Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 784307, 15 pages
doi:10.1155/2012/784307

Research Article

Some Formulae for the Product of Two Bernoulli and Euler Polynomials

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Received 6 March 2012; Accepted 23 April 2012

Academic Editor: Sung Guen Kim

We investigate some formulae for the product of two Bernoulli and Euler polynomials arising from the Euler and Bernoulli basis polynomials.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function as follows:

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.1}$$

(see [1–21]), with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x = 0$, $B_n(0) = B_n$ are called the nth Bernoulli numbers. The Euler polynomials are also defined by the generating function as follows:

$$\frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \tag{1.2}$$

(see [6–11]), with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the nth Euler numbers. From (1.1) and (1.2), we can derive the following recurrence relations for the Bernoulli and Euler numbers:

$$B_0 = 1, \qquad B_n(1) - B_n = \delta_{1,n}, \qquad E_0 = 1, \qquad E_n(1) + E_n = 2\delta_{0,n}, \tag{1.3}$$

where $\delta_{k,n}$ is the Kronecker symbol. By (1.1) and (1.2), we get

$$B_n(x) = \sum_{\ell=0}^{n} x^{n-\ell}\binom{n}{\ell}B_\ell, \qquad E_n(x) = \sum_{\ell=0}^{n} x^{n-\ell}\binom{n}{\ell}E_\ell. \tag{1.4}$$

From (1.4), we can derive

$$\frac{d}{dx}B_n(x) = nB_{n-1}(x), \qquad \frac{d}{dx}E_n(x) = nE_{n-1}(x). \tag{1.5}$$

By (1.4) and (1.5), we get

$$\int_0^1 B_n(x)dx = \frac{\delta_{0,n}}{n+1} = \delta_{0,n}, \qquad \int_0^1 E_n(x)dx = -2\frac{E_{n+1}}{n+1}. \tag{1.6}$$

It is easy to show that

$$e^{tx} = \frac{1}{t}\left(\frac{te^{(x+1)t}}{e^t - 1} - \frac{te^{xt}}{e^t - 1}\right) = \frac{1}{t}\sum_{n=0}^{\infty}(B_n(x+1) - B_n(x))\frac{t^n}{n!}. \tag{1.7}$$

Thus, we have

$$x^n = \frac{1}{n+1}\sum_{\ell=0}^{n}\binom{n+1}{\ell}B_\ell(x) \tag{1.8}$$

(see [11–18]). By the definition of the Euler polynomials, we get

$$e^{tx} = \frac{1}{2}\left(\frac{2e^{(x+1)t}}{e^t + 1} + \frac{2e^{xt}}{e^t + 1}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(E_n(x+1) + E_n(x))\frac{t^n}{n!}. \tag{1.9}$$

From (1.9), we have

$$x^n = E_n(x) + \frac{1}{2}\sum_{\ell=0}^{n-1}\binom{n}{\ell}E_\ell(x) \tag{1.10}$$

(see [1–18]). By (1.8) and (1.10), we see that the set $\{E_0(x), \dots, E_n(x)\}$ and $\{B_0(x), \dots, B_n(x)\}$ are the basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$ (see [1–21]).

From $m, n \in \mathbb{Z}_+ = \mathbb{N} \bigcup \{0\}$, let $I_{m,n} = \int_0^1 B_m(x) x^n dx$. Then, we note that

$$I_{0,n} = \int_0^1 x^n \, dx = \frac{1}{n+1}, \qquad I_{m,0} = \int_0^1 B_m(x) dx = \delta_{0,m}. \tag{1.11}$$

Let us assume that $m, n \geq 1$. Then, we have

$$\begin{aligned} I_{m,n} &= \frac{B_{m+1}}{m+1} - \frac{n}{m+1} I_{m+1,n-1} = \frac{B_{m+1}}{m+1} + (-1) \frac{n}{(m+1)(m+2)} B_{m+2} \\ &\quad + (-1)^2 \frac{n(n-1)}{(m+1)(m+2)} I_{m+2,n-2}. \end{aligned} \tag{1.12}$$

Continuing this process, we get

$$\begin{aligned} I_{m,n} &= \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{n-j+1} \left(\prod_{\ell=1}^{j} \frac{n-\ell+1}{m+\ell} \right) B_{m+j} \\ &\quad + (-1)^{n-1} \frac{n(n-1)\cdots 2}{(m+1)\cdots(m+n-1)} \int_0^1 B_{m+n-1}(x) x \, dx \\ &= \sum_{j=1}^{n} \frac{(-1)^{j-1} \binom{n+1}{j}}{(n+1)\binom{m+j}{m}} B_{m+j}. \end{aligned} \tag{1.13}$$

