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Research Article

On the q-Euler Numbers and Polynomials with Weight 0

T. Kim and J. Choi

Division of General Education-Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

Correspondence should be addressed to T. Kim, tkkim@kw.ac.kr

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The purpose of this paper is to investigate some properties of q-Euler numbers and polynomials with weight 0. From those q-Euler numbers with weight 0, we derive some identities on the q-Euler numbers and polynomials with weight 0.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ for $s, t \in \mathbb{Z}$ with $(p, t) = (p, s) = 1$ and $r \in \mathbb{Q}$. In this paper, we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. As well-known definition, the Euler polynomials are defined by

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.1}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1–15]).

In this special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers (see [1]). Recently, the q-Euler numbers with weight α are defined by

$$\widetilde{E}_{0,q}^{(\alpha)} = 1, \qquad q\left(q^{\alpha} \widetilde{E}_q^{(\alpha)} + 1\right)^n + \widetilde{E}_{n,q}^{(\alpha)} = 0 \quad \text{if } n > 0, \tag{1.2}$$

with the usual convention about replacing $(\widetilde{E}_q^{(\alpha)})^n$ by $\widetilde{E}_{n,q}^{(\alpha)}$ (see [3, 12]). The q-number of x is defined by $[x]_q = (1 - q^x)/(1 - q)$ (see [1–15]). Note that $\lim_{q \to 1} [x]_q = x$. Let us define the notation of q-Euler numbers with weight 0 as $\widetilde{E}_{n,q}^{(0)} = \widetilde{E}_{n,q}$. The purpose of this paper is to investigate some interesting identities on the q-Euler numbers with weight 0.

2. On the Extended q-Euler Numbers of Higher-Order with Weight 0

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \end{aligned} \tag{2.1}$$

(see [1–12]). By (2.1), we get

$$q^n I_q(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \tag{2.2}$$

where $f_n(x) = f(x+n)$ and $n \in \mathbb{N}$ (see [4, 5]).

By (1.2), (2.1), and (2.2), we see that

$$\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_{-q}(x) = \widetilde{E}_{n,q}^{(\alpha)} = \frac{[2]_q}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}. \tag{2.3}$$

In the special case, $n = 1$, we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t+1} = \frac{1+q^{-1}}{e^t+q^{-1}} = \sum_{n=0}^{\infty} H_n\left(-q^{-1}\right) \frac{t^n}{n!}, \tag{2.4}$$

where $H_n(-q^{-1})$ are the nth Frobenius-Euler numbers. From (2.4), we note that the q-Euler numbers with weight 0 are given by

$$\widetilde{E}_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n\left(-q^{-1}\right), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.5}$$

Therefore, by (2.5), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{E}_{n,q} = H_n\left(-q^{-1}\right), \tag{2.6}$$

where $H_n(-q^{-1})$ are called the nth Frobenius-Euler numbers.

Let us define the generating function of the q-Euler numbers with weight 0 as follows:

$$\widetilde{F}_q(t) = \sum_{n=0}^{\infty} \widetilde{E}_{n,q} \frac{t^n}{n!}. \tag{2.7}$$

Then, by (2.3) and (2.7), we get

$$\widetilde{F}_q(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mt} = \frac{1+q}{qe^t+1}. \tag{2.8}$$

Now we define the q-Euler polynomials with weight 0 as follows:

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x) \frac{t^n}{n!} = \frac{1+q}{qe^t+1} e^{xt}. \tag{2.9}$$

Thus, (2.4) and (2.9), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x) \frac{t^n}{n!}. \tag{2.10}$$

From (2.10), we have

$$\sum_{n=0}^{\infty} \widetilde{E}_{n,q}(x) \frac{t^n}{n!} = \left(\frac{1+q^{-1}}{e^t+q^{-1}} \right) e^{xt} = \sum_{n=0}^{\infty} H_n\left(-q^{-1}, x\right) \frac{t^n}{n!}, \tag{2.11}$$

where $H_n(-q^{-1}, x)$ are called the nth Frobenius-Euler polynomials (see [9]).

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(x) = H_n\left(-q^{-1}, x\right), \tag{2.12}$$

where $H_n(-q^{-1}, x)$ are called the nth Frobenius-Euler polynomials.

From (2.2) and Theorem 2.2, we note that

$$q^n H_m\left(-q^{-1}, n\right) + H_m\left(-q^{-1}\right) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l, \tag{2.13}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Therefore, by (2.13), we obtain the following corollary.

