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Research Article

Common Fixed Points of Weak Contractions in Cone Metric Spaces

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Results on common fixed points of mappings in cone metric spaces under weak contractive conditions (B. S. Choudhury and N. Metiya (2010)) are unified and generalized. Also, cone metric versions of some other related results on weak contractions are proved. Examples show that our results are different than the existing ones.

1. Introduction

The idea to use an ordered Banach space instead of the set of real numbers, as the codomain for a metric, goes back to the mid-20th century (see, e.g., Kurepa [1], Kreĭn and Rutman [2], Kantorovič [3]). Fixed point theory in K-metric and K-normed spaces was developed by Perov [4], Vandergraft [5], and others. For more details we refer the reader to survey papers of Zabrejko [6] and Proinov [7]. In 2007, Huang and Zhang [8] reintroduced such spaces under the name of cone metric spaces and gave definitions of convergent and Cauchy sequences in the terms of interior points of the underlying cone, proving some fixed point theorems in such spaces. After that, fixed-points in cone metric spaces have been a subject of intensive research (see [9] for a survey of these results, and also [10–12]).

Fixed point results under so-called weak contractive conditions were first obtained in [13, 14]. They were generalized by various authors (see, e.g., [15]), in particular, using a pair of control functions φ and ψ [16–21]. Note, however, that it was shown in [22] that in a certain sense the usage of function ψ is superfluous. Weak contractions in cone metric spaces were treated in [23–25], results from [24] being the most general ones.

In this paper we generalize and unify the results on weak contractions in cone metric spaces from [24]. Examples show that these generalizations are proper. Further, we extend theorems from [16] and some related results to the case of cone metric spaces and give examples of applications of the obtained results.

2. Preliminaries

Let E be a real Banach space with θ as the zero element, and let P be a subset of E with the interior $\operatorname{int} P$. The subset P is called a *cone* if (a) P is closed, nonempty, and $P \neq \{\emptyset\}$; (b) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $t, u \in P$ imply $at + bu \in P$; (c) $P \cap (-P) = \{\theta\}$. For the given cone P, a partial ordering $\preceq$ with respect to P is introduced in the following way: $t \preceq u$ if and only if $u - t \in P$. If $u - t \in \operatorname{int} P$, we write $t \ll u$.

If $\operatorname{int} P \neq \emptyset$, the cone P is called *solid* (we will always assume that the given cone has this property). It is called *normal* if there is a number $K > 0$, such that, for all $t, u \in E$, $\theta \preceq t \preceq u$ implies $\|t\| \leq K\|u\|$ or, equivalently, if for all$(n) t_n \preceq u_n \preceq v_n$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} v_n = t$ imply $\lim_{n\to\infty} u_n = t$. The cone P is called *regular* if every decreasing sequence in E which is order-bounded from below is convergent, that is, if whenever $\{t_n\}$ is a sequence in E such that $t_1 \succeq t_2 \succeq \cdots \succeq t_n \succeq \cdots \succeq u$ for some $u \in E$, then there exists $t \in E$ such that $t_n \to t$, $n \to \infty$. Finally, P is called *strongly minihedral* if every subset A of E has an infimum in E, provided it is order-bounded from below. It is well known that every strongly minihedral cone is regular and every regular cone is normal. The converses are not true [26].

Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies (d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$; (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (d_3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$. Then d is called a *cone metric* on X, and (X, d) is called a *cone metric space* [8]. The concept of a cone metric space is obviously more general than that of a metric space.

Sometimes the following additional property of the cone metric will be needed:

(d_4) $d(x, y) \gg \theta$, for all $x, y \in X$ with $x \neq y$.

For definitions of notions such as convergent and Cauchy sequences, completeness, and so forth, we refer to [8] and for a survey of fixed point results in such spaces to [9].

Definition 2.1 (see [24]). Let (X, d) be a cone metric space over a solid cone P. Denote by Φ the class of functions $\varphi : \operatorname{int} P \cup \{\theta\} \to \operatorname{int} P \cup \{\theta\}$ satisfying the following conditions:

(1) $\varphi(t) = \theta$ if and only if $t = \theta$;

(2) $\varphi(t) \ll t$ for $t \in \operatorname{int} P$;

(3) for all $t \in \operatorname{int} P$ and $x, y \in X$, either $\varphi(t) \preceq d(x, y)$ or $d(x, y) \ll \varphi(t)$ holds.

Denote by Ψ the class of functions $\psi : P \to P$ satisfying the following conditions:

(4) ψ is strictly increasing; that is, $t_1 \prec t_2$ if and only if $\psi(t_1) \prec \psi(t_2)$;

(5) $\psi(t) = \theta$ if and only if $t = \theta$.

Let two mappings $f, g : X \to X$ and two arbitrary points $x, y \in X$ be given. The following four sets of vectors will be used:

$$\begin{aligned}
M^4_{f,g}(x,y) &= \left\{ d(gx,gy), d(gx,fx), d(gy,fy), \frac{1}{2}(d(gx,fy)+d(gy,fx)) \right\}, \\
M^3_{f,g}(x,y) &= \left\{ d(gx,gy), \frac{1}{2}(d(gx,fx)+d(gy,fy)), \frac{1}{2}(d(gx,fy)+d(gy,fx)) \right\}, \\
N^4_{f,g}(x,y) &= \left\{ d(x,y), d(x,fx), d(y,gy), \frac{1}{2}(d(x,gy)+d(y,fx)) \right\}, \\
N^3_{f,g}(x,y) &= \left\{ d(x,y), \frac{1}{2}(d(x,fx)+d(y,gy)), \frac{1}{2}(d(x,gy)+d(y,fx)) \right\}.
\end{aligned} \tag{2.1}$$

If $i : X \to X$ is the identity mapping, we will write $M^k_f(x,y) := M^k_{f,i}(x,y)$ for $k \in \{3,4\}$.

Let $f, g : X \to X$ be two self-maps on a nonempty set X. Recall that a point $x \in X$ is called a coincidence point of the pair (f,g) and y is its point of coincidence if $fx = gx = y$. The pair (f,g) is said to be *weakly compatible* if for each $x \in X$, $fx = gx$ implies $fgx = gfx$. A classical result of Jungck states that if, two weakly compatible maps have a unique point of coincidence y, then y is their unique common fixed point.

