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*Research Article*

# Integrability and Pseudo-Linearizable Conditions in a Quasi-Analytic System

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This paper deals with the problems of integrability and linearizable conditions at degenerate singular point in a class of quasianalytic septic polynomial differential system. We solve the problems by an indirect method, that is, we transform the quasianalytic system into an analytic system firstly, and the degenerate singular point into an elementary singular point. Then we calculate the singular values at the origin of the analytic system by the known classical methods. We obtain the center conditions and isochronous center conditions. Accordingly, integrability and pseudolinearizable conditions at degenerate singular point in the quasianalytic system are obtained. Especially, when $\lambda = 1$, the system has been studied inWu and Zhang (2010).

## 1. Introduction

In the qualitative theory of planar polynomial differential equations, one of open problems for planar polynomial differential systems

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1.1}$$

is how to characterize their centers and isochronous centers. The characterization of centers for concrete families of differential equations is solved theoretically by computing the so-called *Lyapunov constants*. In most cases the procedure to study all centers is as follows:

compute several Lyapunov constants and when you get one significant, that is, zero, try to prove that the system obtained indeed has a center. Nevertheless, to completely solve this problem, there are two main obstacles. How can you be sure that you have computed enough Lyapunov constants? How do you prove that some system candidate to have a center actually has a center? As far as the case of the center is concerned, a lot of work has been done. Here we will not give an exhaustive bibliography.

In the case of a center, it makes sense to locally define a period function associated with a center, whose value at any point is the minimum period of the periodic orbit through the point. A center is said to be *isochronous* if the associated period function is constant. It is well known that isochronous centers are nondegenerate and systems with an isochronous center can be locally linearized by an analytic change of coordinates in a neighborhood of the center. The problem of characterizing isochronous centers of the origin has attracted the attention of several authors, and many good results have been published. The characterization of isochronous centers has been treated by several authors. However, there is a low number of families of polynomial systems for which there is a complete classification of their isochronous centers. For example, quadratic isochronous center [1]; isochronous centers of a linear center perturbed by third, fourth, and fifth degree homogeneous polynomials [2–4]; the cubic system of Kukles [5, 6]; the class of systems which in complex variable $z = x+iy$ writes as $dz/dt = iP(z) = iz + o(z)$ (all of which have an isochronous center at the origin) and the cubic time-reversible systems with $d\varphi/dt = 1$, see [7]; some isochronous cubic systems with four invariant lines, see [8]; isochronous centers of cubic systems with degenerate infinity [9, 10]; isochronous center conditions of infinity for rational systems [11–13]; and so forth. For more details about centers and isochronous centers, we refer the reader to the [14, 15].

Theory of center focus for a class of higher-degree critical points was established in [16], the authors there considered the following polynomial differential system:

$$\begin{aligned}\frac{dx}{dt} &= (\delta x - y)\left(x^2 + y^2\right)^n + \sum_{k=2n+2}^{\infty} X_k(x, y),\\ \frac{dy}{dt} &= (x + \delta y)\left(x^2 + y^2\right)^n + \sum_{k=2n+2}^{\infty} Y_k(x, y),\end{aligned} \tag{1.2}$$

where

$$X_k(x, y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta}, \qquad Y_k(x, y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta}. \tag{1.3}$$

By using their transformation

$$x = \xi\left(\xi^2 + \eta^2\right)^{n+1}, \qquad y = \eta\left(\xi^2 + \eta^2\right)^{n+1}, \qquad dt = \left(\xi^2 + \eta^2\right)^{-n(2n+3)} d\tau, \tag{1.4}$$

system (1.2) becomes

$$\begin{aligned}\frac{d\xi}{d\tau} = \frac{\delta}{2n+3}\xi - \eta + \sum_{k=2n+2}^{\infty}&\left[\left(\frac{1}{2n+3}\xi^2 + \eta^2\right)X_k(\xi, \eta) - \frac{2n+2}{2n+3}\xi\eta Y_k(\xi, \eta)\right]\\ &\times \left(\xi^2 + \eta^2\right)^{(k-2n-2)(n+1)},\end{aligned}$$

$$\frac{d\eta}{d\tau} = \xi + \frac{\delta}{2n+3}\eta + \sum_{k=2n+2}^{\infty}\left[\left(\xi^2 + \frac{1}{2n+3}\eta^2\right)Y_k(\xi,\eta) - \frac{2n+2}{2n+3}\xi\eta X_k(\xi,\eta)\right] \times \left(\xi^2+\eta^2\right)^{(k-2n-2)(n+1\ )}. \tag{1.5}$$

Furthermore, Liu in [16] gave the definition of singular value and pseudo-isochronous center at a degenerate singular point.

*Definition 1.1.* The degenerate singular point of system $(1.2)_{\delta=0}$ is called a pseudo-isochronous center if the origin of system $(1.5)_{\delta=0}$ is an isochronous center.

The problems of center conditions and pseudo-isochronous center conditions for degenerate singular point are poorly understood in the qualitative theory of ordinary differential equations. There are only a few papers concerning centers of degenerate singular points [17–24].

Recently, the following systems:

$$\begin{gathered}\dot{z} = (\lambda+i)z + (z\overline{z})^{(d-5)/2}\left(Az^{4+j}\overline{z}^{1-j} + Bz^3\overline{z}^2 + Cz^{2-j}\overline{z}^{3+j} + D\overline{z}^5\right), \quad d = 2m+1 \geq 5,\\ \dot{z} = iz + (z\overline{z})^{(d-4)/2}\left(Az^3\overline{z} + Bz^2\overline{z}^2 + C\overline{z}^4\right), \quad d = 2m \geq 4,\\ \dot{z} = (\lambda+i)z + (z\overline{z})^{(d-3)/2}\left(Az^3 + Bz^2\overline{z} + Cz\overline{z}^2 + D\overline{z}^3\right), \quad d = 2m+1 \geq 3,\\ \dot{z} = (\lambda+i)z + (z\overline{z})^{(d-2)/2}\left(Az^2 + Bz\overline{z} + C\overline{z}^2\right), \quad d = 2m \geq 2\end{gathered} \tag{1.6}$$

were investigated by Llibre and Valls, see [25–28]. The conditions of centers and isochronous centers were obtained. But the $d$ is restricted in order to assure the system is polynomial system. In [29], centers and isochronous centers for two classes of generalized seventh and ninth systems were investigated. In [30], linearizable conditions of a time-reversible quartic-like system were obtaied.

For the case of nonanalytic, being difficult, there are very few results. As far as integrability at origin are concerned, several special systems have been studied, see [31–34].

In this paper, we investigate integrability and linearizable conditions at degenerate singular point for a class of quasanalytic polynomial differential system

$$\begin{aligned}\frac{dx}{dt} &= (\delta x - y)\left(x^2+y^2\right)^{\lambda} + X_5(x,y)\left(x^2+y^2\right)^{2(\lambda-1)} - \beta y\left(x^2+y^2\right)^{3\lambda},\\ \frac{dy}{dt} &= (x + \delta y)\left(x^2+y^2\right)^{\lambda} + Y_5(x,y)\left(x^2+y^2\right)^{2(\lambda-1)} + \beta x\left(x^2+y^2\right)^{3\lambda},\end{aligned} \tag{1.7}$$

where

$$X_5(x,y) = \sum_{k+j=5} A_{kj} x^k y^j, \qquad Y_5(x,y) = \sum_{k+j=5} B_{kj} x^k y^j, \tag{1.8}$$

$$\begin{gathered}
A_{50} = \beta_{03} + \beta_{12} + \beta_{21} + \beta_{30}, \qquad A_{41} = -5\alpha_{03} - 3\alpha_{12} - \alpha_{21} + \alpha_{30}, \\
A_{32} = -2(5\beta_{03} + \beta_{12} - \beta_{21} - \beta_{30}), \qquad A_{23} = 2(5\alpha_{03} - \alpha_{12} - \alpha_{21} + \alpha_{30}), \\
A_{14} = 5\beta_{03} - 3\beta_{12} + \beta_{21} + \beta_{30}, \qquad A_{05} = -\alpha_{03} + \alpha_{12} - \alpha_{21} + \alpha_{30}, \\
B_{50} = \alpha_{03} + \alpha_{12} + \alpha_{21} + \alpha_{30}, \qquad B_{41} = 5\beta_{03} + 3\beta_{12} + \beta_{21} - \beta_{30}, \\
B_{32} = -2(5\alpha_{03} + \alpha_{12} - \alpha_{21} - \alpha_{30}), \qquad B_{23} = -2(5\beta_{03} - \beta_{12} - \beta_{21} + \beta_{30}), \\
B_{14} = 5\alpha_{03} - 3\alpha_{12} + \alpha_{21} + \alpha_{30}, \qquad B_{05} = \beta_{03} - \beta_{12} + \beta_{21} - \beta_{30}, \quad \lambda \in R.
\end{gathered} \tag{1.9}$$

When $\lambda = 1$, the system has been invested in [35].