Corollary 2.3. *For $n \in \mathbb{N}$, with $n \equiv 1$ (mod 2) and $m \in \mathbb{Z}_+$, one has*

$$q^n H_m\left(-q^{-1}, n\right) + H_m\left(-q^{-1}\right) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l. \tag{2.14}$$

In particular, $q=1$, we get $E_m(n)+E_m=2\sum_{l=0}^{n-1}(-1)^l l^m$, where E_m and $E_m(n)$ are called the mth Euler numbers and polynomials which are defined by

$$\frac{2}{e^t+1}=\sum_{m=0}^{\infty}E_m\frac{t^m}{m!},\qquad \frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}. \tag{2.15}$$

By (2.2), we easily see that

$$q\int_{\mathbb{Z}_p}f(x+1)d\mu_{-q}(x)+\int_{\mathbb{Z}_p}f(x)d\mu_{-q}(x)=[2]_q f(0). \tag{2.16}$$

Thus, by (2.16), we get

$$\begin{aligned}[2]_q &= q\int_{\mathbb{Z}_p}e^{(x+1)t}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}e^{xt}d\mu_{-q}(x)\\ &=\sum_{n=0}^{\infty}\left(q\int_{\mathbb{Z}_p}(x+1)^n d\mu_{-q}(x)+\int_{\mathbb{Z}_p}x^n d\mu_{-q}(x)\right)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty}\left(qH_n\left(-q^{-1},1\right)+H_n\left(-q^{-1}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.17}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.4. *For $n\in\mathbb{Z}_+$, one has*

$$qH_n\left(-q^{-1},1\right)+H_n\left(-q^{-1}\right)=\begin{cases}1+q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases} \tag{2.18}$$

where $H_n(-q^{-1},x)$ are called the nth Frobenius-Euler polynomials and $H_n(-q^{-1})$ are called the nth Frobenius-Euler numbers. In particular, $q=1$, we have

$$E_n(1)+E_n=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases} \tag{2.19}$$

where E_n are called the nth Euler numbers.

From (2.5) and Theorem 2.2, we note that

$$\widetilde{E}_{n,q}(x)=\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y)=\sum_{l=0}^{n}\binom{n}{l}\int_{\mathbb{Z}_p}y^l d\mu_{-q}(y)x^{n-l}=\sum_{l=0}^{n}\binom{n}{l}\widetilde{E}_{n,q}x^{n-l}=\left(x+\widetilde{E}_q\right)^n, \tag{2.20}$$

where the usual convention about replacing $(\widetilde{E}_q)^l$ by $\widetilde{E}_{l,q}$. By Theorems 2.2 and 2.4, we get

$$q\widetilde{E}_{n,q}(1)+\widetilde{E}_{n,q}=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases} \tag{2.21}$$

From (2.20) and (2.21), we have

$$q\left(\widetilde{E}_q + 1\right)^n + \widetilde{E}_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{2.22}$$

For $n \in \mathbb{N}$, by (2.20) and (2.22), we have

$$\begin{aligned} q^2\widetilde{E}_{n,q}(2) &= q^2\left(\widetilde{E}_q + 1 + 1\right)^n = q^2\sum_{l=1}^{n}\binom{n}{l}\left(\widetilde{E}_q + 1\right)^l + q\left(1 + q - \widetilde{E}_{0,q}\right) = q + q^2 - q\sum_{l=0}^{n}\binom{n}{l}\widetilde{E}_{l,q} \\ &= q + q^2 - q\left(\widetilde{E}_q + 1\right)^n = q + q^2 + \widetilde{E}_{n,q} - q[2]_q\delta_{0,n}. \end{aligned} \tag{2.23}$$

Therefore, by (2.23), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$, one has*

$$q^2\widetilde{E}_{n,q}(2) = q + q^2 + \widetilde{E}_{n,q}. \tag{2.24}$$

For $n \in \mathbb{Z}_+$, we have

$$\widetilde{E}_{n,q^{-1}}(1-x) = \int_{\mathbb{Z}_p}(1 - x + x_1)^n d\mu_{-q^{-1}}(x_1) = (-1)^n\int_{\mathbb{Z}_p}(x_1 + x)^n d\mu_{-q}(x_1) = (-1)^n\widetilde{E}_{n,q}(x). \tag{2.25}$$

Therefore, by (2.25), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{Z}_+$, one has*

$$\widetilde{E}_{n,q^{-1}}(1-x) = (-1)^n\widetilde{E}_{n,q}(x). \tag{2.26}$$

From (2.20), we have

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-q}(x) = (-1)^n\int_{\mathbb{Z}_p}(x-1)^n d\mu_{-q}(x) = (-1)^n\widetilde{E}_{n,q}(-1). \tag{2.27}$$

By Theorem 2.6 and (2.27), we get

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-q}(x) = \widetilde{E}_{n,q^{-1}}(2) = 1 + q + q^2\widetilde{E}_{n,q^{-1}} \quad \text{if } n > 0. \tag{2.28}$$

Therefore, by (2.28), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}$, one has*

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-q}(x) = 1 + q + q^2\widetilde{E}_{n,q^{-1}}. \tag{2.29}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, p-adic analogue of Bernstein operator of order n for f is given by

$$\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} B_{k,n}(x) f\left(\frac{k}{n}\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \tag{2.30}$$

where $n, k \in \mathbb{Z}_+$ (see [1, 6, 7]).