Roughly speaking, there are two types of common fixed point results with weak contractive conditions. Those of the first type use conditions with $d(fx,fy)$ on the left-hand side and some element of the M-set on the right-hand one. The other use conditions with $d(fx,gy)$ on the left-hand side and some element of the N-set on the right-hand side. An example of the first type is the following results in cone metric spaces that were proved by Choudhury and Metiya.

Theorem 2.2 (see [24], Theorems 3.1, 3.2, and 3.3). *Let (X,d) be a cone metric space over a regular cone P such that (d_4) holds. Let $f, g : X \to X$ be such that one of the following inequalities holds for all $x, y \in X$:*

$$\psi(d(fx,fy)) \preceq \psi(d(gx,gy)) - \varphi(d(gx,gy)), \tag{2.2}$$

$$\psi(d(fx,fy)) \preceq \psi\left(\frac{1}{2}[d(fx,gx)+d(fy,gy)]\right) - \varphi(d(gx,gy)), \tag{2.3}$$

$$\psi(d(fx,fy)) \preceq \psi\left(\frac{1}{2}[d(fx,gy)+d(fy,gx)]\right) - \varphi(d(gx,gy)), \tag{2.4}$$

where $\varphi \in \Phi$ and $\psi \in \Psi$ are continuous. If $fX \subset gX$ and gX is complete, then f and g have a unique point of coincidence in X (and so they have a unique common fixed point in X if the pair (f,g) is weakly compatible).

On the other hand, in the case of metric spaces, the following result was proved by Đorić.

Theorem 2.3 (see [16], Theorem 2.1). *Let (X, d) be a complete metric space, $\psi \in \Psi$ be continuous, and $\varphi \in \Phi$ be lower semicontinuous (here $P = [0, +\infty)$). Let $f, g : X \to X$ be two self-maps satisfying the inequality*

$$\psi(d(fx, gy)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \tag{2.5}$$

for all $x, y \in X$, where $m(x, y) = \max N^4_{f,g}(x, y)$. Then f and g have a unique common fixed point in X.

In this paper we generalize and unify results of Theorem 2.2 (we will call the conditions that we use "weak contractive conditions of the first type"). Examples show that these generalizations are proper. Further, we extend Theorem 2.3 and some related results to the case of cone metric spaces (the respective conditions will be called "weak contractive conditions of the second type") and give examples of applications of the obtained results.

3. Auxiliary Results

We will make use of the following result of Choudhury and Metiya.

Lemma 3.1 (see [24]). *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists $\varphi \in \Phi$ (see Definition 2.1). If $\{y_n\}$ is a sequence in X such that $\{d(y_n, y_{n+1})\}$ is decreasing, then $\{d(y_n, y_{n+1})\}$ converges either to θ or to $r \in \operatorname{int}P$.*

Note that $\varphi \in \Phi$ is not supposed to be continuous. It is easy to show that without the existence of function φ the conclusion of Lemma 3.1 may fail to hold.

The following result is a cone metric version of [21, lemma 2.1].

Lemma 3.2. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists $\varphi \in \Phi$ (see Definition 2.1). Let $\{y_n\}$ be a sequence in X such that $\{d(y_n, y_{n+1})\}$ is decreasing w.r.t. $\preceq$ and that*

$$\lim_{n \to \infty} d(y_n, y_{n+1}) = \theta. \tag{3.1}$$

If $\{y_{2n}\}$ is not a Cauchy sequence, then there exists $c \in \operatorname{int}P$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following five sequences tend to $\varphi(c)$ when $k \to \infty$:

$$\begin{gathered} d(y_{2m_k}, y_{2n_k}), \quad d(y_{2m_k+1}, y_{2n_k+1}), \quad d(y_{2m_k}, y_{2n_k+1}), \\ d(y_{2m_k-1}, y_{2n_k}), \quad d(y_{2m_k-1}, y_{2n_k+1}). \end{gathered} \tag{3.2}$$

Proof. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then there exists $c \in \operatorname{int}P$ such that for each $n_0 \in \mathbb{N}$ there exist $n, m \in \mathbb{N}$ with $n > m \geq n_0$ and $\varphi(c) - d(y_{2m}, y_{2n}) \notin \operatorname{int}P$. Hence, by property (2.4) of function φ, $\varphi(c) \preceq d(y_{2m}, y_{2n})$ holds for $n > m \geq n_0$. Therefore, there exist sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that

$$n_k > m_k > k, \quad d(y_{2m_k}, y_{2n_k}) \succeq \varphi(c), \quad d(y_{2m_k}, y_{2n_k-2}) \ll \varphi(c) \tag{3.3}$$

(the last inequality is obtained by taking the smallest possible n_k). Now we have

$$\begin{aligned}\varphi(c) &\preceq d(y_{2m_k}, y_{2n_k}) \preceq d(y_{2m_k}, y_{2n_k-2}) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}) \\ &\ll \varphi(c) + d(y_{2n_k-2}, y_{2n_k-1}) + d(y_{2n_k-1}, y_{2n_k}).\end{aligned} \tag{3.4}$$

Letting $k \to \infty$ and using assumption (3.1) and the normality of the cone, we obtain that

$$\lim_{k\to\infty} d(y_{2m_k}, y_{2n_k}) = \varphi(c). \tag{3.5}$$

Further,

$$\begin{aligned}d(y_{2m_k}, y_{2n_k}) &\preceq d(y_{2m_k}, y_{2m_k+1}) + d(y_{2m_k+1}, y_{2n_k+1}) + d(y_{2n_k+1}, y_{2n_k}), \\ d(y_{2m_k+1}, y_{2n_k+1}) &\preceq d(y_{2m_k+1}, y_{2m_k}) + d(y_{2m_k}, y_{2n_k}) + d(y_{2n_k}, y_{2n_k+1}),\end{aligned} \tag{3.6}$$

where $d(y_{2m_k}, y_{2n_k}) \to \varphi(c)$, imply that

$$\lim_{k\to\infty} d(y_{2m_k+1}, y_{2n_k+1}) = \varphi(c). \tag{3.7}$$

The other three limits can be obtained similarly. □

4. Weak Contractions of the First Type in Cone Metric Spaces

Theorem 4.1. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f, g : X \to X$ be two selfmaps such that $fX \subset gX$ and let one of these subsets of X be complete. Suppose that for all $x, y \in X$ there exists*

$$u = u(x, y) \in M^4_{f,g}(x, y) \tag{4.1}$$

such that

$$d(fx, fy) \preceq u(x, y) - \varphi(u(x, y)) \tag{4.2}$$

holds true. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Remark 4.2. Theorem 4.1 remains true if condition (4.2) is replaced by

$$\psi(d(fx, fy)) \preceq \psi(u(x, y)) - \varphi(u(x, y)) \tag{4.3}$$

for some continuous $\psi \in \Psi$ (see Definition 2.1). The proof is essentially the same, and so, for the sake of simplicity, we stay within the given version. The same remark applies to all other results in the rest of the paper. See also paper [22] where it is shown that practically each weak contractive condition with function ψ can be replaced by an equivalent condition without ψ.