The organization of this paper is as follows. In Section 2, we introduce some preliminary results which are useful throughout this paper. In Section 3, we make two appropriate transformations which let research on the degenerate singular point of system (1.7) be reduced to research on the elementary singular point of a twenty-one degree system. Furthermore, we compute the singular point quantities and derive the center conditions of the origin for the transformed system. Accordingly, the conditions of integrability at the degenerate singular point are obtained. In Section 4, we compute the period constants and discuss isochronous center conditions at the origin of the twenty-one degree system, meanwhile, the pseudolinearizable conditions at degenerate singular point are classified.

All calculations in this paper have been done with the computer algebra system: MATHEMATICA.

## 2. Some Preliminary Results

In [36–38], the authors defined complex center and complex isochronous center for the following complex system:

$$\begin{aligned}
\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w), \\
\frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),
\end{aligned} \tag{2.1}$$

where

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \qquad W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta, \tag{2.2}$$

and gave two recursive algorithms to determine necessary conditions for a center and an isochronous center. We now restate the definitions and algorithms.

By means of transformation

$$z = \rho e^{i\theta}, \quad w = \rho e^{-i\theta}, \quad T = it, \quad i = \sqrt{-1}, \tag{2.3}$$

where $r, \theta$ are complex numbers, system (2.1) can be transformed into

$$\begin{aligned}
\frac{d\rho}{dt} &= \frac{i\rho}{2}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k+2}\left(a_{\alpha,\beta-1} - b_{\beta,\alpha-1}\right)e^{i(\alpha-\beta)\theta}\rho^k, \\
\frac{d\theta}{dt} &= 1 + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k+2}\left(a_{\alpha,\beta-1} + b_{\beta,\alpha-1}\right)e^{i(\alpha-\beta)\theta}\rho^k.
\end{aligned} \tag{2.4}$$

For the complex constant $h$, $|h| \ll 1$, we write the solution of system (2.4) satisfying the initial condition $\rho|_{\theta=0} = h$ as

$$r = \tilde{\rho}(\theta, h) = h + \sum_{k=2}^{\infty}\nu_k(\theta)h^k, \tag{2.5}$$

which could be thought of as the first Poincaré displacement map and denote the period function by

$$\begin{aligned}
\tau(\varphi, h) &= \int_0^{\varphi}\frac{dt}{d\theta}\,d\theta \\
&= \int_0^{\varphi}\left[1 + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k+2}\left(a_{\alpha,\beta-1} + b_{\beta,\alpha-1}\right)e^{i(\alpha-\beta)\theta}\tilde{r}^k(\theta, h)\right]^{-1} d\theta.
\end{aligned} \tag{2.6}$$

*Definition 2.1.* For a sufficiently small complex constant $h$, the origin of system (2.1) is called a complex center if $\tilde{\rho}(2\pi, h) \equiv h$, and it is called a complex isochronous center if

$$\tilde{\rho}(2\pi, h) \equiv h, \qquad \tau(2\pi, h) \equiv 2\pi. \tag{2.7}$$

**Lemma 2.2.** *For system* (2.1), *one can derive uniquely the following formal series:*

$$\xi = z + \sum_{k+j=2}^{\infty}c_{kj}z^k w^j, \qquad \eta = w + \sum_{k+j=2}^{\infty}d_{kj}w^k z^j, \tag{2.8}$$

*where* $c_{k+1,k} = d_{k+1,k} = 0$, $k = 1, 2, \ldots$, *such that*

$$\begin{aligned}
\frac{d\xi}{dT} &= \xi + \sum_{j=1}^{\infty}p_j\xi^{j+1}\eta^j, \\
\frac{d\eta}{dT} &= -\eta - \sum_{j=1}^{\infty}q_j\eta^{j+1}\xi^j.
\end{aligned} \tag{2.9}$$

*Definition 2.3* (see [37, 38]). Let $\mu_0 = 0$, $\mu_k = p_k - q_k$, $\tau_k = p_k + q_k$, $k = 1, 2, \ldots$. $\mu_k$ be called the $k_{\text{th}}$ singular point quantity of the origin of system (2.1) and $\tau_k$ be called the $k_{\text{th}}$ period constant of the origin of system (2.1).

Reeb's criterion (see for instance [39]) says that system (2.1) has a center if and only if there is a nonzero analytic integrating factor (or integral factor) in a neighborhood of the origin. In [16], it is developed an algorithm to compute the focal values through the analytic integrating factor that must exist when we have a center, namely, the following theorem.

**Theorem 2.4** (see [16]). *For system* (2.1)*, one can derive successively the terms of the following formal series:*

$$M(z, w) = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta}, \tag{2.10}$$

*such that*

$$\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} = \sum_{m=1}^{\infty} (m+1)\mu_m (zw)^m, \tag{2.11}$$

*where* $c_{00} = 1$*, for all* $c_{kk} \in R$*,* $k = 1, 2, \ldots$*, and for any integer* $m$*,* $\mu_m$ *is determined by the following recursive formulae:*

$$\begin{aligned}
&c_{00} = 1, \\
&\textit{when } (\alpha = \beta > 0) \textit{ or } \alpha < 0, \textit{ or } \beta < 0, \ c_{\alpha\beta} = 0, \\
&\textit{else} \\
&c_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} \left[ (\alpha+1) a_{k,j-1} - (\beta+1) b_{j,k-1} \right] c_{\alpha-k+1,\beta-j+1}, \\
&\mu_m = \sum_{k+j=3}^{2m+2} \left( a_{k,j-1} - b_{j,k-1} \right) c_{m-k+1,m-j+1}.
\end{aligned} \tag{2.12}$$

**Theorem 2.5** (see [37]). *For system* (2.1)*, one can derive uniquely the following formal series:*

$$f(z, w) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j, \qquad g(z, w) = w + \sum_{k+j=2}^{\infty} d'_{kj} w^k z^j, \tag{2.13}$$

*where* $c'_{k+1,k} = d'_{k+1,k} = 0$*,* $k = 1, 2, \ldots$*, such that*

$$\frac{df}{dT} = f(z, w) + \sum_{j=1}^{\infty} p'_j z^{j+1} w^j, \qquad \frac{dg}{dT} = -g(z, w) - \sum_{j=1}^{\infty} q'_j w^{j+1} z^j, \tag{2.14}$$

*and when $k-j-1\neq 0$, $c'_{kj}$ and $d'_{kj}$ are determined by the following recursive formulae:*

$$c'_{kj} = \frac{1}{j+1-k}\sum_{\alpha+\beta=3}^{k+j+1}\left[(k-\alpha+1)a_{\alpha,\beta-1}-(j-\beta+1)b_{\beta,\alpha-1}\right]c'_{k-\alpha+1,j-\beta+1},$$
$$d'_{kj} = \frac{1}{j+1-k}\sum_{\alpha+\beta=3}^{k+j+1}\left[(k-\alpha+1)b_{\alpha,\beta-1}-(j-\beta+1)a_{\beta,\alpha-1}\right]d'_{k-\alpha+1,j-\beta+1}, \tag{2.15}$$

*and for any positive integer $j$, $p'_j$, and $q'_j$ are determined by the following recursive formulae:*

$$p'_j = \sum_{\alpha+\beta=3}^{2j+2}\left[(j-\alpha+2)a_{\alpha,\beta-1}-(j-\beta+1)b_{\beta,\alpha-1}\right]c'_{j-\alpha+2,j-\beta+1},$$
$$q'_j = \sum_{\alpha+\beta=3}^{2j+2}\left[(j-\alpha+2)b_{\alpha,\beta-1}-(j-\beta+1)a_{\beta,\alpha-1}\right]d'_{j-\alpha+2,j-\beta+1}. \tag{2.16}$$

