_{1,0}\right) + \gamma C_\Gamma \alpha_1\left(\alpha, \beta, r^2\right)\Psi_1\left(Q_1, r^2, P_{1,0}^* P_{1,1}^*\right) + K_1^* \geq 1, \tag{2.83}$$

where $Q_1(t) = \int_t^\beta r(s)q(s)ds$. *If* $x^{(i)}(\beta) = 0$, *for* $i = 0, 1, 2$ *and* $x''(\alpha) = 0$, *then*

$$\Phi_2\left(Q_1, r^2, P_{2,0}\right) + \gamma C_\Gamma \alpha_1\left(\alpha, \beta, r^2\right)\Psi_2\left(Q_1, r^2, P_{2,0}^* P_{2,1}^*\right) + K_2^* \geq 1, \tag{2.84}$$

where $Q_1(t) = \int_\alpha^t r(s)q(s)ds$.

Proof. We prove (2.83). Multiplying (1.1) by $r(t)x''(t)$ and integrating by parts, we have

$$\int_\alpha^\beta \left(r(t)\left(x'''(t)\right)^\gamma\right)' r(t)x''(t)dt = r^2(t)\left(x'''(t)\right)^\gamma x''(t)\Big|_\alpha^\beta - \int_\alpha^\beta r^2(t)\left(x'''(t)\right)^{\gamma+1}dt - \int_\alpha^\beta r(t)r'(t)x''(t)\left(x'''(t)\right)^\gamma dt = -\int_\alpha^\beta r(t)q(t)x''(t)x^\gamma(t)dt. \tag{2.85}$$

Using the boundary conditions $x''(\alpha) = x''(\beta) = 0$ and the assumption $Q_1(t) = \int_t^\beta r(s)q(s)ds$, we have

$$\begin{aligned}\int_\alpha^\beta r^2(t)\left(x'''(t)\right)^{\gamma+1}dt &= -\int_\alpha^\beta r(t)r'(t)x''(t)\left(x'''(t)\right)^\gamma dt + \int_\alpha^\beta q(t)x(t)x''(t)dt \\ &= -\int_\alpha^\beta r(t)r'(t)x''(t)\left(x'''(t)\right)^\gamma dt - \int_\alpha^\beta Q_1'(t)x^\gamma(t)x''(t)dt.\end{aligned} \tag{2.86}$$

Integrating by parts the last term in the right-hand side, we see that

$$\begin{aligned}\int_\alpha^\beta Q_1'(t)x''(t)x^\gamma(t)dt = {} & Q_1(t)x''(t)x^\gamma(t)\big|_\alpha^\beta \\ & -\gamma\int_\alpha^\beta Q_1(t)x^{\gamma-1}(t)x'(t)x''(t)dt \\ & -\int_\alpha^\beta Q_1(t)x^\gamma(t)x'''(t)dt.\end{aligned} \tag{2.87}$$

Using the boundary conditions $x''(\beta) = x''(\alpha) = 0$, we see that

$$\begin{aligned}\int_\alpha^\beta Q_1'(t)x(t)x''(t)dt = {} & -\gamma\int_\alpha^\beta Q_1(t)x^{\gamma-1}(t)x'(t)x''(t)dt \\ & -\int_\alpha^\beta Q_1(t)x^\gamma(t)x'''(t)dt.\end{aligned} \tag{2.88}$$

Substituting (2.88) into (2.86), we have

$$\begin{aligned}\int_\alpha^\beta r^2(t)\left|x'''(t)\right|^{\gamma+1}dt \le {} & \int_\alpha^\beta |Q_1(t)||x(t)|^\gamma \left|x'''(t)\right|dt \\ & +\gamma\int_\alpha^\beta |Q_1(t)||x(t)|^{\gamma-1}\left|x'(t)\right|\left|x''(t)\right|dt \\ & +\int_\alpha^\beta \left|r(t)r'(t)\right|\left|x''(t)\right|\left|x'''(t)\right|^\gamma dt.\end{aligned} \tag{2.89}$$

Applying the inequality (2.6) on the integral

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)|^{\gamma} \left|x'''(t)\right| dt, \tag{2.90}$$

with $\phi(t) = |Q(t)|$, $\vartheta(t) = r(t)$, $m = 1$, $k = 0$, $l = \gamma$, $n = 3$, and $\mu = \gamma + 1$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0, 1, 2$) that