Proof. Starting from arbitrary $x_1 \in X$ and using the assumption $fX \subset gX$, construct a Jungck sequence $\{y_n\}$ satisfying $y_n = fx_n = gx_{n+1}$ for $n \in \mathbb{N}$. If $y_{n_0} = y_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then $gx_{n_0+1} = y_{n_0} = y_{n_0+1} = fx_{n_0+1}$ and f and g have a point of coincidence. Suppose, further, that $y_n \neq y_{n+1}$ for $n \in \mathbb{N}$. Putting $x = x_{n+1}$, $y = x_n$ in (4.2) we obtain that

$$d(y_{n+1}, y_n) = d(fx_{n+1}, fx_n) \preceq u - \varphi(u), \tag{4.4}$$

where

$$\begin{aligned} u = u(x_{n+1}, x_n) \in \Big\{ & d(gx_{n+1}, gx_n), d(gx_{n+1}, fx_{n+1}), d(gx_n, fx_n), \\ & \frac{1}{2}\left(d(gx_{n+1}, fx_n) + d(gx_n, fx_{n+1})\right) \Big\} \\ = \Big\{ & d(y_n, y_{n-1}), d(y_n, y_{n+1}), d(y_{n-1}, y_n), \frac{1}{2} d(y_{n-1}, y_{n+1}) \Big\}. \end{aligned} \tag{4.5}$$

The case $u = d(y_n, y_{n+1})$ is impossible, since it would imply $u \preceq u - \varphi(u)$ and $u = \theta$ (using properties of the function φ), which is already excluded. In all other cases we get that $d(y_{n+1}, y_n) \preceq d(y_n, y_{n-1})$, and, more precisely,

$$d(y_{n+1}, y_n) \preceq u(x_{n+1}, x_n) \preceq d(y_n, y_{n-1}). \tag{4.6}$$

Indeed, the right-hand inequality is trivial in the case when $u = d(y_n, y_{n-1})$, and in the case $u = (1/2)d(y_{n-1}, y_{n+1})$, then $d(y_{n+1}, y_n) \preceq d(y_n, y_{n-1})$ and $u \preceq (1/2)d(y_{n-1}, y_n) + (1/2)d(y_n, y_{n+1}) \preceq (1/2)d(y_{n-1}, y_n) + (1/2)d(y_{n-1}, y_n) = d(y_{n-1}, y_n)$.

We have proved that the sequence $\{d(y_n, y_{n+1})\}$ is decreasing w.r.t. $\preceq$ and so Lemma 3.1 implies that it converges to some r, where either $r = \theta$ or $r \in \operatorname{int} P$. But, if $r \in \operatorname{int} P$, then (4.6) implies that also $u(x_{n+1}, x_n) \to r$ as $n \to \infty$. Hence, passing to the limit in (4.4) we get that $r \preceq r - \varphi(r)$ and $r = \theta$, a contradiction. Thus, $r = \lim_{n\to\infty} d(y_n, y_{n+1}) = \theta$.

Let us prove that $\{y_n\}$ is a Cauchy sequence in X. Suppose that it is not. It follows from monotonicity of the sequence $\{d(y_n, y_{n+1})\}$ and $\lim_{n\to\infty} d(y_n, y_{n+1}) = \theta$ that neither $\{y_{2n}\}$ is a Cauchy sequence. Lemma 3.2 implies that there exist sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the sequences (3.2) all tend to $\varphi(c)$ for some $c \gg 0$. Using (4.6) and putting $x = x_{2n_k}$, $y = x_{2m_k-1}$ in (4.2) we get that

$$d(y_{2n_k+1}, y_{2m_k}) = d(fx_{2n_k}, fx_{2m_k-1}) \preceq u(x_{2n_k}, x_{2m_k-1}) - \varphi(u(x_{2n_k}, x_{2m_k-1})). \tag{4.7}$$

Letting $k \to \infty$ we get that $\varphi(c) \preceq \varphi(c) - \varphi(\varphi(c))$. Properties of function φ imply that $c = \theta$, a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence.

By the assumption, there exists $\lim_{n\to\infty} y_n = gp$ for some $p \in X$. Let us prove that $fp = gp$. Putting $x = x_n$, $y = p$ in (4.2) we get that

$$d(fx_n, fp) \preceq u - \varphi(u), \tag{4.8}$$

where

$$u = u(x_n, p) \in \left\{ d(gx_n, gp), d(gx_n, fx_n), d(gp, fp), \frac{1}{2}(d(gx_n, fp) + d(gp, fx_n)) \right\}. \tag{4.9}$$

In other words, at least one of four possible inequalities holds for infinitely many $n \in \mathbb{N}$. Hence, passing to the limit, we obtain that $d(gp, fp) \preceq \theta - \varphi(\theta) = \theta$ or $d(gp, fp) \preceq d(gp, fp) - \varphi(d(gp, fp))$ or $d(gp, fp) \preceq 1/2d(gp, fp) - \varphi(1/2d(gp, fp)) \preceq 1/2d(gp, fp)$. By the properties of φ it follows that $d(gp, fp) = \theta$ and $gp = fp = q$; hence q is a point of coincidence for the pair (f, g).

To prove that this point of coincidence is unique, assume that there is another $q_1 \in X$ such that $q_1 = fp_1 = gp_1$ for some $p_1 \in X$. Then

$$d(q, q_1) = d(fp, fp_1) \preceq u - \varphi(u), \tag{4.10}$$

where

$$\begin{aligned} u = u(p, p_1) &\in \left\{ d(gp, gp_1), d(gp, fp), d(gp_1, fp_1), \frac{1}{2}(d(gp, fp_1) + d(fp, gp_1)) \right\} \\ &= \{d(q, q_1), \theta\}. \end{aligned} \tag{4.11}$$

In both cases we get that $d(q, q_1) = \theta$, that is, the point of coincidence is unique. □

Obviously, the theorem in [24, Theorem 3.1] (Theorem 2.2 with condition (2.2)) is a special case of Theorem 4.1.