*In the above expression, one has let $c'_{10}=d'_{10}=1$, $c'_{01}=d'_{01}=0$, and if $\alpha<0$ or $\beta<0$, let $a_{\alpha\beta}=b_{\alpha\beta}=c'_{\alpha\beta}=d'_{\alpha\beta}=0$.*

*We introduce double parameter transformation groups*

$$z=\overline{\rho}e^{i\theta}\widetilde{z},\qquad w=\overline{\rho}e^{-i\theta}\widetilde{w}, \tag{2.17}$$

*where $\widetilde{z},\widetilde{w}$ are new variables, $\overline{\rho},\theta$ are complex parameters, and $\overline{\rho}\neq 0$. Denote $z=x+iy$, $w=x-iy$, $\widetilde{z}=\widetilde{x}+i\widetilde{y}$, $\widetilde{w}=\widetilde{x}-i\widetilde{y}$. Transformation (2.17) can be turned into*

$$x=\overline{\rho}(\widetilde{x}\cos\theta-\widetilde{y}\sin\theta),\qquad y=\overline{\rho}(\widetilde{x}\sin\theta+\widetilde{y}\cos\theta). \tag{2.18}$$

*In the case of real variables and real parameters, (2.18) is a transformation of similar rotation. With (2.17) being used, system (2.1) can be transformed into*

$$\frac{d\widetilde{z}}{dT}=\widetilde{z}+\sum_{\alpha+\beta=2}^{\infty}\widetilde{a}_{\alpha\beta}\widetilde{z}^{\alpha}\widetilde{w}^{\beta},$$
$$\frac{d\widetilde{w}}{dT}=-\widetilde{w}-\sum_{\alpha+\beta=2}^{\infty}\widetilde{b}_{\alpha\beta}\widetilde{w}^{\alpha}\widetilde{z}^{\beta}, \tag{2.19}$$

*where $\overline{\rho}$, $\theta$ are parameters, $\widetilde{z}$, $\widetilde{w}$, $T$ are variables, and for all $\alpha\geq 0$, $\beta\geq 0$ one has*

$$\widetilde{a}_{\alpha\beta}=a_{\alpha\beta}\overline{\rho}^{\alpha+\beta-1}e^{i(\alpha-\beta-1)\theta},$$
$$\widetilde{b}_{\alpha\beta}=b_{\alpha\beta}\overline{\rho}^{\alpha+\beta-1}e^{-i(\alpha-\beta-1)\theta}. \tag{2.20}$$

*Under the transformation* (2.17), *suppose that* $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ *is a polynomial of* $a_{\alpha\beta}$, $b_{\alpha\beta}$ *with complex coefficients, and denote*

$$\widetilde{f} = f\left(\widetilde{a}_{\alpha\beta}, \widetilde{b}_{\alpha\beta}\right), \qquad f^{*} = f\left(a^{*}_{\alpha\beta}, b^{*}_{\alpha\beta}\right), \tag{2.21}$$

*where* $a^{*}_{\alpha\beta} = b_{\alpha\beta}$, $b^{*}_{\alpha\beta} = a_{\alpha\beta}$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \geq 2$.

*Definition 2.6* (see [38]). Suppose that there exist constants $\lambda, \sigma$, such that $\widetilde{f} = \overline{\rho}^{\lambda} e^{i\sigma\theta} f$, we say that $\lambda$ is a similar exponent and $\sigma$ a rotation exponent of system (2.1) under the transformation (2.17), which are denoted by $I_s(f) = \lambda, I_\rho(f) = \sigma$.

*Definition 2.7* (see [38]). (i) A polynomial $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ is called a Lie invariant of order $k$, if $\widetilde{f} = \overline{\rho}^{2k} f$.

(ii) An invariant $f$ is called a monomial Lie invariant, if $f$ is both of a Lie invariant and a monomial of $a_{\alpha\beta}, b_{\alpha\beta}$.

(iii) A monomial Lie invariant $f$ is called an elementary Lie invariant, if it can not be expressed as a product of two monomial Lie invariants.

*Definition 2.8* (see [38]). A polynomial $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ is called self-symmetry if $f^{*} = f$. It is called self-antisymmetry if $f^{*} = -f$.

**Theorem 2.9** (see the extended symmetric principle in [38]). *Let $g$ denote an elementary Lie invariant of system* (2.1). *If for all $g$ the symmetric condition $g = g^{*}$ is satisfied, then the origin of system* (2.1) *is a complex center. Namely, all singular point quantities of the origin are zero.*

## 3. Integrability at the Origin of (1.7)

In this section, the integrability at the origin of (1.7) is discussed by an indirect method. By means of transformation

$$u = x + iy, \quad v = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{3.1}$$

system $(1.7)_{\delta=0}$ becomes its concomitant complex system

$$\begin{aligned}\frac{du}{dT} &= u(uv)^{\lambda} + (uv)^{2(\lambda-1)}\left(a_{03}u^{5} + a_{12}vu^{4} + a_{21}u^{3}v^{2} + a_{30}u^{2}v^{3}\right) - \beta u(uv)^{3\lambda},\\ \frac{dv}{dT} &= -v(uv)^{\lambda} - (uv)^{2(\lambda-1)}\left(b_{03}v^{5} + b_{12}v^{4}u + b_{21}u^{2}v^{3} + b_{30}u^{3}v^{2}\right) + \beta v(uv)^{3\lambda},\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}a_{30} &= \alpha_{30} + i\beta_{30}, \quad a_{21} = \alpha_{21} + i\beta_{21}, \quad a_{12} = \alpha_{12} + i\beta_{12}, \quad a_{03} = \alpha_{03} + i\beta_{03},\\ b_{30} &= \alpha_{30} - i\beta_{30}, \quad b_{21} = \alpha_{21} - i\beta_{21}, \quad b_{12} = \alpha_{12} - i\beta_{12}, \quad b_{03} = \alpha_{03} - i\beta_{03}.\end{aligned} \tag{3.3}$$

Then, by using transformation

$$\xi = u^{(\lambda+1)/2} v^{(\lambda-1)/2}, \qquad \eta = v^{(\lambda+1)/2} u^{(\lambda-1)/2}, \tag{3.4}$$

system (3.2) can be transformed into the following system:

$$\begin{aligned}
\frac{d\xi}{dT} &= \xi^2\eta + \frac{1}{2}a_{03}(1+\lambda)\xi^5 + \frac{1}{2}(a_{12}+b_{30}+a_{12}\lambda-b_{30}\lambda)\xi^4\eta + \frac{1}{2}(a_{21}+b_{21}+a_{21}\lambda-b_{21}\lambda)\xi^3\eta^2 \\
&\quad + \frac{1}{2}(a_{30}+b_{12}+a_{30}\lambda-b_{12}\lambda)\eta^3\xi^2 + \frac{1}{2}b_{03}(-1+\lambda)\xi\eta^4 + \beta\xi^4\eta^3, \\
\frac{d\eta}{dT} &= -\eta^2\xi - \frac{1}{2}b_{03}(1+\lambda)\eta^5 - \frac{1}{2}(b_{12}+a_{30}+b_{12}\lambda-a_{30}\lambda)\eta^4\xi - \frac{1}{2}(b_{21}+a_{21}+b_{21}\lambda-a_{21}\lambda)\eta^3\xi^2 \\
&\quad - \frac{1}{2}(b_{30}+a_{12}+b_{30}\lambda-a_{12}\lambda)\eta^2\xi^3 - \frac{1}{2}a_{03}(-1+\lambda)\eta\xi^4 + \beta\xi^3\eta^4.
\end{aligned} \tag{3.5}$$

At last, by means of transformation $(1.4)_{n=1}$, system (3.5) is reduced to

$$\begin{aligned}
\frac{dz}{d\tau} &= z + \frac{1}{10}w^3z^4\Big(-b_{03}(-5+\lambda)w^4 + (-b_{12}(-5+\lambda)+a_{30}(5+\lambda))w^3z \\
&\qquad + (-b_{21}(-5+\lambda)+a_{21}(5+\lambda))w^2z^2 + (-b_{30}(-5+\lambda)+a_{12}(5+\lambda))wz^3 \\
&\qquad + a_{03}(5+\lambda)z^4\Big) + \beta w^{10}z^{11}, \\
\frac{dw}{d\tau} &= -w - \frac{1}{10}w^4z^3\Big(b_{03}(-5+\lambda)w^4 + (b_{12}(5+\lambda)-a_{30}(-5+\lambda))w^3z \\
&\qquad + (b_{21}(-5+\lambda)-a_{21}(-5+\lambda))w^2z^2 + (b_{30}(5+\lambda)-a_{12}(-5+\lambda))wz^3 \\
&\qquad - a_{03}(-5+\lambda)z^4\Big) + \beta z^{10}w^{11}.
\end{aligned} \tag{3.6}$$

By those transformations, we transform the quasanalytic system into an analytic system firstly, and the degenerate singular point into an elementary singular point. Under the conjugate condition (3.6): it is obvious that the origin of system (3.5) to be integrability (linearizable) is equivalent to the degenerate singular point of system (1.7) to be integrability (pseudolinearizable).