Let $J_{m,n} = \int_0^1 E_m(x) x^n dx$ for $m, n \in \mathbb{Z}_+$. Then, we have

$$J_{0,n} = \int_0^1 x^n \, dx = \frac{1}{n+1}, \qquad J_{m,0} = \int_0^1 E_m(x) dx = -\frac{2}{m+1} E_{m+1}. \tag{1.14}$$

Assume that $m, n \geq 1$. Then, we get

$$\begin{aligned} J_{m,n} &= \int_0^1 E_m(x) x^n \, dx = -\frac{E_{m+1}}{m+1} - \frac{n}{m+1} \int_0^1 E_{m+1}(x) x^{n-1} \, dx \\ &= -\frac{1}{m+1} E_{m+1} - \frac{n}{m+1} J_{m+1,n-1} \\ &= -\frac{E_{m+1}}{m+1} + (-1)^2 \frac{n}{(m+1)(m+2)} E_{m+2} + (-1)^2 \frac{n(n-1)}{(m+1)(m+2)} J_{m+2,n-2}. \end{aligned} \tag{1.15}$$

Continuing this process, we get

$$J_{m,n} = \sum_{j=1}^{n-1}(-1)^j \frac{1}{n-j+1}\left(\prod_{\ell=1}^{j}\frac{n-\ell+1}{m+\ell}\right)E_{m+j} + (-1)^{n-1}\frac{n(n-1)\cdots 2}{(m+1)\cdots(m+n-1)}J_{m+n-1,1}, \tag{1.16}$$

$$J_{m+n-1,1} = \int_0^1 E_{m+n-1}(x)x\,dx = -\frac{E_{m+n}}{m+n} + (-1)^2\frac{2E_{m+n+1}}{(m+n)(m+n+1)}.$$

By (1.16), we get

$$J_{m,n} = \frac{1}{n+1}\sum_{j=1}^{n}(-1)^j\frac{\binom{n+1}{j}}{\binom{m+j}{m}}E_{m+j} + 2\frac{(-1)^{n+1}E_{n+m+1}}{(n+m+1)\binom{n+m}{m}}. \tag{1.17}$$

From the properties of the Bernoulli and Euler basis for the space of the polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, we derive some identities for the product of two Bernoulli and Euler polynomials.

2. Some Identities for the Bernoulli and Euler Numbers

Let us consider the polynomial $p(x) = \sum_{k=0}^{n} B_k(x)x^{n-k}$, with $n \in \mathbb{N}$. Then, we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!}\sum_{\ell=k}^{n}B_{\ell-k}(x)x^{n-\ell} \quad (k = 0,1,2,\ldots,n). \tag{2.1}$$

From the properties of the Bernoulli basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, $p(x)$ is given by

$$p(x) = \sum_{k=0}^{n}a_k B_k(x). \tag{2.2}$$

Thus, by (2.2), we get

$$a_0 = \int_0^1 p(t)dt = \sum_{k=0}^{n}\int_0^1 B_k(t)x^{n-k}\,dt = \sum_{k=0}^{n}I_{k,n-k} = I_{0,n} + \sum_{k=1}^{n-1}I_{k,n-k} + I_{n,0} = \frac{1}{n+1} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k}\frac{(-1)^{j-1}\binom{n-k+1}{j}}{(n-k+1)\binom{k+j}{k}}B_{k+j} + \delta_{0,n}. \tag{2.3}$$

By (2.1) and (2.2), we get

$$a_k = \left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) = \frac{(n+1)!}{k!(n-k+2)!}\sum_{\ell=k-1}^{n}\left\{B_{\ell-k+1}(1) - B_{\ell-k+1}0^{n-\ell}\right\}$$
$$= \frac{\binom{n+2}{k}}{n+2}\left(\sum_{\ell=k-1}^{n-1}B_{\ell-k+1}+1\right). \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$, one has*

$$\sum_{k=0}^{n}B_k(x)x^{n-k} = \frac{1}{n+1} + \sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\frac{(-1)^{j-1}\binom{n-\ell+1}{j}}{(n-\ell+1)\binom{\ell+j}{\ell}}B_{\ell+j}$$
$$+ \frac{1}{n+2}\sum_{k=1}^{n}\binom{n+2}{k}\left(\sum_{\ell=k-1}^{n-1}B_{\ell-k+1}+1\right)B_k(x). \tag{2.5}$$