For $n, k \in \mathbb{Z}_+$, p-adic Bernstein polynomial of degree n is defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in \mathbb{Z}_p \tag{2.31}$$

(see [1, 6, 7]).

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for one Bernstein polynomials in (2.31) as follows:

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} x^k (1-x)^{n-k} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \widetilde{E}_{k+l,q}.
\end{aligned} \tag{2.32}$$

By simple calculation, we easily get

$$\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) &= \int_{\mathbb{Z}_p} B_{n-k,n}(1-x) d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \int_{\mathbb{Z}_p} (1-x)^{n-l} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} \left(1 + q + q^2 \widetilde{E}_{n-l,q^{-1}}\right) \\
&= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 \widetilde{E}_{n-l,q^{-1}} + [2]_q \binom{n}{k} (-1)^k \delta_{0,k} \quad \text{if } n > k.
\end{aligned} \tag{2.33}$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{Z}_+$ with $n > k > 0$, one has*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \widetilde{E}_{k+l,q} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 \widetilde{E}_{n-l,q^{-1}}. \tag{2.34}$$

In particular, $k = 0$, we get

$$\sum_{l=0}^{n} \binom{n}{l} (-1)^l \widetilde{E}_{l,q} = q^2 \widetilde{E}_{n,q^{-1}} + [2]_q. \tag{2.35}$$

By Theorems 2.1 and 2.2, we get

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l H_{k+l}\left(-q^{-1}\right) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k+l} q^2 H_{n-l}(-q), \tag{2.36}$$

where $n, k \in \mathbb{Z}_+$ with $n > k > 0$.

References

[1] S. Araci, D. Erdal, and J. J. Seo, "A Study on the fermionic p-adic $q$integral on $\mathbb{Z}_p$ associated with weighted q-Bernstein and q-Genocchi polynomials," *Abstract and Applied Analysis*, vol. 2011, Article ID 649248, 10 pages, 2011.

[2] D. Erdal, J. J. Seo, and S. Araci, "New construction weighted (h, g)-Genocchi numbers and polynomials related to Zeta type function," *Discrete Dynamics in Nature and Society*, vol. 2011, Article ID 487490, 7 pages, 2011.

[3] T. Kim, B. Lee, J. Choi, Y. H. Kim, and S. H. Rim, "On the q-Euler numbers and weighted q-Bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, pp. 13–18, 2011.

[4] T. Kim, "Some identities on the q-Euler polynomials of higher order and q-stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 4, pp. 484–491, 2009.

[5] T. Kim, J. Y. Choi, and J. Y. Sug, "Extended q-euler numbers and polynomials associated with fermionic p-adic q-integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 14, no. 2, pp. 160–163, 2007.

[6] T. Kim, "A note on q-Bernstein polynomials," *Russian Journal of Mathematical Physic*, vol. 18, no. 4, pp. 73–82, 2011.

[7] L. C. Jang, W.-J. Kim, and Y. Simsek, "A study on the p-adic integral representation on $\mathbb{Z}_p$ associated with Bernstein and Bernoulli polynomials," *Advances in Difference Equations*, vol. 2010, Article ID 163217, 6 pages, 2010.

[8] L. C. Jang, K.-W. Hwang, and Y.-H. Kim, "A note on (h, q)-Genocchi polynomials and numbers of higher order," *Advances in Difference Equations*, vol. 2010, Article ID 309480, 6 pages, 2010.

[9] M. Can, M. Cenkci, V. Kurt, and Y. Simsek, "Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler l-functions," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 135–160, 2009.

[10] Y. Simsek, "Special functions related to Dedekind-type DC-sums and their applications," *Russian Journal of Mathematical Physics*, vol. 17, no. 4, pp. 495–508, 2010.

[11] Y. Simsek, O. Yurekli, and V. Kurt, "On interpolation functions of the twisted generalized Frobenius-Euler numbers," *Advanced Studies in Contemporary Mathematics*, vol. 15, pp. 187–194, 2007.

[12] C. S. Ryoo, "A note on the weighted q-Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, pp. 47–54, 2011.

[13] S. H. Rim, S. J. Lee, E. J. Moon, and J. H. Jin, "On the q-Genocchi numbers and polynomials associated with Q-zeta function," *Proceedings of the Jangjeon Mathematical Society*, vol. 12, no. 3, pp. 261–267, 2009.

[14] C. S. Ryoo, "On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity," *Proceedings of the Jangjeon Mathematical Society*, vol. 13, no. 2, pp. 255–263, 2010.

[15] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.