Using the recursive formulae of Theorem 2.4 to compute the singular point quantities at the origin of system (3.6) (for detailed recursive formulae, see Appendix A) and simplify them with the constructive theorem of singular point quantities, we get the following.

**Theorem 3.1.** *The first 55 singular point quantities at the origin of system* (3.6) *are as follows:*

$$\mu_5 = \frac{1}{5}(a_{21} - b_{21})\lambda,$$
$$\mu_{10} = -\frac{1}{5}(a_{30}a_{12} - b_{30}b_{12})\lambda. \tag{3.7}$$

*Case 1.* $a_{12}b_{12} \neq 0$, then there exist $k$ to make $a_{30} = kb_{12}$, $b_{30} = ka_{12}$,

$$\begin{aligned}
\mu_{15} &= \frac{\lambda}{40}\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)(-1+3k)(2+2k-\lambda+k\lambda),\\
\mu_{20} &= \frac{\lambda^2}{10(\lambda+2)}b_{21}\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)(3k-1),\\
\mu_{25} &= \frac{\lambda^2}{240(\lambda+2)^3}(3k-1)\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)\\
&\quad\times\left(-32a_{03}b_{03} - 28a_{03}b_{03}\lambda + 32a_{12}b_{12}\lambda^2 + 5a_{03}b_{03}\lambda^3 + a_{03}b_{03}\lambda^4 + 192\beta + 192\lambda\beta + 48\lambda^2\beta\right),\\
\mu_{30} &= 0,\\
\mu_{35} &= -\frac{\lambda^2}{19200(\lambda+2)^5}(3k-1)\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)\\
&\quad\times\left(1024a_{03}^2b_{03}^2 + 1920a_{03}^2b_{03}^2\lambda - 13824a_{12}a_{03}b_{12}b_{03}\lambda^2 + 224a_{03}^2b_{03}^2\lambda^2 - 11392a_{12}a_{03}b_{12}b_{03}\lambda^3\right.\\
&\qquad - 1584a_{03}^2b_{03}^2\lambda^3 + 12800a_{12}^2b_{12}^2\lambda^4 + 2432a_{12}a_{03}b_{12}b_{03}\lambda^4 - 1056a_{03}^2b_{03}^2\lambda^4\\
&\qquad\left. + 4064a_{12}a_{03}b_{12}b_{03}\lambda^5 + 864a_{12}a_{03}b_{12}b_{03}\lambda^6 + 206a_{03}^2b_{03}^2\lambda^6 + 69a_{03}^2b_{03}^2\lambda^7 + 7a_{03}^2b_{03}^2\lambda^8\right),\\
\mu_{40} &= -\frac{7\lambda^2}{2880(\lambda+2)^5}(3k-1)\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)\left(a_{03}b_{12}^2 + b_{03}a_{12}^2\right)(\lambda+1)\\
&\quad\times\left(-32a_{03}b_{03} - 24a_{03}b_{03}\lambda + 32a_{12}b_{12}\lambda^2 + 4a_{03}b_{03}\lambda^2 + 6a_{03}b_{03}\lambda^3 + a_{03}b_{03}\lambda^4\right).
\end{aligned} \tag{3.8}$$

If $a_{12}b_{12} = -a_{03}b_{03}(-2+\lambda)(2+\lambda)^2(4+\lambda)/32\lambda^2$,

$$\mu_{45} = \frac{24131}{67184640}(3k-1)a_{03}^3b_{03}^3\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right). \tag{3.9}$$

If $a_{03}b_{12}^2 + b_{03}a_{12}^2 = 0$, then there exist $m$ to make $a_{03} = ma_{12}^2$, $b_{03} = -mb_{12}^2$,

$$\mu_{45} = \frac{7\lambda^2}{4838400000(\lambda+2)^3}(3k-1)(\lambda+1)a_{12}b_{12}a_{03}^2b_{03}^2\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)$$
$$\times\left(-23257088a_{12}b_{12}m^2 + 6577280a_{12}b_{12}m^2\lambda - 23257088\lambda^2 + 26650064a_{12}b_{12}m^2\lambda^2\right.$$
$$+ 75164416\lambda^3 - 2764244a_{12}b_{12}m^2\lambda^3 + 8304896\lambda^4 - 10922212a_{12}b_{12}m^2\lambda^4 - 18884608\lambda^5$$
$$\left.-916685a_{12}b_{12}m^2\lambda^5 + 1341691a_{12}b_{12}m^2\lambda^6 + 255854a_{12}b_{12}m^2\lambda^7\right),$$
$$\mu_{50} = 0,$$
$$\mu_{55} = -\frac{1099511627776\lambda^{19}}{8859375m^8(\lambda+2)^{17}}(3k-1)\left(a_{03}b_{12}^2 - b_{03}a_{12}^2\right)(\lambda+1). \tag{3.10}$$

*Case 2.* $a_{12} = b_{12} = 0$,

$$\mu_{15} = \frac{3\lambda}{40}\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)(\lambda+2),$$
$$\mu_{20} = \frac{3}{10}\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)b_{21},$$
$$\mu_{25} = -\frac{1}{40}\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)(4a_{30}b_{30} - 3a_{03}b_{03} + 24\beta),$$
$$\mu_{30} = 0, \tag{3.11}$$
$$\mu_{35} = -\frac{1}{400}\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)\left(11a_{03}^2b_{03}^2 - 19a_{03}b_{03}a_{30}b_{30} - 50a_{30}^2b_{30}^2\right),$$
$$\mu_{40} = \frac{7}{480}\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)\left(a_{03}a_{30}^2 + b_{03}b_{30}^2\right)(a_{03}b_{03} - a_{30}b_{30}),$$
$$\mu_{45} = -\frac{1}{1344000}a_{30}^3b_{30}^3\left(a_{03}a_{30}^2 - b_{03}b_{30}^2\right)$$
$$\times\left(102334400 + 28122192a_{30}b_{30}m^2 - 26626536a_{30}^2b_{30}^2m^4 + 2330697a_{30}^3b_{30}^3m^6\right),$$

where $\mu_k = 0$, $k \neq 5i$, $i \leq 11$, $i \in N$. In the above expression of $\mu_k$, we have already let $\mu_1 = \cdots = \mu_{k-1} = 0$, $k = 2, 3, \ldots, 45$.

From Theorem 3.1, we get the following.

**Theorem 3.2.** *For system* (3.6)*, the first 55 singular point quantities are zero if and only if one of the following conditions holds:*

$$a_{21} = b_{21}, \quad a_{12} = b_{12} = 0, \quad a_{30}^2 a_{03} = b_{03} b_{30}^2, \tag{3.12}$$

$$a_{21} = b_{21}, \quad a_{30} = \frac{1}{3} b_{12}, \quad b_{30} = \frac{1}{3} a_{12}, \quad a_{12} b_{12} \neq 0, \tag{3.13}$$

$$a_{21} = b_{21}, \quad a_{30} a_{12} = b_{30} b_{12}, \quad a_{12}^2 b_{03} = b_{12}^2 a_{03}, \quad a_{12} b_{12} \neq 0, \tag{3.14}$$

$$\begin{gathered} \lambda = -1,\ \beta = 0, \quad a_{21} = b_{21} = 0, \quad a_{30} = -3b_{12}, \quad b_{30} = -3a_{12}, \quad a_{03} b_{03} = 4a_{12} b_{12}, \\ a_{12} b_{12} \neq 0. \end{gathered} \tag{3.15}$$

In order to obtain the integrability conditions of the origin, we have to find out all the elementary Lie invariants of system (3.6). According to Definitions 2.6, 2.7 and 2.8, we have the following.