From the properties of the Euler basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, $p(x)$ is given by

$$p(x) = \sum_{k=0}^{n}b_kE_k(x). \tag{2.6}$$

By (2.1) and (2.6), we get

$$b_k = \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right)$$
$$= \frac{1}{2k!}\frac{(n+1)!}{(n-k+1)!}\sum_{\ell=k}^{n}\left\{B_{\ell-k}(1) + B_{\ell-k}0^{n-\ell}\right\}$$
$$= \frac{\binom{n+1}{k}}{2}\left(\sum_{\ell=k}^{n}B_{\ell-k} + 1 - \delta_{k,n} + B_{n-k}\right) \tag{2.7}$$
$$= \begin{cases}\dfrac{\binom{n+1}{k}}{2}\left(\displaystyle\sum_{\ell=k}^{n}B_{\ell-k} + 1 + B_{n-k}\right) & \text{if } k \neq n,\\ n+1 & \text{if } k = n.\end{cases}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{k=0}^{n}B_k(x)x^{n-k} = \frac{1}{2}\sum_{k=0}^{n-1}\binom{n+1}{k}\left\{\sum_{\ell=k}^{n}B_{\ell-k} + 1 + B_{n-k}\right\}E_k(x) + (n+1)E_n(x). \tag{2.8}$$

Let us take polynomial $p(x)$ with $p(x) = \sum_{k=0}^{n} E_k(x)x^{n-k}$. Then, we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!}\sum_{\ell=k}^{n} E_{\ell-k}(x)x^{n-\ell} \quad (k = 0,1,2,\dots,n). \tag{2.9}$$

By the basis set $\{B_0(x),\dots,B_n(x)\}$ for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, we see that $p(x)$ is given by

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{2.10}$$

From (2.10), we note that

$$\begin{aligned}
a_0 &= \int_0^1 p(t)dt = \sum_{\ell=0}^{n}\int_0^1 E_\ell(t)x^{n-\ell}dt = \sum_{\ell=0}^{n} J_{\ell,n-\ell} \\
&= J_{0,n} + \sum_{\ell=1}^{n-1} J_{\ell,n-\ell} + J_{n,0} = \frac{1}{n+1} + \sum_{\ell=1}^{n-1} J_{\ell,n-\ell} - \frac{2}{n+1}E_{n+1} \\
&= \left(\sum_{j=1}^{n}(-1)^j \frac{\binom{n+1}{j}}{n+1}E_j + \frac{2(-1)^{n+1}}{n+1}E_{n+1}\right) \\
&\quad + \sum_{\ell=1}^{n-1}\left\{\sum_{j=1}^{n-\ell}(-1)^j \frac{\binom{n-\ell+1}{j}}{(n-\ell+1)\binom{\ell+j}{\ell}}E_{\ell+j} + \frac{2(-1)^{n-\ell+1}}{(n+1)\binom{n}{\ell}}E_{n+1}\right\} - \frac{2}{n+1}E_{n+1} \\
&= \sum_{\ell=0}^{n}\left\{\sum_{j=1}^{n-\ell}(-1)^j \frac{\binom{n-\ell+1}{j}}{(n-\ell+1)\binom{\ell+j}{\ell}}E_{\ell+j} + \frac{2(-1)^{n-\ell+1}}{(n+1)\binom{n}{\ell}}E_{n+1}\right\}.
\end{aligned} \tag{2.11}$$

Note that

$$\begin{aligned}
\sum_{\ell=0}^{n}\frac{(-1)^{\ell-1}}{(n+1)\binom{n}{\ell}} &= -\sum_{\ell=0}^{n}\frac{(n-\ell)!\ell!}{(n+1)!}(-1)^\ell = -\sum_{\ell=0}^{n} B(n-\ell+1,\ell+1)(-1)^\ell \\
&= -\frac{1+(-1)^n}{n+2},
\end{aligned} \tag{2.12}$$

where $B(\alpha,\beta)$ is the beta function.

From (2.11) and (2.12), we have

$$a_0 = \sum_{\ell=0}^{n}\sum_{j=1}^{n-\ell}\frac{(-1)^j\binom{n-\ell+1}{j}}{(n-\ell+1)\binom{\ell+j}{\ell}}E_{\ell+j} + \frac{4(-1)^{n+1}}{n+2}E_{n+1}. \tag{2.13}$$

For $k = 1, 2, \ldots, n$, by (2.9) and (2.10), we get

$$a_k = \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) = \frac{(n+1)!}{k!(n-k+2)!}\sum_{\ell=k-1}^{n}\left\{E_{\ell-k+1}(1) - E_{\ell-k+1}0^{n-\ell}\right\}$$
$$= \frac{1}{n+2}\binom{n+2}{k}\left\{-\sum_{\ell=k-1}^{n}E_{\ell-k+1} + 2 - E_{n-k+1}\right\}. \tag{2.14}$$