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)|^{\gamma} \left|x'''(t)\right| dt \le \Phi_1\left(|Q_1(t)|, r^2, P_{1,0}\right) \left[\int_{\alpha}^{\beta} r^2(t) \left|x'''(t)\right|^2 dt\right], \tag{2.91}$$

where $\Phi_1(Q_1, r^2, P_{1,0})$ is defined as in (2.24) by replacing Q by Q_1 and r by r^2. Applying the inequality (2.13) on the integral

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt, \tag{2.92}$$

with $\phi(t) = |Q_1(t)|$, $\vartheta(t) = r^2(t)$, $n = 2$, $r_0 = \gamma - 1$, $r_1 = 1$, $r_2 = 1$, $\sigma + r_2 = \gamma + 1$, and $r = \gamma + 1$, we see that

$$\int_{\alpha}^{\beta} |Q_1(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt \le \Psi_1\left(Q_1, r^2, P^*_{1,0} P^*_{1,1}\right) \int_{\alpha}^{\beta} r^2(t) \left|x''(t)\right|^{\gamma+1} dt, \tag{2.93}$$

where $\Psi_1(Q_1, r^2, P^*_{1,0} P^*_{1,1})$ is defined as in (2.26) after replacing Q by Q_1 and r by r^2. Applying the Hardy inequality (2.1) on the term

$$\int_{\alpha}^{\beta} r^2(t) \left|x''(t)\right|^{\gamma+1} dt, \tag{2.94}$$

with $y(t) = x''(t)$ where $x''(\alpha) = 0 = x''(\beta)$, we see that

$$\int_{\alpha}^{\beta} r^2(t) \left|x''(t)\right|^{\gamma+1} dt \le C_{\Gamma} \alpha_1\left(\alpha, \beta, r^2\right) \int_{\alpha}^{\beta} r^2(t) \left|x'''(t)\right|^{\gamma+1} dt, \tag{2.95}$$

where C_{Γ} and $\alpha_1(\alpha, \beta, r^2)$ are defined as in (2.45) and (2.80). Substituting (2.95) into (2.93), we have

$$\begin{aligned}\int_{\alpha}^{\beta} |Q(t)||x(t)|^{\gamma-1} \left|x'(t)\right| \left|x''(t)\right| dt \le C_{\Gamma} \alpha_1\left(\alpha, \beta, r^2\right) \Psi_1\left(Q, r^2, P^*_{1,0} P^*_{1,1}\right) \\ \times \int_{\alpha}^{\beta} r(t) \left|x'''(t)\right|^{\gamma+1} dt.\end{aligned} \tag{2.96}$$

Applying the inequality (2.6) on the integral

$$\int_{\alpha}^{\beta} \left|r(t)r'(t)\right|\left|x''(t)\right|\left|x'''(t)\right|^{\gamma} dt, \tag{2.97}$$

with $\phi(t) = |r(t)r'(t)|$, $\vartheta(t) = r^2(t)$, $m = \gamma$, $k = 2$, $l = 1$, $n = 3$, and $\mu = \gamma + 1$, we get (note that $x^{(i)}(\alpha) = 0$, for $i = 0, 1, 2$) that

$$\int_{\alpha}^{\beta} \left|r(t)r'(t)\right|\left|x''(t)\right|\left|x'''(t)\right|^{\gamma} dt \le K_1^*\left[\int_{\alpha}^{\beta} r^2(t)\left|x'''(t)\right|^{\gamma+1} dt\right], \tag{2.98}$$

where K_1^* is defined as in (2.81). Substituting (2.91), (2.96), and (2.98) into (2.89) and cancelling the term $\int_{\alpha}^{\beta} r^2(t)|x'''(t)|^{\gamma+1} dt$, we have

$$\Phi_1\left(Q_1, r^2, P_{1,0}\right) + \gamma C_{\Gamma}\alpha_1\left(\alpha, \beta, r^2\right)\Psi_1\left(Q_1, r^2, P_{1,0}^* P_{1,1}^*\right) + K_1^* \ge 1, \tag{2.99}$$

which is the desired inequality (2.83). The proof of (2.84) is similar to (2.83) by using the integration by parts and $\Phi_1(Q_1, r^2, P_{1,0}), \Psi_1(Q_1, r^2, P_{1,0}^* P_{1,1}^*)$; K_1^* are replaced by $\Phi_2(Q_1, r^2, P_{2,0})$, $\Psi_2(Q_1, r^2, P_{2,0}^* P_{2,1}^*)$; K_2^* are defined by (2.25), (2.27), and (2.81) by replacing r by r^2. The proof is complete. □