**Lemma 3.3.** *All the elementary Lie invariants of system* (3.6) *are as follows:*

$$\begin{gathered} \beta, a_{21}, b_{21}, a_{30} b_{30}, a_{12} b_{12}, a_{03} b_{03}, a_{30} a_{12}, b_{30} b_{12}, \\ a_{30}^2 a_{03}, a_{30} b_{12} a_{03}, b_{12}^2 a_{03}, b_{30}^2 b_{03}, b_{30} a_{12} b_{03}, a_{12}^2 b_{03}. \end{gathered} \tag{3.16}$$

The following result holds.

**Theorem 3.4.** *For system* (3.6)*, all the singular point quantities at the origin are zero if and only if the first 55 singular point quantities are zero, that is, one of the conditions in Theorem 3.2 holds. Correspondingly, the conditions in Theorem 3.2 are the integrability conditions of the origin.*

*Proof.* If condition (3.12) or (3.14) holds, system $(3.5)_{\delta=0}$ satisfies the conditions of Theorem 2.9. If condition (3.13) holds, system (3.6) has the first integral

$$\begin{gathered} zw e^{3a_{03} z^7 w^3 + 4a_{12} z^6 w^4 + 6b_{21} z^5 w^5 + 4b_{12} z^4 w^6 + 3b_{03} z^3 w^7 + 3z^{10} w^{10} \beta - 3}, \quad \lambda = -2, \\ (zw)^{5(\lambda+2)/\lambda} f_1, \quad \lambda \neq -2, \end{gathered} \tag{3.17}$$

where

$$\begin{aligned} f_1 = {} & \left(-12a_{03} + 3a_{03}\lambda^2\right) z^7 w^3 + \left(4a_{41}\lambda^2 - 16a_{12}\right) z^6 w^4 + \left(6b_{21}\lambda^2 - 24b_{21}\right) z^5 w^5 \\ & + \left(4b_{12}\lambda^2 + 16b_{12}\right) z^4 w^6 + \left(3b_{03}\lambda^2 - 12b_{03}\right) z^3 w^7 + (24\beta - 12r\beta) z^{10} w^{10} - 24 + 12\lambda. \end{aligned} \tag{3.18}$$

If condition (3.15) holds, system (3.6) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{1}{10}\left(10 + 4a_{03}z^7w^3 - 14a_{12}z^6w^4 - 6b_{12}z^4w^6 + 6b_{03}z^3w^7\right),\\ \frac{dw}{dT} &= -\frac{1}{10}\left(10 + 4b_{03}z^3w^7 - 14b_{12}w^6z^4 - 6a_{12}w^4z^6 + 6a_{03}w^3z^7\right),\end{aligned} \tag{3.19}$$

there exists a transformation

$$z = \frac{u}{(uv)^{2/5}}, \qquad w = \frac{v}{(uv)^{2/5}}, \tag{3.20}$$

system (3.19) is changed into

$$\begin{aligned}\frac{du}{dT} &= u + b_{03}v^3 + b_{12}v^2u - 3a_{12}u^3 = U,\\ \frac{dv}{dT} &= -\left(v + a_{03}u^3 + a_{12}u^2v - 3b_{12}v^3\right) = V,\end{aligned} \tag{3.21}$$

system (3.21) has the integral factor $f_2^{-5/6}$, where

$$\begin{aligned}f_2 &= 1 - 6\left(b_{12}u^2 + a_{12}v^2\right)\\ &\quad + 3\left(3b_{12}^2u^4 - 2a_{12}b_{03}u^3v + 2a_{12}b_{12}u^2v^2 - 2b_{12}a_{03}v^3u + 3a_{12}^4v^4\right)\\ &\quad + \frac{1}{2}(2a_{12}u - a_{03}v)(2b_{12}v - b_{03}u)\left(b_{03}u^4 - 2b_{12}u^3v - 2a_{12}v^3u + a_{03}v^4\right),\\ \frac{df_2}{dt} &= -12\left(b_{12}u^2 - a_{12}v^2\right)f_2 = \frac{6}{5}\left(\frac{\partial U}{\partial u} - \frac{\partial V}{\partial v}\right)f_2.\end{aligned} \tag{3.22}$$

□

Synthesizing all the above cases, we have the following.

**Theorem 3.5.** *The system* (1.7) *is integrability at the origin if and only if one of conditions in Theorem 3.2 holds.*

## 4. Linearizable Conditions at the Origin of (1.7)

In this section we classify the pseudolinearizable conditions at the origin of (1.7). We discuss the linearizable conditions for system (3.6) firstly. According to Theorem 2.5, we get the recursive formulae to compute period constants (detailed recursive formulae, see Appendix B). Denote $a_{21} = b_{21} = r_{21}$, from the integrability conditions given in Section 4, we investigate the following three cases.

*Case 1.* Integrability condition (3.12) holds.

If $a_{30} = b_{30} = 0$, we easily obtain the first 30 period constants

$$\begin{aligned}
\tau_5 &= 2r_{21},\\
\tau_{10} &= \frac{1}{2}\left((\lambda-2)a_{03}b_{03} + 4\beta\right),\\
\tau_{15} &= 0,\\
\tau_{20} &= -\frac{1}{32}a_{03}^2b_{03}^2(-2+\lambda)^2(-2+3\lambda),\\
\tau_{25} &= 0,\\
\tau_{30} &= \frac{1}{25600}a_{03}^2b_{03}^2(-2+\lambda)^2\left(50+365\lambda-1962\lambda^2+2153\lambda^3\right).
\end{aligned} \tag{4.1}$$

If $a_{30}b_{30} \neq 0$, from condition (3.12), there exists an arbitrary complex constant $s$, such that

$$a_{03} = sb_{30}^2, \qquad b_{03} = sa_{30}^2, \tag{4.2}$$

then we get the first 30 period constants

$$\begin{aligned}
\tau_5 &= 2r_{21},\\
\tau_{10} &= \frac{1}{2}\left(-2a_{30}b_{30} - 2a_{30}^2b_{30}^2s^2 - 2a_{30}b_{30}\lambda + a_{30}^2b_{30}^2s^2\lambda + 4\beta\right),\\
\tau_{15} &= \frac{3}{4}a_{30}^2b_{30}^2s(\lambda+2),\\
\tau_{20} &= \frac{1}{4}a_{30}^2b_{30}^2(1+\lambda)^2(3\lambda+1),\\
\tau_{25} &= 0,\\
\tau_{30} &= -\frac{1}{800}a_{30}^3b_{30}^3(1+\lambda)^2\left(-25+35\lambda+981\lambda^2+2153\lambda^3\right).
\end{aligned} \tag{4.3}$$

In expressions (4.1) and (4.3), $\tau_k = 0$, $k \neq 5i$, $i \leq 4$, $i \in N$, and we have already let $\tau_1 = \cdots = \tau_{k-1} = 0$, $k = 2, 3, \ldots, 30$.

From expressions (4.1) and (4.3), we have the following.

**Theorem 4.1.** *The first 30 period constants at the origin of system* (3.6) *are zero if and only if one of the following conditions holds:*

$$\beta = a_{21} = b_{21} = a_{12} = b_{12} = a_{30} = b_{30} = 0, \quad \lambda = 2. \tag{4.4}$$

$$\beta = a_{21} = b_{21} = a_{12} = b_{12} = a_{03} = b_{03} = 0, \quad \lambda = -1, \quad a_{30}b_{30} \neq 0. \tag{4.5}$$

**Theorem 4.2.** *Under integrability condition* (3.12)*, the origin of system* (3.6) *is a complex isochronous center if and only if one of the conditions in Theorem 4.1 holds.*

*Proof.* When condition (4.4) is satisfied, system (3.6) becomes

$$\frac{dz}{dT} = \frac{1}{10}z\left(10 + 7a_{03}z^7w^3 + 3b_{03}z^3w^7\right),$$
$$\frac{dw}{dT} = -\frac{1}{10}w\left(10 + 3a_{03}z^7w^3 + 7b_{03}z^3w^7\right). \tag{4.6}$$

There exists a transformation

$$u = \frac{z(1 + b_{03}z^3w^7)^{3/40}}{(1 + a_{03}z^7w^3)^{7/40}}, \qquad v = \frac{w(1 + a_{03}z^7w^3)^{3/40}}{(1 + b_{03}z^3w^7)^{7/40}}, \tag{4.7}$$

such that system (4.6) is reduced to a linear system.