Therefore, by (2.13) and (2.14), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{k=0}^{n}E_k(x)x^{n-k} = \sum_{k=0}^{n-1}\sum_{j=1}^{n-\ell}\frac{(-1)^j\binom{n-\ell+1}{k}}{(n-\ell+1)\binom{\ell+j}{\ell}}E_{\ell+j} + \frac{4(-1)^{n+1}}{n+2}E_{n+1}$$
$$+ \frac{1}{n+2}\sum_{k=1}^{n}\binom{n+2}{k}\left\{-\sum_{\ell=k-1}^{n}E_{\ell-k+1} + 2 - E_{n-k+1}\right\}B_k(x). \tag{2.15}$$

From the Euler basis $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, we note that $p(x)$ can be written as follows:

$$p(x) = \sum_{k=0}^{n}b_kE_k(x). \tag{2.16}$$

Thus, we have

$$b_k = \frac{1}{2k!}\left(p^{(k)}(1) + p^{(k)}(0)\right) = \frac{1}{2k!}\frac{(n+1)!}{(n-k+1)!}\sum_{\ell=k}^{n}\left(E_{\ell-k}(1) + E_{\ell-k}0^{n-\ell}\right)$$
$$= \frac{1}{2}\binom{n+1}{k}\left(-\sum_{\ell=k}^{n-1}E_{\ell-k} + 2\right). \tag{2.17}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{k=0}^{n}E_k(x)x^{n-k} = \frac{1}{2}\sum_{k=0}^{n-1}\binom{n+1}{k}\left\{-\sum_{\ell=k}^{n-1}E_{\ell-k} + 2\right\}E_k(x) + (n+1)E_n(x). \tag{2.18}$$

Let us consider the polynomial $p(x) = \sum_{k=0}^{n}(B_k(x)x^{n-k})/(k!(n-k)!) = \sum_{k=0}^{n} a_k B_k(x)$. Then, we have

$$\begin{aligned} a_0 = \int_0^1 p(t)dt &= \sum_{\ell=0}^{n} \frac{1}{\ell!(n-\ell)!} I_{\ell,n-\ell} = \frac{1}{n!} I_{0,n} + \sum_{\ell=1}^{n-1} \frac{I_{\ell,n-\ell}}{l!(n-\ell)!} + \frac{I_{n,0}}{n!} \\ &= \frac{1}{(n+1)!} + \frac{1}{(n+1)!} \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-\ell} (-1)^{j-1} \binom{n+1}{\ell+j} B_{\ell+j}. \end{aligned} \tag{2.19}$$

It is easy to show that

$$p^{(k)}(x) = 2^k \sum_{\ell=k}^{n} \frac{1}{(\ell-k)!(n-\ell)!} B_{\ell-k}(x) x^{n-\ell}. \tag{2.20}$$

For $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} a_k &= \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) \\ &= \frac{2^{k-1}}{k!} \sum_{\ell=k-1}^{n} \frac{1}{(\ell-k+1)!(n-\ell)!}\left(B_{\ell-k+1}(1) - B_{\ell-k+1}0^{n-\ell}\right) \\ &= \frac{2^{k-1}}{k!}\left\{\sum_{\ell=k-1}^{n} \frac{B_{\ell-k+1} + \delta_{1,\ell-k+1}}{(\ell-k+1)!(n-\ell)!} - \frac{B_{n-k+1}}{(n-k+1)!}\right\} \\ &= \frac{2^{k-1}}{k!}\left(\sum_{\ell=k-1}^{n-1} \frac{B_{\ell-k+1}}{(\ell-k+1)!(n-\ell)!} + \frac{1}{(n-k)!}\right). \end{aligned} \tag{2.21}$$

Therefore, by (2.19) and (2.21), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} \sum_{k=0}^{n} \frac{B_k(x)}{k!(n-k)!} x^{n-k} &= \frac{1}{(n+1)!} + \frac{1}{(n+1)!} \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-\ell} (-1)^{j-1} \binom{n+1}{\ell+j} B_{\ell+j} \\ &\quad + \sum_{k=1}^{n} \frac{2^{k-1}}{k!}\left(\sum_{\ell=k-1}^{n-1} \frac{B_{\ell-k+1}}{(\ell-k+1)!(n-\ell)!} + \frac{1}{(n-k)!}\right) B_k(x). \end{aligned} \tag{2.22}$$

Remark 2.6. If $p(x) = \sum_{k=0}^{n} b_k E_k(x)$, by the same method, we get

$$\begin{aligned} \sum_{k=0}^{n} \frac{B_k(x)x^{n-k}}{k!(n-k)!} &= \sum_{k=0}^{n-1} \frac{2^{k-1}}{k!}\left\{\sum_{\ell=k}^{n} \frac{B_{\ell-k}}{(\ell-k)!(n-\ell)!} + \frac{1}{(n-k-1)!} + \frac{B_{n-k}}{(n-k)!}\right\} E_k(x) \\ &\quad + \frac{2^n}{n!} E_n(x). \end{aligned} \tag{2.23}$$