In the following, we assume that there exists $\tau \in (\alpha, \beta)$ such that

$$\int_{\alpha}^{t} \frac{ds}{r^{2/\gamma}(s)} = \int_{t}^{\beta} \frac{ds}{\mathrm{r}^{2/\gamma}(s)}, \tag{2.100}$$

denoted by $P^{\alpha\beta}$. In this case, we denote

$$K^*\left(r^2\right) = K_1^* = K_2^*, \tag{2.101}$$

where

$$K^*\left(r^2\right) := \left(\frac{\gamma P^{\alpha\beta}}{\gamma+1}\right)^{\gamma/(\gamma+1)}\left[\int_{\alpha}^{\beta} \frac{\left|r(t)r'(t)\right|^{\gamma+1}}{r^{2\gamma}(t)} dt\right]^{1/(\gamma+1)}. \tag{2.102}$$

We also assume that there exists $\tau \in (\alpha, \beta)$ such that

$$\int_{\alpha}^{\tau} \frac{(\tau - s)^{2(\gamma+1)/\gamma}}{r^{2/\gamma}(s)} ds = \int_{\tau}^{\beta} \frac{(s - \tau)^{2(\gamma+1)/\gamma}}{r^{2/\gamma}(s)} ds, \tag{2.103}$$

denoted by $P_{r^2}(\alpha,\beta)$. By using r^2 instead of r in $\Phi(Q,r)$ and $\Psi(Q,r)$, we have

$$\Phi\left(Q,r^2\right) := \frac{\Lambda}{2^{\gamma}}\left[\int_{\alpha}^{\beta}\frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{2/\gamma}(t)}P_{r^2}^{\gamma}(\alpha,\beta)dt\right]^{\gamma/(\gamma+1)},$$
$$\Psi\left(Q,r^2\right) := \Lambda\left[\int_{\alpha}^{\beta}\frac{|Q(t)|^{(\gamma+1)/\gamma}}{r^{1/\gamma}(t)}P_{r^2}^{*}(\alpha,\beta)dt\right]^{\gamma/(\gamma+1)}, \tag{2.104}$$

where $P_{r^2}^{*}(\alpha,\beta)$ is obtained from (2.103). Using these new values and proceeding as in the proof of Theorem 2.15, we have the following result.

Theorem 2.16. *Assume that $Q'(t) = q(t)$ and $x(t)$ is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$, for $i = 0, 1, 2$, then*

$$\Phi\left(Q,r^2\right) + \gamma C_{\Gamma}\alpha_1\left(\alpha,\beta,r^2\right)\Psi\left(Q,r^2\right) + K^*\left(r^2\right) \geq 1, \tag{2.105}$$

where C_{Γ} and $\alpha_1(\alpha,\beta,r)$ are defined as in (2.45) *and* (2.80) *and $K^*(r^2)$, $\Phi(Q,r^2)$, and $\Psi(r^2,Q)$ are defined as in* (2.102), (2.104).

In the proofs of Theorems 2.13 and 2.15 if we apply the Wirtinger inequality (2.20) instead of the Hardy inequality (2.1), then we have the following results.

Theorem 2.17. *Assume that $Q'(t) = q(t)$ and $x(t)$ is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$, for $i = 0, 1, 2$, then*

$$\Phi\left(Q,r^2\right) + \gamma(\gamma+1)^{\gamma+1}\Psi\left(Q,r^2\right) + K^*\left(r^2\right) \geq 1, \tag{2.106}$$

where C_{Γ} and $\alpha_1(\alpha,\beta,r)$ are defined as in (2.45) *and* (2.80) *and $K^*(r^2)$, $\Phi(Q,r^2)$, and $\Psi(r^2,Q)$ are defined as in* (2.102), (2.104).