When condition (4.5) is satisfied, system (3.6) becomes

$$\frac{dz}{dT} = \frac{1}{10}z\left(10 + 6b_{30}z^6w^4 + 4a_{30}z^4w^6\right),$$
$$\frac{dw}{dT} = -\frac{1}{10}w\left(10 + 4b_{30}z^4w^6 + 6a_{30}z^6w^4\right). \tag{4.8}$$

There exists a transformation

$$u = \frac{z(1 + a_{30}z^4w^6)^{1/5}}{(1 + b_{30}z^6w^4)^{3/10}}, \qquad v = \frac{w(1 + b_{30}z^6w^4)^{1/5}}{(1 + a_{30}z^4w^6)^{3/10}}, \tag{4.9}$$

such that system (4.8) is reduced to a linear system. □

*Case 2.* Integrability condition (3.13) holds.

Substituting condition (3.13) into the recursive formulae in Appendix B, we obtain the first 40 period constants

$$\tau_5 = 2r_{21},$$
$$\tau_{10} = \frac{1}{18}(-18a_{03}b_{03} - 32a_{12}b_{12} + 9a_{03}b_{03}\lambda + 16a_{12}b_{12}\lambda + 36\beta),$$
$$\tau_{15} = \frac{1}{3}\left(a_{12}^2b_{03} + a_{03}b_{12}^2\right)(-4 + \lambda)(-1 + \lambda),$$
$$\tau_{20} = -\frac{1}{2592}a_{12}^2b_{12}^2(\lambda - 2)(3\lambda + 2)\left(-512 - 1152a_{12}b_{12}s^2 - 162a_{12}^2b_{12}^2s^4 + 256\lambda + 81a_{12}^2b_{12}^2s^4\lambda\right),$$
$$\tau_{25} = 0,$$

$$\begin{aligned}
\tau_{30} &= \frac{1}{9\left(256+81a_{12}^2b_{12}^2s^4\right)^5} 2048a_{12}^5b_{12}^5s^4\left(256+720a_{12}b_{12}s^2+81a_{12}^2b_{12}^2s^4\right) \\
&\quad \times\left(256+1152a_{12}b_{12}s^2+81a_{12}^2b_{12}^2s^4\right)\Big(-262144-2801664a_{12}b_{12}s^2 \\
&\qquad +4147200a_{12}^2b_{12}^2s^4-2192832a_{12}^3b_{12}^3s^6 \\
&\qquad +183708a_{12}^4b_{12}^4s^8+59049a_{12}^5b_{12}^5s^{10}\Big), \\
\tau_{35} &= 0, \\
\tau_{40} &= \frac{1}{2025\left(256+81a_{12}^2b_{12}^2s^4\right)^7} 64a_{12}^6b_{12}^6s^4 \\
&\quad \times\Big(10923931256149875 0976+23935157090596222240256a_{12}b_{12}s^2 \\
&\qquad +1662753414453165608140 8a_{12}^2b_{12}^2s^4+4580866611183608030822 4a_{12}^3b_{12}^3s^6 \\
&\qquad +3448731449986070976921 6a_{12}^4b_{12}^4s^8-4367189859685381649203 2a_{12}^5b_{12}^5s^{10} \\
&\qquad -3165616584070076458598 4a_{12}^6b_{12}^6s^{12}+4919336728253449843507 2a_{12}^7b_{12}^7s^{14} \\
&\qquad -2655678216972439388160a_{12}^8b_{12}^8s^{16}-8653122543235415408640a_{12}^9b_{12}^9s^{18} \\
&\qquad -539890165260393578496a_{12}^{10}b_{12}^{10}s^{20}+408937711778677748736a_{12}^{11}b_{12}^{11}s^{22} \\
&\qquad +82987106695640213760a_{12}^{12}b_{12}^{12}s^{24}+5972974608886179588a_{12}^{13}b_{12}^{13}s^{26} \\
&\qquad +151421489259386859a_{12}^{14}b_{12}^{14}s^{28}\Big),
\end{aligned} \tag{4.10}$$

where $\tau_k = 0$, $k \neq 5i$, $i \leq 8$, $i \in N$. In the above expression of $\tau_k$, we have already let $\tau_1 = \cdots = \tau_{k-1} = 0$, $k = 2, 3, \ldots, 40$.

From expressions (4.10), we have the following.

**Theorem 4.3.** *The first 40 period constants at the origin of system* (3.6) *are zero if and only if one of the following conditions holds:*

$$\lambda = 2, \quad \beta = a_{21} = b_{21} = a_{03} = b_{03} = 0, \quad a_{30} = \frac{1}{3}b_{12}, \quad b_{30} = \frac{1}{3}a_{12}. \tag{4.11}$$

**Theorem 4.4.** *Under integrability condition* (3.13)*, the origin of system* (3.6) *is a complex isochronous center if and only if the condition in Theorem 4.3 holds.*

*Proof.* When condition (4.11) is satisfied, system (3.6) becomes

$$\begin{aligned}
\frac{dz}{dT} &= \frac{1}{10}z\left(10+8a_{12}z^6w^4+\frac{16}{3}b_{12}z^4w^6\right), \\
\frac{dw}{dT} &= -\frac{1}{10}w\left(10+8b_{12}z^4w^6+\frac{16}{3}a_{12}z^6w^4\right).
\end{aligned} \tag{4.12}$$

There exists a transformation

$$u = \frac{z\left(3 + 4b_{12}z^4w^6\right)^{1/5}}{\left(3 + 4a_{12}z^6w^4\right)^{3/10}}, \qquad v = \frac{w\left(3 + 4a_{12}z^6w^4\right)^{1/5}}{\left(3 + 4b_{12}z^4w^6\right)^{3/10}}, \tag{4.13}$$

such that system (4.12) is reduced to a linear system. □

*Case 3.* Integrability condition (3.14) holds.

Because $a_{12}b_{12} \neq 0$, we can let

$$a_{30} = kb_{12}, \quad b_{30} = ka_{12}, \quad a_{03} = ma_{12}^2, \quad b_{03} = mb_{12}^2, \tag{4.14}$$

where $k, m$ are arbitrary complex constants. Substituting (4.14) into the recursive formulae in Appendix B, we obtain the first 30 period constants

$$\begin{aligned}
\tau_5 &= 2r_{21}, \\
\tau_{10} &= \frac{1}{2}\Big(-2a_{12}b_{12} - 4a_{12}b_{12}k - 2a_{12}b_{12}k^2 - 2a_{12}^2b_{12}^2m^2 \\
&\qquad +2a_{12}b_{12}\lambda - 2a_{12}b_{12}k^2\lambda + a_{12}^2b_{12}^2m^2\lambda + 4\beta\Big), \\
\tau_{15} &= \frac{1}{4}a_{12}^2b_{12}^2m(3 + 3k - 4\lambda)(2 + 2k - \lambda + k\lambda).
\end{aligned} \tag{4.15}$$

If $m = 0$,

$$\begin{aligned}
\tau_{20} &= \frac{1}{4}a_{12}^2b_{12}^2(1 + k)(1 + k - \lambda + k\lambda)^2(1 + k - 3\lambda + 3k\lambda), \\
\tau_{25} &= 0, \\
\tau_{30} &= \frac{1}{(1 + 3\lambda)^3}2a_{12}^3b_{12}^3(1 + k)\lambda^3(1 + k - \lambda + k\lambda)^2.
\end{aligned} \tag{4.16}$$

If $k = (1/3)(-3 + 4\lambda)$,

$$\begin{aligned}
\tau_{20} = -\frac{1}{2592}a_{12}^2b_{12}^2\Big(&-648a_{12}^2b_{12}^2m^4 + 1620a_{12}^2b_{12}^2m^4\mu - 1152a_{12}b_{12}m^2\lambda^2 \\
&- 1134a_{12}^2b_{12}^2m^4\lambda^2 + 4608a_{12}b_{12}m^2\lambda^3 + 243a_{12}^2b_{12}^2m^4\lambda^3 \\
&+1792r^4 - 4608a_{12}b_{12}m^2\lambda^4 - 8704\lambda^5 + 13312\lambda^6 - 6144\lambda^7\Big), \\
\tau_{25} = -\frac{1}{1944}a_{12}^3b_{12}^3m\lambda^2\Big(&-864a_{12}b_{12}m^2 - 459a_{12}b_{12}m^2\lambda + 576\lambda^2 + 6075a_{12}b_{12}m^2\lambda^2 \\
&- 256\lambda^3 - 5670a_{12}b_{12}m^2\lambda^3 - 5888\lambda^4 \\
&+1296a_{12}b_{12}m^2\lambda^4 + 11264\lambda^5 - 6144\lambda^6\Big),
\end{aligned}$$

$$\tau_{30} = -\frac{1}{492075\left(-32-17\lambda+225\lambda^2-210\lambda^3+48\lambda^4\right)^3}4a_{12}^3b_{12}^3\lambda^6(1+\lambda)(-1+2\lambda)^2$$
$$\times\Big(16223998464-25771071744\lambda-266392474620\lambda^2+779771224771\lambda^3+502091493918\lambda^4$$
$$-4814334558957\lambda^5+7235802457920\lambda^6-2358342484708\lambda^7-5235799647872\lambda^8$$
$$+7279836287888\lambda^9-3998287419904\lambda^{10}+800816397376\lambda^{11}+199732916736\lambda^{12}$$
$$-135353438208\lambda^{13}+20141015040\lambda^{14}\Big). \tag{4.17}$$