Let us consider the polynomial $p(x) = \sum_{k=0}^{n}(E_k(x)x^{n-k})/(k!(n-k)!) = \sum_{k=0}^{n} a_k B_k(x)$. Then, we have

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{\ell=0}^{n} \frac{1}{\ell!(n-\ell)!} J_{\ell,n-\ell} \\ &= \sum_{\ell=0}^{n} \frac{1}{\ell!(n-\ell)!} \left\{ \sum_{j=1}^{n-\ell} (-1)^j \frac{\binom{n-\ell+1}{j}}{(n-\ell+1)\binom{\ell+j}{\ell}} E_{\ell+j} + \frac{2(-1)^{n-\ell+1}}{(n+1)\binom{n}{\ell}} E_{n+1} \right\} \\ &= \sum_{\ell=0}^{n} \sum_{j=1}^{n-\ell} \frac{(-1)^j E_{\ell+j}}{(\ell+j)!(n+1-\ell-j)!} + \frac{2E_{n+1}(-1)^{n+1}}{(n+1)!} \sum_{\ell=0}^{n} (-1)^{\ell} \\ &= \sum_{\ell=0}^{n} \sum_{j=1}^{n-\ell} \frac{(-1)^j E_{\ell+j}}{(\ell+j)!(n+1-\ell-j)!} + \frac{2E_{n+1}(-1)^{n+1}}{(n+1)!}. \end{aligned} \tag{2.24}$$

It is easy to show that

$$p^{(k)}(x) = 2^k \sum_{\ell=k}^{n} \frac{1}{(\ell-k)!(n-\ell)!} E_{\ell-k}(x) x^{n-\ell}. \tag{2.25}$$

For $k = 1, 2, \ldots, n$, we have

$$\begin{aligned} a_k &= \frac{1}{k!}\left(p^{(k-1)}(1) - p^{(k-1)}(0)\right) \\ &= \frac{2^{k-1}}{k!} \sum_{\ell=k-1}^{n} \frac{1}{(\ell-k+1)!(n-\ell)!} \left(E_{\ell-k+1}(1) - E_{\ell-k+1} 0^{n-\ell}\right) \\ &= \frac{2^{k-1}}{k!} \left\{ -\sum_{\ell=k-1}^{n} \frac{E_{\ell-k+1}}{(\ell-k+1)!(n-\ell)!} + \frac{2}{(n-k+1)!} - \frac{E_{n-k+1}}{(n-k+1)!} \right\}. \end{aligned} \tag{2.26}$$

Therefore, by (2.24) and (2.26), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{Z}_+$, one has*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} E_k(x) x^{n-k} &= \sum_{\ell=0}^{n} \sum_{j=1}^{n-\ell} \frac{(-1)^j \binom{n+1}{\ell+j}}{n+1} E_{\ell+j} + \frac{2(-1)^{n+1}}{(n+1)} E_{n+1} \\ &\quad + \sum_{k=1}^{n} \left\{ -\frac{1}{n+1} \sum_{\ell=k-1}^{n} 2^{k-1} \binom{n+1}{k} \binom{n-k+1}{\ell-k+1} E_{\ell-k+1} \right. \\ &\quad \left. + \frac{2}{(n+1)} \binom{n+1}{k} - \frac{2^{k-1}}{n+1} \binom{n+1}{k} E_{n-k+1} \right\} B_k(x). \end{aligned} \tag{2.27}$$

Let us consider the polynomial $p(x) = \sum_{k=0}^{n}(E_k(x)x^{n-k})/(k!(n-k)!) = \sum_{k=0}^{n} b_k E_k(x)$. By the same method, we obtain the following identity:

$$\sum_{k=0}^{n}\frac{E_k(x)x^{n-k}}{k!(n-k)!} = \sum_{k=0}^{n}\frac{2^{k-1}}{k!}\left\{-\sum_{\ell=k}^{n}\frac{E_{\ell-k}}{(\ell-k)!(n-\ell)!} + \frac{2}{(n-k)!} + \frac{E_{n-k}}{(n-k)!}\right\}E_k(x). \tag{2.28}$$

Let us take $p(x) = \sum_{k=1}^{n-1}(1/(k(n-k)))B_k(x)x^{n-k}$. Then, the kth derivative of $p(x)$ is given by

$$p^{(k)}(x) = C_k\left(x^{n-k} + B_{n-k}(x)\right) + (n-1)\cdots(n-k)\sum_{\ell=k+1}^{n-1}\frac{B_{\ell-k}(x)x^{n-\ell}}{(n-\ell)(\ell-k)}, \tag{2.29}$$

where

$$C_k = \frac{\sum_{j=1}^{k}(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k)}{n-k} \quad (k = 1, 2, \ldots, n-1),\ C_0 = 0. \tag{2.30}$$

Note that

$$p^{(n)}(x) = \left(p^{(n-1)}(x)\right)' = C_{n-1}(x + B_1(x))' = 2C_{n-1} = 2(n-1)!H_{n-1}, \tag{2.31}$$

where $H_{n-1} = \sum_{j=1}^{n-1}(1/j)$.