Theorem 2.18. *Assume that $Q'(t) = q(t)$ and $x(t)$ is a nontrivial solution of* (1.1). *If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$, for $i = 0, 1, 2$, then*

$$\Phi\left(Q,r^2\right) + \gamma(\gamma+1)^{\gamma+1}\Psi\left(Q,r^2\right) + K^*\left(r^2\right) \geq 1, \tag{2.107}$$

where $K^*(r^2)$, $\Phi(Q,r^2)$, $\Psi(r^2,Q)$, and K^* are defined as in (2.102), (2.104).

3. Discussions and Examples

In this section, we establish some special cases of the results obtained in Section 2 and also give some illustrative examples. We begin with Theorem 2.8 and consider the case when $r(t) = 1$. In this case, (1.1) becomes

$$((x'''(t)^{\gamma})' + q(t)x^{\gamma}(t) = 0, \quad t \in [\alpha,\beta]. \tag{3.1}$$

When $r(t) = 1$, we see that

$$\Phi_1(Q,1,P_{1,0}) \le \frac{1}{2^\gamma}\left(\frac{1}{\gamma+1}\right)^{1/(\gamma+1)}\left[\frac{\gamma(\beta-\alpha)^{3\gamma+3}}{(3\gamma+3)(3\gamma+2)}\right]^{\gamma/(\gamma+1)}\max_{t\in[\alpha,\beta]}Q(t),$$

$$\Psi_1\left(Q,1,P^*_{1,0}P^*_{1,1}\right) \le \max_{t\in[\alpha,\beta]}|Q(t)|\left(\frac{1}{\gamma+1}\right)^{1/(\gamma+1)}\left(\frac{\gamma}{2\gamma+1}\right)^{\gamma(\gamma-1)/(\gamma+1)} \quad (3.2)$$

$$\times\left(\frac{\gamma}{(2\gamma+1)(\gamma-1)+2\gamma}\right)^{\gamma/(\gamma+1)}(\beta-\alpha)^{((2\gamma+1)(\gamma-1)+2\gamma)/(\gamma+1)},$$

where $Q(t) = \int_t^\beta q(s)ds$. The same will be for $\Phi_2(Q,1,P_{1,0})$ and $\Psi_2(Q,1,P^*_{1,0}P^*_{1,1})$, but in this case we assume that $Q(t) = \int_\alpha^t q(s)ds$. This gives us the following result for (3.1).

Theorem 3.1. *Suppose that x is a nontrivial solution of* (3.1). *If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then*

$$M_\gamma \max_{t\in[\alpha,\beta]}|Q(t)|(\beta-\alpha)^{\gamma((3\gamma+3)/(\gamma+1))} + \gamma N_\gamma \max_{t\in[\alpha,\beta]}|Q(t)|(\beta-\alpha)^{((2\gamma+1)(\gamma-1)+2\gamma)/(\gamma+1)} \ge 1, \quad (3.3)$$

where $Q(t) = \int_t^\beta q(s)ds$, and

$$M_\gamma := \frac{1}{2^\gamma}\left(\frac{1}{\gamma+1}\right)^{1/(\gamma+1)}\left[\frac{\gamma}{(3\gamma+3)(3\gamma+2)}\right]^{\gamma/(\gamma+1)},$$

$$N_\gamma := (\gamma+1)^{\gamma+1}\left(\frac{\gamma}{2\gamma+1}\right)^{\gamma(\gamma-1)/(\gamma+1)}\left(\frac{\gamma}{(2\gamma+1)(\gamma-1)+2\gamma}\right)^{\gamma/(\gamma+1)}\left(\frac{1}{\gamma+1}\right)^{1/(\gamma+1)}. \quad (3.4)$$

If $x^{(i)}(\beta) = 0$, for $i = 0,1,2$ and $x''(\alpha) = 0$, then (3.3) *holds with $Q(t) = \int_\alpha^t q(s)ds$.*

As a special case of Theorem 3.1, if $\gamma = 1$, we have the following result.