If $k=(\lambda-2)/(\lambda+2)$,

$$\tau_{20} = -\frac{1}{96(\lambda+2)^4}a_{12}^2b_{12}^2\Big(-384a_{12}^2b_{12}^2m^4+192a_{12}^2b_{12}^2m^4\lambda$$
$$-1536a_{12}b_{12}m^2\lambda^2+672a_{12}^2b_{12}^2m^4\lambda^2-1280a_{12}b_{12}m^2\lambda^3$$
$$+48a_{12}^2b_{12}^2m^4\lambda^3+1920\lambda^4+384a_{12}b_{12}m^2\lambda^4-264a_{12}^2b_{12}^2m^4\lambda^4$$
$$+576a_{12}b_{12}m^2\lambda^5-60a_{12}^2b_{12}^2m^4\lambda^5+128a_{12}b_{12}m^2\lambda^6$$
$$+30a_{12}^2b_{12}^2m^4\lambda^6+9a_{12}^2b_{12}^2m^4\lambda^7\Big),$$
$$\tau_{25} = -\frac{1}{12(2+\lambda)^4}a_{12}^3b_{12}^3m\lambda^2\Big(96a_{12}b_{12}m^2+76a_{12}b_{12}m^2\lambda-96\lambda^2-40a_{12}b_{12}m^2\lambda^2-32\lambda^3$$
$$-33a_{12}b_{12}m^2\lambda^3+5a_{12}b_{12}m^2\lambda^4+4a_{12}b_{12}m^2\lambda^5\Big),$$
$$\tau_{30} = \frac{2a_{12}^3b_{12}^3\lambda^7}{75(2+\lambda)^6(24-5\lambda-11\lambda^2+4\lambda^3)^3}\Big(-8200224-114773490\lambda+250179993\lambda^2$$
$$-320032585\lambda^3-178032915\lambda^4+94932055\lambda^5$$
$$+18512302\lambda^6-24125520\lambda^7+4769504\lambda^8\Big), \tag{4.18}$$

where $\tau_k=0$, $k\neq 5i$, $i\leq 6$, $i\in N$. In the above expression of $\tau_k$, we have already let $\tau_1=\cdots=\tau_{k-1}=0$, $k=2,3,\ldots,30$.

From expressions (4.15), (4.16), (4.17) and (4.18), we have the following.

**Theorem 4.5.** *The first 30 period constants at the origin of system* (3.6) *are zero if and only if one of the following conditions holds:*

$$\beta = a_{21} = b_{21} = a_{03} = b_{03} = 0, \quad a_{30} = -b_{12}, \quad b_{30} = -a_{12}, \tag{4.19}$$

$$\beta = a_{21} = b_{21} = a_{03} = b_{03} = 0, \quad a_{30} = \frac{-1+\lambda}{1+\lambda} b_{12}, \quad b_{30} = \frac{-1+\lambda}{1+\lambda} a_{12}. \tag{4.20}$$

**Theorem 4.6.** *Under integrability condition* (3.14)*, the origin of system* (3.6) *is a complex isochronous center if and only if one of the conditions in Theorem 4.5 holds.*

*Proof.* When condition (4.19) is satisfied, system (3.6) becomes

$$\begin{aligned} \frac{dz}{dT} &= \frac{1}{10} z\left(10 + 2a_{12}\lambda z^6 w^4 - 2b_{12}\lambda z^4 w^6\right), \\ \frac{dw}{dT} &= -\frac{1}{5} w\left(10 + 2b_{12}\lambda z^4 w^6 - 2a_{12}\lambda z^6 w^4\right), \end{aligned} \tag{4.21}$$

we have for system (4.21) that

$$\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{1}{z}\frac{dz}{dT} - \frac{1}{w}\frac{dw}{dT}\right) = 1. \tag{4.22}$$

When condition (4.20) is satisfied, system (3.6) becomes

$$\begin{aligned} \frac{dz}{dT} &= \frac{1}{5(1+\lambda)} z\left(5(1+\lambda) + 4a_{12}\lambda z^6 w^4 + 6b_{12}\lambda z^4 w^6\right), \\ \frac{dw}{dT} &= -\frac{1}{5(1+\lambda)} w\left(5(1+\lambda) + 4b_{12}\lambda z^4 w^6 + 6a_{12}\lambda z^6 w^4\right). \end{aligned} \tag{4.23}$$

There exists a transformation

$$u = \frac{z\left(1+\lambda+2b_{12}\lambda z^4 w^6\right)^{1/5}}{\left(1+\lambda+2a_{12}\lambda z^6 w^4\right)^{3/10}}, \qquad v = \frac{w\left(1+\lambda+2a_{12}\lambda z^6 w^4\right)^{1/5}}{\left(1+\lambda+2b_{12}\lambda z^4 w^6\right)^{3/10}}, \tag{4.24}$$

such that system (4.23) is reduced to a linear system. □

*Case 4.* Integrability condition (3.15) holds.

Substituting condition (3.15) into the recursive formulae in Appendix B, we obtain the first 10 period constants

$$\begin{aligned} \tau_5 &= 2r_{21}, \\ \tau_{10} &= -2a_{12}b_{12}. \end{aligned} \tag{4.25}$$

Because $\tau_{10} = a_{12}b_{12} \neq 0$, under integrability condition (3.15), the origin of system $(3.5)_{\delta=0}$ is not a complex isochronous center.

Synthesizing all the above cases, we get the main result of this paper.

**Theorem 4.7.** *The degenerate singular point (origin) of system system* $(1.7)_{\delta=0}$ *(*$(3.5)_{\delta=0}$*) is pseudo-linearizable (linearizable) if and only if one of conditions* (4.4), (4.5), (4.11), (4.19), (4.20) *holds.*

## Appendices

### A.

The recursive formulae to compute the singular point quantities at the origin of system (3.6):

$c[0,0] = 1$ ,

when ($k = j > 0$) or $k < 0$, or $j < 0$,

$c[k,j] = 0$ .

Else

$$
\begin{aligned}
c[k,j] = \frac{-1}{10(j-k)} & \big(10j\beta c[-10+k,-10+j] - 10k\beta c[-10+k,-10+j] + 5a_{03}jc[-7+k,-3+j] \\
& - 5a_{03}kc[-7+k,-3+j] - 2a_{03}\lambda c[-7+k,-3+j] - a_{03}j\lambda c[-7+k,-3+j] \\
& - a_{03}k\lambda c[-7+k,-3+j] + 5a_{12}jc[-6+k,-4+j] + 5b_{30}jc[-6+k,-4+j] \\
& - 5a_{12}kc[-6+k,-4+j] - 5b_{30}kc[-6+k,-4+j] - 2a_{12}\lambda c[-6+k,-4+j] \\
& + 2b_{30}\lambda c[-6+k,-4+j] - a_{12}j\lambda c[-6+k,-4+j] + b_{30}j\lambda c[-6+k,-4+j] \\
& - a_{12}k\lambda c[-6+k,-4+j] + b_{30}k\lambda c[-6+k,-4+j] + 5a_{21}jc[-5+k,-5+j] \\
& + 5b_{21}jc[-5+k,-5+j] - 5a_{21}kc[-5+k,-5+j] - 5b_{21}kc[-5+k,-5+j] \\
& - 2a_{21}\lambda c[-5+k,-5+j] + 2b_{21}\lambda c[-5+k,-5+j] - a_{21}j\lambda c[-5+k,-5+j] \\
& + b_{21}j\lambda c[-5+k,-5+j] - a_{21}k\lambda c[-5+k,-5+j] + b_{21}k\lambda c[-5+k,-5+j] \\
& + 5a_{30}jc[-4+k,-6+j] + 5b_{12}jc[-4+k,-6+j] - 5a_{30}kc[-4+k,-6+j] \\
& - 5b_{12}kc[-4+k,-6+j] - 2a_{30}\lambda c[-4+k,-6+j] + 2b_{12}\lambda c[-4+k,-6+j] \\
& - a_{30}j\lambda c[-4+k,-6+j] + b_{12}j\lambda c[-4+k,-6+j] - a_{30}k\lambda c[-4+k,-6+j] \\
& + b_{12}k\lambda c[-4+k,-6+j] + 5b_{03}jc[-3+k,-7+j] - 5b_{03}kc[-3+k,-7+j] \\
& + 2b_{03}\lambda c[-3+k,-7+j] + b_{03}j\lambda c[-3+k,-7+j] + b_{03}k\lambda c[-3+k,-7+j]\big), \\
\mu_k = \frac{1}{5}\lambda & \big(a_{03}c[-7+k,-3+k] + a_{12}c[-6+k,-4+k] - b_{30}c[-6+k,-4+k] \\
& + a_{21}c[-5+k,-5+k] - b_{21}c[-5+k,-5+k] + a_{30}c[-4+k,-6+k] \\
& - b_{12}c[-4+k,-6+k] - b_{03}c[-3+k,-7+k]\big).
\end{aligned}
\tag{A.1}
$$