By the properties of the Bernoulli basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, $p(x)$ is given by

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{2.32}$$

Thus, by (2.32), we get

$$\begin{aligned} a_0 &= \sum_{\ell=1}^{n-1}\frac{1}{\ell(n-\ell)}\int_0^1 B_\ell(t)x^{n-\ell}dt = \sum_{\ell=0}^{n-1}\frac{1}{\ell(n-\ell)}I_{\ell,n-\ell} \\ &= \sum_{\ell=0}^{n-1}\frac{1}{\ell(n-\ell)}\left\{\sum_{j=1}^{n-\ell}\frac{(-1)^{j-1}\binom{n-\ell+1}{j}B_{\ell+j}}{(n-\ell+1)\binom{\ell+j}{\ell}}\right\} \\ &= \frac{1}{n(n^2-1)}\sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\frac{(-1)^{j-1}\binom{n+1}{j+\ell}}{\binom{n-2}{\ell-1}}B_{\ell+j}. \end{aligned} \tag{2.33}$$

From (2.29), (2.31), and (2.32), we note that

$$\begin{aligned} a_k &= \frac{1}{k!}\left(p^{(k-1)}(1)-p^{(k-1)}(0)\right) \\ &= \frac{C_{k-1}}{k!}\{B_{n-k+1}(1)-B_{n-k+1}\}+\left\{1-0^{n-k+1}\right\} \\ &\quad+\frac{\binom{n}{k}}{n}\sum_{\ell=k}^{n-1}\frac{1}{(n-\ell)(\ell-k+1)}\left\{B_{\ell-k+1}(1)-B_{\ell-k+1}0^{n-\ell}\right\} \\ &= \frac{C_{k-1}}{k!}(\delta_{1,n-k+1}+1)+\frac{\binom{n}{k}}{n}\sum_{\ell=k}^{n-1}\frac{B_{\ell-k+1}+\delta_{1,\ell-k+1}}{(\ell-k+1)(n-\ell)} \\ &= \frac{C_{k-1}}{k!}(\delta_{1,n-k+1}+1)+\frac{\binom{n}{k}}{n}\left\{\sum_{\ell=k}^{n-1}\frac{B_{\ell-k+1}}{(\ell-k+1)(n-\ell)}+\frac{1}{n-k}\right\} \\ &= \begin{cases} \dfrac{C_{k-1}}{k!}+\dfrac{\binom{n}{k}}{n}\left\{\displaystyle\sum_{\ell=k}^{n-1}\frac{B_{\ell-k+1}}{(\ell-k+1)(n-\ell)}+\frac{1}{n-k}\right\} & \text{if } 1\le k\le n-1, \\ \dfrac{2H_{n-1}}{n} & \text{if } k=n. \end{cases} \end{aligned} \tag{2.34}$$

From (2.30), we have

$$\begin{aligned} \frac{C_{k-1}}{k!} &= \frac{1}{k!}\sum_{j=1}^{k-1}\frac{(n-1)!}{(n-k+1)(n-k)!(n-j)} \\ &= \left(\frac{n!}{k!(n-k)!}\right)\left(\frac{1}{n(n-k+1)}\right)\sum_{j=1}^{k-1}\frac{1}{n-j} \\ &= \binom{n}{k}\frac{1}{n(n-k+1)}\left(\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{j=1}^{n-k}\frac{1}{j}\right) \\ &= \frac{\binom{n}{k}}{n(n-k+1)}(H_{n-1}-H_{n-k}). \end{aligned} \tag{2.35}$$

Therefore, by (2.32), (2.34), and (2.35), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$, one has*

$$\begin{aligned} \sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)x^{n-k} &= \frac{1}{n(n^2-1)}\sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\frac{(-1)^{j-1}\binom{n+1}{\ell+j}B_{\ell+j}}{\binom{n-2}{\ell-1}}+\frac{1}{n}\sum_{k=1}^{n-1}\binom{n}{k} \\ &\quad\times\left(\frac{H_{n-1}-H_{n-k}}{n-k+1}+\sum_{\ell=k}^{n-1}\frac{B_{\ell-k+1}}{(\ell-k+1)(n-\ell)}+\frac{1}{n-k}\right) \\ &\quad\times B_k(x)+\frac{2H_{n-1}B_n(x)}{n}. \end{aligned} \tag{2.36}$$