Remark 4.3. The previous theorem can be modified so that continuity of φ is substituted by its lower semicontinuity; however, in this case it has to be assumed that the cone P is strongly minihedral. For details see [10]. The same applies to other assertions to the end of the paper.

The following example shows that there are cases when the existence of a common fixed point can be deduced using Theorem 4.1, but cannot be obtained using the theorem in [24, Theorems 3.1, 3.2, and 3.3] (Theorem 2.2 with either of the conditions (2.2), (2.3), or (2.4)).

Example 4.4. Let $X = [0,1] \cup [3/2,2]$, $E = \mathbb{R}^2$ $(\theta = (0,0))$, $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$ and $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha > 0$ is fixed. d is obviously a cone metric satisfying property (d_4) and the cone P is regular (even minihedral). Function $\varphi : \operatorname{int} P \cup \{\theta\} \to \operatorname{int} P \cup \{\theta\}$ defined by $\varphi(\theta) = \theta$ and $\varphi(t_1, t_2) = ((1/4)t_1, (1/4)t_2)$ for $t_1 > 0$, $t_2 > 0$ belongs to the respective class Φ. Consider the mappings $f, g : X \to X$ defined by: $gx = x$ for $x \in X$, $fx = 3/4$ for $x \in [0,1]$ and $fx = 1/4$ for $x \in [3/2, 2]$. We will show that, taking, for example, $\psi(t) = t$, neither of conditions (2.2), (2.3), (2.4) is satisfied; hence neither of Theorems 3.1, 3.2, and 3.3 from [24] can be used to conclude that there exists a common fixed point of f and g (which is obviously $p = 3/4$).

Indeed, for $x = 1$, $y = 3/2$, we get that $d(fx, fy) = (1/2, (1/2)\alpha)$. On the other hand $d(gx, gy) - \varphi(d(gx, gy)) = d(1, 3/2) - \varphi(d(1, 3/2)) = (1/2, (1/2)\alpha) - 1/4(1/2, (1/2)\alpha) = (3/8, (3/8)\alpha) \prec (1/2, (1/2)\alpha)$. Hence, condition (2.2) is not satisfied. Further, for $x = 3/4$, $y = 3/2$, we obtain that $d(fx, fy) = (1/2, (1/2)\alpha)$ and $(1/2)(d(gx, fx) + d(gy, fy)) - \varphi(d(gx, gy)) = (1/2)(d(3/4, 3/4) + d(3/2, 1/4)) - \varphi(d(3/4, 3/2)) = (1/2)(5/4, (5/4)\alpha) - (1/4)(3/4, (3/4)\alpha) = (7/16, (7/16)\alpha) \prec (1/2, (1/2)\alpha)$. Hence condition (2.3) is not satisfied. Finally, taking $x = 1/4$, $y = 3/2$, we have that $d(fx, fy) = (1/2, (1/2)\alpha)$, but $(1/2)(d(fx, gy) + d(fy, gx)) - \varphi(d(gx, gy)) = (1/2)(d(3/4, 3/2) + d(1/4, 1/4)) - \varphi(d(1/4, 3/2)) = (1/2)(3/4, (3/4)\alpha) - (1/4)(5/4, (5/4)\alpha) = (1/16, (1/16)\alpha) \prec (1/2, (1/2)\alpha)$. Hence, neither (2.4) is satisfied.

On the other hand, condition (4.2) of Theorem 4.1 is satisfied. Indeed, if $x, y \in [0, 1]$ or $x, y \in [1, 3/2]$, then $d(fx, fy) = \theta$ and the condition is trivially satisfied. If $x \in [0, 1]$ and $y \in [3/2, 2]$ (or vice versa), take $u = u(x, y) = d(gy, fy) = (y - 1/4, (y - 1/4)\alpha)$ and we obtain that $d(fx, fy) = (1/2, (1/2)\alpha)$ and $u - \varphi(u) = (y - 1/4, (y - 1/4)\alpha) - (1/4)(y - 1/4, (y - 1/4)\alpha) = 3/4(y - 1/4, (y - 1/4)\alpha) \succeq (3/4)(5/4, (5/4)\alpha) = (15/16, (15/16)\alpha) \succ (1/2, (1/2)\alpha)$. Thus, conclusion about the existence of a common fixed point can be deduced from Theorem 4.1.

In the next theorem the set $M^3_{f,g}(x, y)$ is used instead of $M^4_{f,g}(x, y)$. The proof is essentially the same as for Theorem 4.1 and so is omitted.

Theorem 4.5. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f, g : X \to X$ be two selfmaps such that $fX \subset gX$ and let one of these subsets of X be complete. Suppose that for all $x, y \in X$ there exists*

$$u = u(x, y) \in M^3_{f,g}(x, y), \tag{4.12}$$

such that

$$d(fx, fy) \preceq u(x, y) - \varphi(u(x, y)) \tag{4.13}$$

holds true. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

In the following theorem we unify Theorems 3.1, 3.2, and 3.3 of [24] (Theorem 2.2 with conditions (2.2), (2.3), or (2.4)).

Theorem 4.6. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f, g : X \to X$ be two selfmaps such that $fX \subset gX$ and let one of these subsets of X be complete. Suppose that for all $x, y \in X$ there exists*

$$u = u(x, y) \in M^3_{f,g}(x, y), \tag{4.14}$$

such that

$$d(fx, fy) \preceq u(x, y) - \varphi(d(gx, gy)) \tag{4.15}$$

holds true. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Proof. As usual, form a Jungck sequence by $y_n = fx_n = gx_{n+1}$. If $y_{n_0} = y_{n_0+1}$, then it can be proved as in Theorem 4.1 that (f,g) has a point of coincidence. Assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Then

$$d(y_{n+1},y_n) = d(fx_{n+1},fx_n) \preceq u(x_{n+1},x_n) - \varphi(d(y_{n+1},y_n)), \tag{4.16}$$

where

$$u(x_{n+1},x_n) \in \left\{d(y_n,y_{n_1}), \frac{1}{2}(d(y_n,y_{n+1}) + d(y_{n-1},y_n)), \frac{1}{2}(\theta + d(y_{n-1},y_{n+1}))\right\}. \tag{4.17}$$

In each of the three possible cases it is easy to obtain that $\{d(y_{n+1},y_n)\}$ is a decreasing sequence and that

$$d(y_{n+1},y_n) \preceq u(x_{n+1},x_n) \preceq d(y_n,y_{n-1}). \tag{4.18}$$

Hence, all these three terms tend to some $r \in \mathrm{int}P \cup \{\theta\}$. Passing to the limit in relation (4.16) we get that $r \preceq r - \varphi(r)$, wherefrom it follows that $r = \theta$.