Theorem 3.2. *Suppose that x is a nontrivial solution of*

$$x''''(t) + q(t)x(t) = 0, \quad t \in [\alpha,\beta]. \quad (3.5)$$

If $x^{(i)}(\alpha) = 0$, for $i = 0,1,2$ and $x''(\beta) = 0$, then

$$\frac{1}{4\sqrt{15}}\max_{t\in[\alpha,\beta]}\left|\int_t^\beta q(s)ds\right|\left[(\beta-\alpha)^3 + 8\sqrt{15}(\beta-\alpha)\right] \ge 1. \quad (3.6)$$

If $x^{(i)}(\beta) = 0$, *for* $i = 0, 1, 2$ *and* $x''(\alpha) = 0$, *then*

$$\frac{1}{4\sqrt{15}} \max_{t\in[\alpha,\beta]} \left| \int_{\alpha}^{t} q(s)ds \right| \left[(\beta-\alpha)^3 + 8\sqrt{15}(\beta-\alpha) \right] \geq 1. \tag{3.7}$$

As a special case of Theorem 2.11, if $r(t) = 1$, we have the following result.

Theorem 3.3. *Suppose that* x *is a nontrivial solution of* (3.1). *If* $x^{(i)}(0) = 0$, *for* $i = 0, 1, 2$ *and* $x''(\beta) = 0$, *then*

$$\max_{t\in[0,\beta]} \left| \int_{0}^{t} q(s)ds \right| \left[\frac{1}{4\sqrt{15}} \beta^{\gamma((3\gamma+3)/(\gamma+1))} + \frac{1}{42\sqrt{3}} \beta^{2\gamma-1+2/\gamma} \right] \geq 1. \tag{3.8}$$

Example 3.4. Consider the equation

$$x^{(4)}(t) + \cos(\alpha t)x(t) = 0, \quad 0 \leq t \leq \beta, \tag{3.9}$$

where λ and α are positive constants. Theorem 3.3 gives that if the solution of (3.9) satisfies $x^{(i)}(0) = 0$, for $i = 0, 1, 2$ and $x''(\beta) = 0$, then

$$\max_{t\in[\alpha,\beta]} \left| \int_{0}^{t} q(t)dt \right| = \max_{t\in[0,\beta]} \left| \int_{0}^{t} \cos(\alpha t)dt \right| := \frac{1}{\alpha} \geq \frac{4\sqrt{15}+42\sqrt{3}}{\beta^3}. \tag{3.10}$$

That is $\beta \geq (4\sqrt{15}+42\sqrt{3})^{1/3}\alpha^{1/3}$.

Using the definitions of the functions $P_{1,0}$ and $P_{2,0}$ and putting $r(t) = 1$, we see after simplifications that

$$\begin{aligned} P_{1,0}(t) &:= \int_{\alpha}^{t} (t-s)^{2(\gamma+1)/\gamma} ds = \frac{(t-\alpha)^{(3\gamma+2)}}{(3\gamma+2)}, \\ P_{2,0}(t) &:= \int_{t}^{\beta} (s-t)^{2(\gamma+1)/\gamma} ds = \frac{(\beta-t)^{(3\gamma+2)}}{(3\gamma+2)}. \end{aligned} \tag{3.11}$$

The condition (2.57) is satisfied when $(\tau-\alpha)^{(3\gamma+2)/\gamma} = (\beta-\tau)^{(3\gamma+2)/\gamma}$. This in fact is satisfied when $\tau = (\alpha+\beta)/2$. In this case, we see that

$$P(\alpha,\beta) = \frac{\gamma}{2^{(3\gamma+2)/\gamma}(3\gamma+2)} (\beta-\alpha)^{(3\gamma+2)/\gamma}. \tag{3.12}$$

Also, when $r(t) = 1$, one can get that

$$P^*(\alpha,\beta) = \left(\frac{\gamma}{2\gamma+1}\right)^{\gamma-1} \left(\frac{\beta-\alpha}{2}\right)^{(2\gamma+1)(\gamma-1)/\gamma+1}. \tag{3.13}$$

In this case, we have that

$$\Phi(Q,1) := \frac{\Lambda}{2^{\gamma}}\left[\int_{\alpha}^{\beta} |Q(t)|^{(\gamma+1)/\gamma} P^{\gamma}(\alpha,\beta)\,dt\right]^{\gamma/(\gamma+1)},$$
$$\Psi(Q,1) := \Lambda\left[\int_{\alpha}^{\beta} |Q(t)|^{(\gamma+1)/\gamma} P^{*}(\alpha,\beta)\,dt\right]^{\gamma/(\gamma+1)}. \tag{3.14}$$

As a special case of Theorem 2.15, if $r(t) = 1$, then we have the following result.