We assume that $p(x) = \sum_{k=1}^{n-1}(1/k(n-k))B_k(x)x^{n-k} = \sum_{k=0}^{n} b_k E_k(x)$. For $k = 0,1,2,\ldots,n-1$, we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right) \\ &= \frac{C_k}{2k!}\left\{B_{n-k}(1)+1+B_{n-k}0^{n-k}\right\} \\ &\quad + \frac{(n-1)\cdots(n-k)}{2k!}\sum_{\ell=k+1}^{n-1}\frac{1}{(\ell-k)(n-\ell)}\left\{B_{\ell-k}(1)+B_{\ell-k}0^{n-\ell}\right\} \\ &= \begin{cases} \dfrac{C_k}{2k!}(2B_{n-k}+1)+\dfrac{\binom{n-1}{k}}{2}\left(\displaystyle\sum_{\ell=k+1}^{n-1}\frac{B_{\ell-k}}{(\ell-k)(n-\ell)}+\frac{1}{n-k-1}\right) & \text{if } 0\le k\le n-2, \\ \dfrac{C_{n-1}}{2(n-1)!}=\dfrac{1}{2}H_{n-1} & \text{if } k=n-1. \end{cases} \end{aligned} \tag{2.37}$$

Finally,

$$b_n = \frac{1}{2n!}\left(p^{(n)}(1)+p^{(n)}(0)\right) = \frac{4C_{n-1}}{2n!} = \frac{2C_{n-1}}{n!} = \frac{2H_{n-1}}{n}. \tag{2.38}$$

By the same method, we obtain the following identity:

$$\begin{aligned} &\sum_{k=1}^{n-1}\frac{1}{k(n-k)}B_k(x)x^{n-k} \\ &\quad = \frac{1}{2}\sum_{k=0}^{n-2}\left\{\frac{\binom{n}{k}}{n}(H_{n-1}-H_{n-k-1})(2B_{n-k}+1)+\binom{n-1}{k}\left(\sum_{\ell=k+1}^{n-1}\frac{B_{\ell-k}}{(\ell-k)(n-\ell)}+\frac{1}{n-k-1}\right)\right\} \\ &\quad\quad \times E_k(x)+\frac{1}{2}H_{n-1}E_{n-1}(x)+\frac{2H_{n-1}}{n}E_n(x). \end{aligned} \tag{2.39}$$

Let us take $p(x) = \sum_{k=1}^{n-1}(1/k(n-k))E_k(x)x^{n-k}$. Then, for $k = 0,1,2,\ldots,n-1$, we have

$$p^{(k)}(x) = C_k\left(E_{n-k}(x)+x^{n-k}\right)+(n-1)\cdots(n-k)\sum_{\ell=k+1}^{n-1}\frac{E_{\ell-k}(x)x^{n-\ell}}{(\ell-k)(n-\ell)}, \tag{2.40}$$

where $C_k = (\sum_{j=1}^{k}(n-1)\cdots(n-j+1)(n-j-1)\cdots(n-k))/(n-k)$.

Note that

$$p^{(n)}(x) = \left(p^{(n-1)}(x)\right)' = (C_{n-1}(E_1(x)+x))' = 2C_{n-1} = 2(n-1)!H_{n-1}. \tag{2.41}$$

By the properties of the Bernoulli basis for the space of polynomials of degree less than or equal to n with coefficients in $\mathbb{Q}$, $p(x)$ can be written as

$$p(x) = \sum_{k=0}^{n} a_k B_k(x). \tag{2.42}$$

Thus, by (2.42), we get

$$\begin{aligned} a_0 &= \int_0^1 p(t)dt = \sum_{\ell=1}^{n-1} \frac{1}{\ell(n-\ell)} \int_0^1 E_\ell(t) t^{n-\ell} dt = \sum_{\ell=1}^{n-1} \frac{1}{\ell(n-\ell)} J_{\ell,n-\ell} \\ &= \frac{1}{n(n^2-1)} \sum_{\ell=1}^{n-1} \sum_{j=1}^{n-\ell} \frac{(-1)^j \binom{n+1}{j+\ell}}{\binom{n-2}{\ell-1}} E_{\ell+j} + \frac{2(-1)^{n+1}}{n(n^2-1)} E_{n+1} \sum_{\ell=1}^{n-1} \frac{(-1)^\ell}{\binom{n-2}{\ell-1}}. \end{aligned} \tag{2.43}$$

It is easy to show that

$$\begin{aligned} \sum_{\ell=1}^{n-1} \frac{(-1)^\ell}{(n-1)\binom{n-2}{\ell-1}} &= \sum_{\ell=1}^{n-1} (-1)^\ell B(\ell, n-\ell) \\ &= \left(\frac{4}{n(n+1)(n+2)} - \frac{2(n-1)}{n^2(n+1)^2} \right) E_{n+1}. \end{aligned} \tag{2.44}$$