That $\{y_n\}$ is a Cauchy sequence can be proved using Lemma 3.2 similarly as in the proof of Theorem 4.1. Hence, there exists $p \in X$ such that $y_n = fx_n = gx_{n+1} \to gp$ when $n \to \infty$. Let us prove that gp is a point of coincidence of the pair (f,g).

Putting $x = x_n$, $y = p$ in (4.15) we get that

$$d(fx_n,fp) \preceq u(x_n,p) - \varphi(d(gx_n,gp)), \tag{4.19}$$

where

$$u(x_n,p) \in \left\{d(gx_n,gp), \frac{1}{2}(d(gx_n,fx_n) + d(gp,fp)), \frac{1}{2}(d(gx_n,fp) + d(fx_n,gp))\right\}. \tag{4.20}$$

Passing to the limit (more precisely, considering one of the inequalities that holds for infinitely many n as in the proof of Theorem 4.1) we get either $d(gp,fp) \preceq \theta$ or $d(gp,fp) \preceq (1/2)d(gp,fp)$ and both are possible only if $fp = gp$. Hence gp is a point of coincidence of (f,g).

The proof that the point of coincidence is unique is essentially the same as in Theorem 4.1. □

The next example illustrates how Theorem 4.6 can be used to prove the existence of a common fixed point, while either Theorem 3.2 or 3.3 of [24] cannot.

Example 4.7. Let $X = [0,1]$, and let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0, y \geq 0\}$ and $d(x,y) = (|x-y|, \alpha|x-y|)$ be as in the previous example. Take function $\varphi \in \Phi$ defined by $\varphi(\theta) = \theta$ and $\varphi(t_1,t_2) = ((1/2)t_1^2, (1/2)t_2^2)$. Mappings $f,g : X \to X$ are defined as $fx = x - (1/2)x^2$ and $gx = x$.

Condition (2.3) is not satisfied. Indeed, take $x = 1$ and $y = 0$ to obtain that $d(fx, fy) = d(1/2, 0) = (1/2, (1/2)\alpha)$ and $(1/2)(d(gx, fx) + d(gy, fy)) - \varphi(d(gx, gy)) = (1/2)d(1, 1/2) - \varphi(d(1, 0)) = (1/4, (1/4)\alpha) - (1/2, (1/2)\alpha) = (-1/4, -(1/4)\alpha) \prec (1/2, (1/2)\alpha)$. Similarly, condition (2.4) is not satisfied, for taking again $x = 1$ and $y = 0$ one obtains that $d(fx, fy) = d(1/2, 0) = (1/2, (1/2)\alpha)$, but $(1/2)(d(gx, fy) + d(gy, fx)) - \varphi(d(gx, gy)) = (1/2)(d(1, 0) + d(0, 1/2)) - \varphi(d(1, 0)) = (3/4, (3/4)\alpha) - (1/2, (1/2)\alpha) = (1/4, (1/4)\alpha) \prec (1/2, (1/2)\alpha)$.

We show that, however, condition (4.15) is satisfied and so Theorem 4.6 can be used to conclude that there exists a common fixed point of f and g (which is obviously $p = 0$) (note that this can also be done using condition (2.2)). Indeed, take $u = u(x, y) = d(gx, gy) = d(x, y) \in M^3_{f,g}(x, y)$. In order to prove inequality (4.15) it is enough to consider the first coordinates of respective vectors, that is, we have to prove that

$$\left| x - \frac{1}{2}x^2 - y + \frac{1}{2}y^2 \right| \leq \left| x - y \right| - \frac{1}{2}|x - y|^2 \tag{4.21}$$

holds for all $x, y \in [0, 1]$. But, it is an easy consequence of $|x - y| \leq x + y$.

Finally, we state (proof can be deduced similarly as for the previous theorems) the following cone metric version of [18, Theorems 3.1 and 4.1] (see also [21, Theorem 3.6]).

Theorem 4.8. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f : X \to X$ be a selfmap such that for all $x, y \in X$ there exist*

$$u(x, y) \in M^4_f(x, y), \qquad v(x, y) \in \{d(x, y), d(y, fy)\}, \tag{4.22}$$

such that $d(fx, fy) \preceq u(x, y) - \varphi(v(x, y))$. Then f has a fixed point.

Note that in this case fixed point of f need not be unique. It is enough to consider the identity mapping $f = i_X$ and take $v = \theta$.

5. Weak Contractions of the Second Type in Cone Metric Spaces

In this section we consider weak contractions which we have called "of the second type" (see the end of Section 2).

Theorem 5.1. *Let (X, d) be a complete cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f, g : X \to X$ be two mappings such that for all $x, y \in X$ there exists*

$$u(x, y) \in N^4_{f,g}(x, y) \tag{5.1}$$

such that

$$d(fx, gy) \preceq u(x, y) - \varphi(u(x, y)). \tag{5.2}$$

Then f and g have a unique common fixed point.

Proof. Let us prove first that the common fixed point of f and g is unique (if it exists). Suppose that $p \neq q$ are two distinct common fixed points of f and g. Then (5.2) implies that

$$d(p,q) = d(fp,gq) \preceq u(p,q) - \varphi(u(p,q)), \tag{5.3}$$

where $u(p,q) \in N^4_{f,g}(p,q) = \{d(p,q),\theta,\theta,d(p,q)\} = \{\theta,d(p,q)\}$. Checking both possible cases and using the properties of function φ, we readily obtain that $d(p,q) = \theta$, that is, $p = q$.

In order to prove the existence of a common fixed point, proceed this time constructing a Jungck sequence by $x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, for arbitrary $x_0 \in X$. Consider the two possible cases.

Suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$. Then $x_{n+1} = x_{n+2}$ and it follows that the sequence is eventually constant, and so convergent. Indeed, let, for example, $n = 2k$ (in the case $n = 2k+1$ the proof is similar). Then, putting $x = x_{2k}$, $y = x_{2k+1}$ in (5.2), we get that there exists

$$\begin{aligned} u &\in \left\{ d(x_{2k},x_{2k+1}), d(x_{2k},fx_{2k}), d(x_{2k+1},gx_{2k+1}), \frac{1}{2}\left(d(x_{2k},gx_{2k+1}) + d(x_{2k+1},fx_{2k})\right) \right\} \\ &= \left\{ \theta, d(x_{2k+1},x_{2k+2}), \frac{1}{2}d(x_{2k},x_{2k+2}) \right\}, \end{aligned} \tag{5.4}$$

such that $d(x_{2k+1},x_{2k+2}) \preceq u - \varphi(u)$. Consider the three possible cases:

(1°) $u = \theta$; it trivially follows that $x_{2k} = x_{2k+1}$.

(2°) $u = d(x_{2k+1},x_{2k+2})$; it follows that

$$d(x_{2k+1},x_{2k+2}) \preceq d(x_{2k+1},x_{2k+2}) - \varphi(d(x_{2k+1},x_{2k+2})), \tag{5.5}$$

and by the properties of function φ that $x_{2k} = x_{2k+1}$.

(3°) $u = (1/2)d(x_{2k},x_{2k+2})$; since $u \preceq (1/2)(d(x_{2k},x_{2k+1}) + d(x_{2k+1},x_{2k+2})) = (1/2)d(x_{2k+1}, x_{2k+2})$, it follows that

$$d(x_{2k+1},x_{2k+2}) \preceq \frac{1}{2}d(x_{2k+1},x_{2k+2}) - \varphi\left(\frac{1}{2}d(x_{2k},x_{2k+2})\right) \preceq \frac{1}{2}d(x_{2k+1},x_{2k+2}), \tag{5.6}$$

wherefrom $d(x_{2k+1},x_{2k+2}) \preceq (1/2)d(x_{2k+1},x_{2k+2})$ which is only possible if $x_{2k} = x_{2k+1}$.

Suppose now that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Putting $x = x_{2n}$, $y = x_{2n-1}$ in (5.2), we get that there exists

$$\begin{aligned} u &\in \left\{ d(x_{2n},x_{2n-1}), d(x_{2n},fx_{2n}), d(x_{2n-1},gx_{2n-1}), \frac{1}{2}\left(d(x_{2n},gx_{2n-1}) + d(x_{2n-1},fx_{2n})\right) \right\} \\ &= \left\{ d(x_{2n},x_{2n-1}), d(x_{2n},x_{2n+1}), \frac{1}{2}d(x_{2n-1},x_{2n+1}) \right\}, \end{aligned} \tag{5.7}$$

such that $d(x_{2n+1}, x_{2n}) \preceq u - \varphi(u)$. Consider the three possible cases:

(1°) $u = d(x_{2n}, x_{2n-1})$; it follows that

$$d(x_{2n+1}, x_{2n}) \preceq d(x_{2n}, x_{2n-1}) - \varphi(d(x_{2n}, x_{2n-1})) \prec d(x_{2n}, x_{2n-1}) \tag{5.8}$$

and $d(x_{2n+1}, x_{2n}) \prec d(x_{2n}, x_{2n-1})$.

(2°) $u = d(x_{2n}, x_{2n+1})$; it follows that

$$d(x_{2n+1}, x_{2n}) \preceq d(x_{2n}, x_{2n+1}) - \varphi(d(x_{2n}, x_{2n+1})) \prec d(x_{2n}, x_{2n+1}), \tag{5.9}$$

which is impossible.

(3°) $u = (1/2)d(x_{2n-1}, x_{2n+1})$; it follows that

$$d(x_{2n+1}, x_{2n}) \preceq \frac{1}{2}d(x_{2n-1}, x_{2n+1}) - \varphi\left(\frac{1}{2}d(x_{2n-1}, x_{2n+1})\right). \tag{5.10}$$

By the properties of function φ we obtain that $d(x_{2n+1}, x_{2n}) \preceq (1/2)(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))$ and $d(x_{2n+1}, x_{2n}) \preceq d(x_{2n}, x_{2n-1})$.

Hence, in any possible case, $d(x_{2n+1}, x_{2n}) \preceq d(x_{2n}, x_{2n-1})$ and, similarly, $d(x_{2n+2}, x_{2n+1}) \preceq d(x_{2n+1}, x_{2n})$. Thus, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing; moreover,

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) \preceq u(x_{2n+1}, x_{2n}) \preceq d(x_{2n+1}, x_{2n}), \\ d(x_{2n+1}, x_{2n}) \preceq u(x_{2n}, x_{2n-1}) \preceq d(x_{2n}, x_{2n-1}). \end{aligned} \tag{5.11}$$

We prove now that

$$d(x_n, x_{n+1}) \longrightarrow \theta, \quad n \longrightarrow \infty. \tag{5.12}$$

Indeed, passing to the limit in (5.11) when $n \to \infty$ (and using regularity of the cone), we obtain that $d(x_{n+1}, x_n) \to r$ and $u(x_{n+1}, x_n) \to r$ $(n \to \infty)$ for some $r \in \operatorname{int}P \cup \{\theta\}$. If $r \in \operatorname{int}P$, then passing to the limit in

$$d(x_{2n+1}, x_{2n+2}) \preceq u(x_{2n}, x_{2n+1}) - \varphi(u(x_{2n}, x_{2n+1})), \tag{5.13}$$

we obtain that $r \preceq r - \varphi(r)$ and $r = \theta$ by the properties of function $\varphi \in \Phi$. Hence, (5.12) holds.

We next prove that $\{x_n\}$ is a Cauchy sequence. According to monotonicity of $\{d(x_n, x_{n+1})\}$ and (5.12), it is sufficient to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence. Suppose that this is not the case. Applying Lemma 3.2 we obtain that there exist $c \gg \theta$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that the sequences

$$d(x_{2m_k}, x_{2n_k}), \quad d(x_{2m_k}, x_{2n_k+1}), \quad d(x_{2m_k-1}, x_{2n_k}), \quad d(x_{2m_k-1}, x_{2n_k+1}) \tag{5.14}$$

all tend to $\varphi(c)$ when $k \to \infty$.