# B.

The recursive formulae to compute the period constants of the origin of system (3.6):

$c'[1,0] = d'[1,0] = 1;\ c'[0,1] = d'[0,1] = 0,$

if $k < 0$ or $j < 0$ or ($j > 0$ and $k = j + 1$) then $c'[k,j] = 0$, $d'[k,j] = 0$.

Else

$$\begin{aligned}
c'[k,j] = \frac{1}{j+1-k}\Bigg( &(-(-10+j)\beta + (-10+k)\beta)c[-10+k,-10+j] \\
&+ \left(\frac{1}{10}a_{03}(-3+j)(-5+\lambda) + \frac{1}{10}a_{03}(-7+k)(5+\lambda)\right)c[-7+k,-3+j] \\
&+ \left(\frac{1}{10}(-6+k)(5a_{12}+5b_{30}+a_{12}\lambda-b_{30}\lambda) - \frac{1}{10}(-4+j)\right. \\
&\quad \left.\times(5a_{12}+5b_{30}-a_{12}\lambda+b_{30}\lambda)\right)c[-6+k,-4+j] \\
&+ \left(\frac{1}{10}(-5+k)(5a_{21}+5b_{21}+a_{21}\lambda-b_{21}\lambda) - \frac{1}{10}(-5+j)\right. \\
&\quad \left.\times(5a_{21}+5b_{21}-a_{21}\lambda+b_{21}\lambda)\right)c[-5+k,-5+j] \\
&+ \left(\frac{1}{10}(-4+k)(5a_{30}+5b_{12}+a_{30}\lambda-b_{12}\lambda) - \frac{1}{10}(-6+j)\right. \\
&\quad \left.\times(5a_{30}+5b_{12}-a_{30}\lambda+b_{12}\lambda)\right)c[-4+k,-6+j] \\
&+ \left(-\left(\frac{1}{10}\right)b_{03}(-3+k)(-5+\lambda) - \frac{1}{10}b_{03}(-7+j)(5+\lambda)\right) \\
&\times c[-3+k,-7+j]\Bigg),
\end{aligned}$$

$$\begin{aligned}
d'[k,j] = \frac{1}{j+1-k}\Bigg( &(-(-10+j)\beta + (-10+k)\beta)d[-10+k,-10+j] \\
&+ \left(\frac{1}{10}b_{03}(-3+j)(-5+\lambda) + \frac{1}{10}b_{03}(-7+k)(5+\lambda)\right)d[-7+k,-3+j] \\
&+ \left(-\left(\frac{1}{10}\right)(-4+j)(5a_{30}+5b_{12}+a_{30}\lambda-b_{12}\lambda) + \frac{1}{10}(-6+k)\right. \\
&\quad \left.\times(5a_{30}+5b_{12}-a_{30}\lambda+b_{12}\lambda)\right)d[-6+k,-4+j] \\
&+ \left(-\left(\frac{1}{10}\right)(-5+j)(5a_{21}+5b_{21}+a_{21}\lambda-b_{21}\lambda) + \frac{1}{10}(-5+k)\right. \\
&\quad \left.\times(5a_{21}+5b_{21}-a_{21}\lambda+b_{21}\lambda)\right)d[-5+k,-5+j]
\end{aligned}$$

$$+\left(-\left(\frac{1}{10}\right)(-6+j)(5a_{12}+5b_{30}+a_{12}\lambda-b_{30}\lambda)+\frac{1}{10}(-4+k)\right.$$

$$\left.\times(5a_{12}+5b_{30}-a_{12}\lambda+b_{30}\lambda)\right)d[-4+k,-6+j]$$

$$+\left(-\frac{1}{10}a_{03}(-3+k)(-5+\lambda)-\frac{1}{10}a_{03}(-7+j)(5+\lambda)\right)$$

$$\left.\times d[-3+k,-7+j]\right),$$

$$\tau[m]=(-(-10+j)\beta+(-9+j)\beta)c[-9+j,-10+j]$$

$$+\left(\frac{1}{10}a_{03}(-3+j)(-5+\lambda)+\frac{1}{10}a_{03}(-6+j)(5+\lambda)\right)c[-6+j,-3+j]$$

$$+\left(\frac{1}{10}(-5+j)(5a_{12}+5b_{30}+a_{12}\lambda-b_{30}\lambda)-\frac{1}{10}(-4+j)\right.$$

$$\left.\times(5a_{12}+5b_{30}-a_{12}\lambda+b_{30}\lambda)\right)c[-5+j,-4+j]$$

$$+\left(\frac{1}{10}(-4+j)(5a_{21}+5b_{21}+a_{21}\lambda-b_{21}\lambda)-\frac{1}{10}(-5+j)\right.$$

$$\left.\times(5a_{21}+5b_{21}-a_{21}\lambda+b_{21}\lambda)\right)c[-4+j,-5+j]$$

$$+\left(\frac{1}{10}(-3+j)(5a_{30}+5b_{12}+a_{30}\lambda-b_{12}\lambda)-\frac{1}{10}(-6+j)\right.$$

$$\left.\times(5a_{30}+5b_{12}-a_{30}\lambda+b_{12}\lambda)\right)c[-3+j,-6+j]$$

$$+\left(-\frac{1}{10}b_{03}(-2+j)(-5+\lambda)-\frac{1}{10}b_{03}(-7+j)(5+\lambda)\right)$$

$$\times c[-2+j,-7+j]+(-(-10+j)\beta+(-9+j)\beta)d[-9+j,-10+j]$$

$$+\left(\frac{1}{10}b_{03}(-3+j)(-5+\lambda)+\frac{1}{10}b_{03}(-6+j)(5+\lambda)\right)d[-6+j,-3+j]$$

$$+\left(-\frac{1}{10}(-4+j)(5a_{30}+5b_{12}+a_{30}\lambda-b_{12}\lambda)+\frac{1}{10}(-5+j)\right.$$

$$\left.\times(5a_{30}+5b_{12}-a_{30}\lambda+b_{12}\lambda)\right)d[-5+j,-4+j]$$

$$+\left(-\frac{1}{10}(-5+j)(5a_{21}+5b_{21}+a_{21}\lambda-b_{21}\lambda)+\frac{1}{10}(-4+j)\right.$$

$$\left.\times(5a_{21}+5b_{21}-a_{21}\lambda+b_{21}\lambda)\right)d[-4+j,-5+j]$$

$$+\left(-\frac{1}{10}(-6+j)(5a_{12}+5b_{30}+a_{12}\lambda-b_{30}\lambda)+\frac{1}{10}(-3+j)\right.$$

$$
\begin{aligned}
&\times(5a_{12}+5b_{30}-a_{12}\lambda+b_{30}\lambda)\Big)d\left[-3+j,-6+j\right]\\
&-\left(\frac{1}{10}a_{03}(-2+j)(-5+\lambda)+\frac{1}{10}a_{03}(-7+j)(5+\lambda)\right)\\
&\times d\left[-2+j,-7+j\right].
\end{aligned}
\tag{B.1}
$$

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