For $k = 1, 2, \dots, n$, one has

$$\begin{aligned} a_k &= \frac{1}{k!} \left(p^{(k-1)}(1) - p^{(k-1)}(0) \right) \\ &= \frac{C_{k-1}}{k!} \left(E_{n-k+1}(1) + 1 - E_{n-k+1} - 0^{n-k+1} \right) \\ &\quad + \frac{(n-1)\cdots(n-k+1)}{k!} \sum_{\ell=k}^{n-1} \frac{1}{(\ell-k+1)(n-\ell)} \left(E_{\ell-k+1}(1) - E_{\ell-k+1} 0^{n-\ell} \right) \\ &= \frac{1}{k!} C_{k-1} (-2E_{n-k+1} + 1) - \frac{(n-1)\cdots(n-k+1)}{k!} \sum_{\ell=k}^{n-1} \frac{E_{\ell-k+1}}{(\ell-k+1)(n-\ell)} \\ &= \frac{\binom{n}{k}(H_{n-1} - H_{n-k})}{n(n-k+1)} (-2E_{n-k+1} + 1) - \frac{\binom{n}{k}}{n} \sum_{\ell=k}^{n-1} \frac{E_{\ell-k+1}}{(\ell-k+1)(n-\ell)}. \end{aligned} \tag{2.45}$$

Therefore, by (2.42), (2.44), and (2.45), we obtain the following theorem.

Theorem 2.9. *For $n \in \mathbb{N}$, one has*

$$\sum_{k=1}^{n}\frac{E_k(x)x^{n-k}}{k(n-k)}$$
$$=\frac{1}{n(n^2-1)}\sum_{\ell=1}^{n-1}\sum_{j=1}^{n-\ell}\frac{(-1)^j\binom{n+1}{\ell+j}E_{\ell+j}}{\binom{n-2}{\ell-1}}+\frac{2(-1)^{n+1}}{n(n+1)}\left(\frac{4}{n(n+1)(n+2)}-\frac{2(n-1)}{n^2(n+1)^2}\right)E_{n+1}$$
$$+\frac{1}{n}\left\{\sum_{k=1}^{n}\binom{n}{k}\left(\frac{H_{n-1}-H_{n-k}}{n-k+1}\right)(1-2E_{n-k+1})-\binom{n}{k}\sum_{\ell=k}^{n-1}\frac{E_{\ell-k+1}}{(\ell-k+1)(n-\ell)}\right\}B_k(x). \tag{2.46}$$

We may assume that $p(x) = \sum_{k=0}^{n}(1/(k(n-k)))E_k(x)x^{n-k} = \sum_{k=0}^{n} b_kE_k(x)$. Then, we note that

$$b_k = \frac{1}{2k!}\left(p^{(k)}(1)+p^{(k)}(0)\right)\quad (k=0,1,2,\dots,n-1). \tag{2.47}$$

Thus, we have

$$\begin{aligned}
b_k &= \frac{C_k}{2k!}\left\{E_{n-k}(1)+1+E_{n-k}+0^{n-k}\right\}\\
&\quad+\frac{\binom{n-1}{k}}{2}\sum_{\ell=k+1}^{n-1}\frac{1}{(\ell-k)(n-\ell)}\left\{E_{\ell-k}(1)+E_{\ell-k}0^{n-\ell}\right\}\\
&=\frac{C_k}{2k!}+\frac{\binom{n-1}{k}}{2}\sum_{\ell=k+1}^{n-1}\frac{-E_{\ell-k}}{(\ell-k)(n-\ell)}\\
&=\frac{C_k}{2k!}-\frac{\binom{n-1}{k}}{2}\sum_{\ell=k+1}^{n-1}\frac{E_{\ell-k}}{(\ell-k)(n-\ell)},\\
b_n &= \frac{1}{2n!}\left(p^{(n)}(1)+p^{(n)}(0)\right)=\frac{1}{2n!}4C_{n-1}=\frac{2C_{n-1}}{n!}=\frac{2H_{n-1}}{n}.
\end{aligned} \tag{2.48}$$

By the same method, we obtain the following identity:

$$\begin{aligned}
\sum_{k=1}^{n-1}\frac{1}{k(n-k)}E_k(x)x^{n-k} &= \sum_{k=0}^{n-1}\left\{\frac{1}{2n}\binom{n}{k}(H_{n-1}-H_{n-k-1})-\frac{\binom{n-1}{k}}{2}\sum_{\ell=k+1}^{n-1}\frac{E_{\ell-k}}{(\ell-k)(n-\ell)}\right\}\\
&\quad\times E_k(x)+\frac{2}{n}H_{n-1}E_n(x).
\end{aligned} \tag{2.49}$$

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