Theorem 3.5. *Assume that $r(t) > 0$, $Q'(t) = q(t)$, and $x(t)$ is a nontrivial solution of* (3.1)*. If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$ for $i = 0,1,2$, then*

$$\Phi(Q,1) + \gamma(\gamma+1)^{\gamma+1}\Psi(Q,1) \geq 1. \tag{3.15}$$

As a special case when $\gamma = 1$, we see that

$$\Phi(Q,1) \leq \frac{1}{80}\sqrt{5}\max_{t\in[\alpha,\beta]}\left|\int^{t} q(t)\,dt\right| \times (\beta-\alpha)^{3},$$
$$\Psi(Q,1) \leq \frac{1}{2}\max_{t\in[\alpha,\beta]}\left|\int^{t} q(t)\,dt\right|(\beta-\alpha). \tag{3.16}$$

This gives us the following result for (3.5).

Theorem 3.6. *Assume that $r(t) > 0$, $Q'(t) = q(t)$, and $x(t)$ is a nontrivial solution of* (3.5)*. If $x^{(i)}(\alpha) = 0 = x^{(i)}(\beta)$ for $i = 0,1,2$, then*

$$\max_{t\in[\alpha,\beta]}\left|\int^{t} q(t)\,dt\right|\left[\frac{1}{80}\sqrt{5}(\beta-\alpha)^{3} + 2(\beta-\alpha)\right] \geq 1. \tag{3.17}$$

One can also use the rest of theorems to get some new results and due to the limited space the details are left to the reader. The following example illustrates the result.

Example 3.7. Consider the equation

$$x^{(4)}(t) + \lambda\cos^{2}(\alpha t)x(t) = 0, \quad 0 \leq t \leq \pi, \tag{3.18}$$

where λ and α are positive constants. If $x(t)$ is a solution of (3.18), which satisfies $x^{(i)}(0) = x^{(i)}(\pi) = 0$ for $i = 0, 1, 2$, then

$$\begin{aligned}\lambda \max_{t\in[0,\pi]} \int_0^t \cos^2(\alpha t)dt &= \lambda \max_{t\in[0,\pi]} \left[\frac{1}{2}t + \frac{1}{4\alpha}\sin(2\alpha t)\right] \\ &\quad\times \left[\frac{1}{80}\sqrt{5}\pi^3 + 2\pi\right] \\ &= \left(\frac{\lambda\pi}{2} + \frac{\lambda}{4\alpha}\right) \times \left[\frac{1}{80}\sqrt{5}\pi^3 + 2\pi\right].\end{aligned} \tag{3.19}$$

That is, $\lambda(\pi/2 + 1/4\alpha) \geq 1/[(1/80)\sqrt{5}\pi^3 + 2\pi]$.

It will be interesting to establish some new results related to some boundary value problems in bending of beams, see [21, 22].

Problem 1. In particular, one can consider the boundary conditions

$$x(\alpha) = x'(\alpha) = x(\beta) = x'(\beta) = 0, \tag{3.20}$$

which correspond to a beam clamped at each end and establish some new Lyapunov's type inequalities. The main problem in this case that has been appeared when I tried to treat it is the integral $\int (x''')^{\gamma} dt$. Note that this integral is trivial if $\gamma = 1$. So to complete the proof, one should give a relation between this integral and $(x'')^{\gamma}$.

Problem 2. One can also consider the boundary conditions

$$x(\alpha) = x'(\alpha) = x''(\beta) = x'''(\beta) = 0, \tag{3.21}$$

which correspond to a beam clamped at $t = \alpha$ and free at $t = \beta$.

Remark 3.8. The study of the boundary conditions $x(\beta) = x'(\beta) = x''(\alpha) = x'''(\alpha) = 0$, which correspond to a beam clamped at $t = \beta$ and free at $t = \alpha$, and the boundary conditions $x(\alpha) = x''(\alpha) = x(\beta) = x''(\beta) = 0$, which correspond to a beam hinged or supported at both ends will be similar to the proof of the boundary conditions (3.20)-(3.21) and will be left to the interested reader. For more discussions of boundary conditions of the bending of beams, we refer to [21, 22].