Now, from (5.11) and the obtained limits, we have that

$$\lim_{k\to\infty} u(x_{2m_k-1}, x_{2n_k}) = \varphi(c), \tag{5.15}$$

for any $u(x_{2m_k-1}, x_{2n_k}) \in N^4_{f,g}(x_{2m_k-1}, x_{2n_k})$. Letting $k \to \infty$, utilizing (5.15) and the obtained limits, we get

$$\varphi(c) \preceq \varphi(c) - \varphi(\varphi(c)), \tag{5.16}$$

which is a contradiction if $c \gg \theta$. This shows that $\{x_{2n}\}$ is a Cauchy sequence and hence $\{x_n\}$ is a Cauchy sequence.

Since the space (X, d) is complete, there exists $p \in X$ such that $\lim_{n\to\infty} x_n = p$. Then also $x_{2n+1} = f x_{2n} \to p$ and $x_{2n} = g x_{2n-1} \to p$ $(n \to \infty)$. Putting $x = x_{2n}$ and $y = p$ in (5.2), we get $d(f x_{2n}, gp) \preceq u - \varphi(u)$, where

$$u \in \left\{ d(x_{2n}, p), d(x_{2n}, f x_{2n}), d(p, gp), \frac{1}{2}\left(d(x_{2n}, gp) + d(p, f x_{2n})\right) \right\}. \tag{5.17}$$

So, in this case we have four possibilities:

(1°) $d(f x_{2n}, gp) \preceq d(x_{2n}, p) - \varphi(d(x_{2n}, p))$;

(2°) $d(f x_{2n}, gp) \preceq d(x_{2n}, f x_{2n}) - \varphi(d(x_{2n}, f x_{2n}))$;

(3°) $d(f x_{2n}, gp) \preceq d(p, gp) - \varphi(d(p, gp))$;

(4°) $d(f x_{2n}, gp) \preceq (1/2)(d(x_{2n}, gp) + d(p, f x_{2n})) - \varphi((1/2)(d(x_{2n}, gp) + d(p, f x_{2n})))$.

Passing to the limit when $n \to \infty$ in these four relations, we obtain one of the next three inequalities:

$$\begin{gathered} d(p, gp) \preceq \theta - \varphi(\theta), \\ d(p, gp) \preceq d(p, gp) - \varphi(d(p, gp)), \\ d(p, gp) \preceq \frac{1}{2} d(p, gp) - \varphi\left(\frac{1}{2} d(p, gp)\right) \preceq \frac{1}{2} d(p, gp). \end{gathered} \tag{5.18}$$

In each of the cases it easily follows that $gp = p$.

Now, putting $x = y = p$ in (5.2), one gets $d(fp, gp) \preceq u - \varphi(u)$, where $u \in \{\theta, d(p, fp), (1/2)d(p, fp)\}$ and in each of the possible three cases it easily follows that $fp = p$. Hence, p is a common fixed point of f and g. □

Putting $g = f$ in Theorem 5.1, one obtains

Corollary 5.2. *Let (X, d) be a complete cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f : X \to X$ be such that for all $x, y \in X$ there exists*

$$u(x, y) \in M^4_f(x, y) \tag{5.19}$$

such that

$$d(fx,fy) \preceq u(x,y) - \varphi(u(x,y)). \tag{5.20}$$

Then f has a unique common fixed point.

Note that putting $E = \mathbb{R}$ and $P = [0,+\infty)$ in Theorem 5.1 and Corollary 5.2 we obtain as corollaries [16, Theorems 2.1 and 2.2], [15, Theorem 2.1], and [21, Theorem 4.1 and Corollary 4.2].

Adapting an example from [16] we give an example when Theorem 5.1 (modified to use a function $\psi \in \Psi$ according to Remark 4.2) can be used to deduce the existence of a common fixed point.

Example 5.3. Let $X = [0,1]$, and let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x \geq 0,\ y \geq 0\}$ and $d(x,y) = (|x-y|, \alpha|x-y|)$ be as in Examples 4.4 and 4.7. Take $\varphi \in \Phi$ defined by $\varphi(\theta) = \theta$ and $\varphi(t_1,t_2) = (t_1,t_2)$ for $t_1,t_2 > 0$; take $\psi \in \Psi$ defined by $\psi(t_1,t_2) = (3t_1,3t_2)$ for $t_1,t_2 \geq 0$ (they satisfy the conditions of Definition 2.1). Consider the mappings $f,g : X \to X$ given as $fx = (1/3)x$ and $gx = 0$. Condition

$$\psi(d(fx,gy)) \preceq \psi(u(x,y)) - \varphi(u(x,y)), \tag{5.21}$$

reduced to the first coordinates of respective vectors, has the form which was checked to be true in [16]. Hence, the existence of a common fixed point ($p = 0$) of mappings f and g follows from Theorem 5.1.

The next is a kind of Hardy-Rogers-type result with weak condition. It can be considered as a cone metric version of results from [19, 21]. For the sake of simplicity we take only one mapping $f : X \to X$ and for $x,y \in X$ denote

$$\Theta_f^5(x,y) = Ad(x,y) + Bd(x,fx) + Cd(y,fy) + Dd(x,fy) + Ed(y,fx), \tag{5.22}$$

where $A > 0$, $B,C,D,E \geq 0$, $A+B+C+D+E \leq 1$.

Theorem 5.4. *Let (X,d) be a complete cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f : X \to X$ and suppose that for all $x,y \in X$,*

$$d(fx,fy) \preceq \Theta_f^5(x,y) - \varphi\left(\Theta_f^5(x,y)\right), \tag{5.23}$$

holds. Then f has a unique fixed point.