Acknowledgment

This project was supported by King Saud University, Deanship of Scientific Research, College of Science Research Centre.

References

[1] Z. Nehari, "Green's functions and disconjugacy," *Archive for Rational Mechanics and Analysis*, vol. 62, no. 1, pp. 53–76, 1976.

[2] U. Elias, "Necessary conditions and sufficient conditions for disfocality and disconjugacy of a differential equation," *Pacific Journal of Mathematics*, vol. 81, no. 2, pp. 379–397, 1979.
[3] P. Almenar and L. Jódar, "A Lyapunov inequality for a second order nonlinear differential equation," *Applied Mathematics Letters*, vol. 24, no. 4, pp. 524–527, 2011.
[4] D. Çakmak, "Lyapunov-type integral inequalities for certain higher order differential equations," *Applied Mathematics and Computation*, vol. 216, no. 2, pp. 368–373, 2010.
[5] L. S. Chen, "A Lyapunov inequality and forced oscillations in general nonlinear nth order differential-difference equations," *Glasgow Mathematical Journal*, vol. 18, no. 2, pp. 161–166, 1977.
[6] S. B. Eliason, "Lyapunov type inequalities for certain second order functional differential equations," *SIAM Journal on Applied Mathematics*, vol. 27, pp. 180–199, 1974.
[7] N. Parhi and S. Panigrahi, "On Liapunov-type inequality for third-order differential equations," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 2, pp. 445–460, 1999.
[8] B. G. Pachpatte, "On the zeros of solutions of certain differential equations," *Demonstratio Mathematica*, vol. 25, no. 4, pp. 825–833, 1992.
[9] B. G. Pachpatte, *An Inequality Suggested by Lyapunov's Inequality*, vol. 26, Centre de Recherches en Mathématiques Pures, Chambery, France, 1995.
[10] W. T. Patula, "On the distance between zeroes," *Proceedings of the American Mathematical Society*, vol. 52, pp. 247–251, 1975.
[11] S. H. Saker, "Lyapunov inequalities for half-linear dynamic equations on time scales and disconjugacy," *Dynamics of Continuous, Discrete & Impulsive Systems B*, vol. 18, no. 2, pp. 149–161, 2011.
[12] S. H. Saker, "Applications of Opial and Wirtinger inequalities on zeros of third order differential equations," *Dynamic Systems and Applications*, vol. 20, pp. 479–494, 2011.
[13] S. H. Saker, "Lyapunov type inequalities for a second order differential equations with a damping term," *Annales Polonici Mathematici*, vol. 103, pp. 37–57, 2012.
[14] S. H. Saker, "Opial's type inequalities on time scales and some applications," *Annales Polonici Mathematici*, vol. 104, pp. 243–260, 2012.
[15] X. Yang, "On Liapunov-type inequality for certain higher-order differential equations," *Applied Mathematics and Computation*, vol. 134, no. 2-3, pp. 307–317, 2003.
[16] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, River Edge, NJ, USA, 2003.
[17] A. Kufner, L. Maligranda, and L. Persson, *The Hardy Inequalities: About Its History and Some Related Results*, Pilsen, 2007.
[18] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, vol. 320, Kluwer Academic, Dordrecht, The Netherlands, 1995.
[19] S. H. Saker, "Applications of Wirtinger inequalities on the distribution of zeros of the Riemann zeta-function," *Journal of Inequalities and Applications*, vol. 2010, Article ID 215416, 15 pages, 2010.
[20] R. P. Agarwal, M. Bohner, D. O'regan, and S. H. Saker, "Some dynamic wirtinger-type inequalities and their applications," *Pacific Journal of Mathematics*, vol. 252, no. 1, pp. 1–18, 2011.
[21] R. Courant and D. Hilbert, *Methods of Mathematical Physics. Vol. I*, John Wiley & Sons, New York, NY, USA, 1989.
[22] D. G. Zill, W. S. Wright, and M. R. Cullen, *Advanced Engineering Mathematics*, Jones and Bartlett, LLC, Canada, 4th edition, 2011.