Proof. The given condition (5.23) and properties of function φ imply that

$$d(fx,fy) \preceq \Theta_f^5(x,y) \tag{5.24}$$

for each $x, y \in X$. Starting with arbitrary $x_0 \in X$ construct the Picard sequence by $x_{n+1} = fx_n$. Condition (5.24) implies that

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(fx_n, fx_{n+1}) \\ &\preceq Ad(x_n, x_{n+1}) + Bd(x_n, x_{n+1}) + Cd(x_{n+1}, x_{n+2}) \\ &\quad + Dd(x_n, x_{n+2}) + Ed(x_{n+1}, x_{n+1}) \\ &\preceq (A + B + D)d(x_n, x_{n+1}) + (C + D)d(x_{n+1}, x_{n+2}), \end{aligned} \tag{5.25}$$

wherefrom

$$(1 - C - D)d(x_{n+1}, x_{n+2}) \preceq (A + B + D)d(x_n, x_{n+1}), \tag{5.26}$$

and, similarly,

$$(1 - B - E)d(x_{n+2}, x_{n+1}) \preceq (A + C + E)d(x_{n+1}, x_n). \tag{5.27}$$

Adding up, one obtains that

$$d(x_{n+1}, x_{n+2}) \preceq \lambda d(x_n, x_{n+1}), \tag{5.28}$$

where $\lambda = (2A+B+C+D+E)/(2-B-C-D-E) \leq 1$. It follows that $\{d(x_{n+1}, x_n)\}$ is a decreasing sequence which (by the regularity of cone P) tends to some $r \in \operatorname{int}P \cup \{\theta\}$. In order to prove that $r = \theta$, put $x = x_{n+1}$ and $y = x_n$ in (5.23) to obtain

$$d(x_{n+2}, x_{n+1}) \preceq \Theta_f^5(x_{n+1}, x_n) - \varphi\Big(\Theta_f^5(x_{n+1}, x_n)\Big), \tag{5.29}$$

where

$$\begin{aligned} \Theta_f^5(x_{n+1}, x_n) &= Ad(x_{n+1}, x_n) + Bd(x_{n+1}, x_{n+2}) + Cd(x_n, x_{n+1}) \\ &\quad + Dd(x_{n+1}, x_{n+1}) + Ed(x_n, x_{n+2}) \\ &\preceq (A + C + E)d(x_n, x_{n+1}) + (B + E)d(x_{n+1}, x_{n+2}). \end{aligned} \tag{5.30}$$

Similarly,

$$\Theta_f^5(x_n, x_{n+1}) \preceq (A + C + D)d(x_n, x_{n+1}) + (B + D)d(x_{n+1}, x_{n+2}). \tag{5.31}$$

On the other hand, (5.24) implies that

$$\Theta_f^5(x_{n+1}, x_n) \succeq d(x_{n+1}, x_{n+2}). \tag{5.32}$$

In the case when $D = E$, passing to the limit when $n \to \infty$, we obtain that $\lim_{n\to\infty}\Theta_f^5(x_{n+1}, x_n) = r$; the same conclusion is obtained if $D < E$ (or $D > E$). Hence, passing to the limit in (5.29), we get that $r \preceq r - \varphi(r)$, wherefrom $r = \theta$.

As in some previous proofs, in order to obtain that $\{x_n\}$ is a Cauchy sequence, suppose that it is not the case and using Lemma 3.2 deduce that there exist $c \gg \theta$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and the sequences

$$d(x_{2m_k}, x_{2n_k}), \quad d(x_{2m_k}, x_{2n_k+1}), \quad d(x_{2m_k-1}, x_{2n_k}), \quad d(x_{2m_k-1}, x_{2n_k+1}) \tag{5.33}$$

all tend to $\varphi(c)$. Putting $x = x_{2n_k}$ and $y = x_{2m_k-1}$ in (5.23) gives

$$\begin{aligned} d(x_{2n_k+1}, x_{2m_k}) &= d(fx_{2n_k}, fx_{2m_k-1}) \\ &\preceq \Theta_f^5(x_{2n_k}, x_{2m_k-1}) - \varphi\Big(\Theta_f^5(x_{2n_k}, x_{2m_k-1})\Big). \end{aligned} \tag{5.34}$$

Here

$$\begin{aligned} \Theta_f^5(x_{2n_k}, x_{2m_k-1}) = {}& Ad(x_{2n_k}, x_{2m_k-1}) + Bd(x_{2n_k}, x_{2n_k+1}) \\ &+ Cd(x_{2m_k-1}, x_{2n_k}) + Dd(x_{2n_k}, x_{2m_k}) + Ed(x_{2m_k-1}, x_{2n_k+1}) \\ &\longrightarrow A\varphi(c) + B\cdot\theta + C\cdot\theta + D\varphi(c) + E\varphi(c) = (A + D + E)\varphi(c), \end{aligned} \tag{5.35}$$

when $k \to \infty$. Since also $d(x_{2n_k+1}, x_{2m_k}) \to \varphi(c)$ when $k \to \infty$, we obtain that

$$\varphi(c) \preceq (A + D + E)\varphi(c) - \varphi\big((A + D + E)\varphi(c)\big) \preceq \varphi(c) - \varphi\big((A + D + E)\varphi(c)\big), \tag{5.36}$$

implying that $\varphi(c) = \theta$ (because $A > 0$).

Thus, the sequence $\{x_n\}$ converges to some z in the complete metric space X. In order to prove that $fz = z$, suppose the contrary and put $x = x_n$ and $y = z$ in (5.24). It follows that

$$d(fx_n, fz) \preceq Ad(x_n, z) + Bd(x_n, x_{n+1}) + Cd(z, fz) + Dd(x_n, fz) + Ed(z, x_{n+1}). \tag{5.37}$$

Passing to the limit when $n \to \infty$ gives that

$$d(z, fz) \preceq (C + D)d(z, fz) \prec (A + B + C + D + E)d(z, fz) \preceq d(z, fz), \tag{5.38}$$

a contradiction, since $A > 0$.

The proof that the fixed point of f is unique is standard. □

In a similar way one can obtain a version of the previous theorem containing two selfmaps f and g (see [21, Theorem 5.2]).

At the end, we again state a cone metric version of a result from [18, Theorems 3.2 and 4.2] (see also [21, Theorem 3.7]).

Theorem 5.5. *Let (X, d) be a cone metric space over a regular cone P such that (d_4) holds and suppose that there exists a continuous function $\varphi \in \Phi$. Let $f, g : X \to X$ be two selfmaps such that for all $x, y \in X$ there exist*

$$u(x,y) \in N_f^4(x,y), \qquad v(x,y) \in \{d(x,y), d(x,fx), d(y,gy)\}, \tag{5.39}$$

such that $d(fx, gy) \preceq u(x,y) - \varphi(v(x,y))$. Then f and g have a common fixed point.

Here also common fixed point of f and g need not be unique.

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