Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 765965, 13 pages
doi:10.1155/2012/765965

Research Article

Modified Poisson Integral and Green Potential on a Half-Space

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Received 16 February 2012; Revised 5 May 2012; Accepted 16 May 2012

Academic Editor: Stefan Siegmund

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We discuss the behavior at infinity of modified Poisson integral and Green potential on a half-space of the n-dimensional Euclidean space, which generalizes the growth properties of analytic functions, harmonic functions and superharmonic functions.

1. Introduction and Main Results

Let $\mathbf{R}^n\,(n \geq 2)$ denote the n-dimensional Euclidean space with points $x = (x_1, x_2, \ldots, x_{n-1}, x_n) = (x', x_n)$, where $x' \in \mathbf{R}^{n-1}$ and $x_n \in \mathbf{R}$. The boundary and closure of an open Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$, respectively. The upper half-space is the set $H = \{x = (x', x_n) \in \mathbf{R}^n : x_n > 0\}$, whose boundary is ∂H. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1} \times \mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1} \times \{0\}$, writing typical points $x, y \in \mathbf{R}^n$ as $x = (x', x_n)$, $y = (y', y_n)$, where $x' = (x_1, x_2, \ldots, x_{n-1})$, $y' = (y_1, y_2, \ldots, y_{n-1}) \in \mathbf{R}^{n-1}$ and putting $x \cdot y = \sum_{j=1}^{n} x_j y_j = x' \cdot y' + x_n y_n$, $|x| = \sqrt{x \cdot x}$, $|x'| = \sqrt{x' \cdot x'}$.

For $x \in \mathbf{R}^n$ and $r > 0$, let $B_n(x, r)$ denote the open ball with center at x and radius r in $\mathbf{R}^n$.

Set

$$E_\alpha(x) = \begin{cases} -\log|x| & \text{if } \alpha = n = 2, \\ |x|^{\alpha-n} & \text{if } 0 < \alpha < n. \end{cases} \tag{1.1}$$

Let $G_\alpha(x,y)$ be the green function of order α for H, that is,

$$G_\alpha(x,y) = E_\alpha(x-y) - E_\alpha(x-y^*) \quad x,\, y \in \overline{H},\ x \neq y,\ 0<\alpha\leq n, \tag{1.2}$$

where * denotes reflection in the boundary plane ∂H just as $y^* = (y_1, y_2, \ldots, y_{n-1}, -y_n)$.

We define the Poisson kernel $P_\alpha(x,y')$ when $x \in H$ and $y' \in \partial H$ by

$$P_\alpha(x,y') = \left.\frac{\partial G_\alpha(x,y)}{\partial y_n}\right|_{y_n=0} = C_\alpha \frac{x_n}{\left|x-y'\right|^{n-\alpha+2}}, \tag{1.3}$$

where $C_\alpha = 2(n-\alpha)$ if $0<\alpha<n$ and $=2$ if $\alpha = n = 2$. It has the expansion

$$P(x,y') = \sum_{k=0}^{\infty} \frac{C_\alpha x_n |x|^k}{\left|y'\right|^{n-\alpha+2+k}} C_k^{(n-\alpha+2)/2}\left(\frac{x\cdot y'}{|x||y'|}\right), \tag{1.4}$$

where $C_k^{(n-\alpha+2)/2}(t)$ is a Gegenbauer polynomial [1]. The series converges for $|y'|>|x|$. Each term in the series is a harmonic function of x and vanishes on ∂H.

In case $\alpha = n = 2$, we consider the modified kernel function defined by

$$E_{n,m}(x-y) = \begin{cases} E_n(x-y) & \text{if } |y|<1, \\ E_n(x-y) + \Re\left(\log y - \displaystyle\sum_{k=1}^{m-1}\left(\frac{x^k}{ky^k}\right)\right) & \text{if } |y|\geq 1. \end{cases} \tag{1.5}$$

In case $0<\alpha<n$, we define

$$E_{\alpha,m}(x-y) = \begin{cases} E_\alpha(x-y) & \text{if } |y|<1, \\ E_\alpha(x-y) - \displaystyle\sum_{k=0}^{m-1}\frac{|x|^k}{\left|y\right|^{n-\alpha+k}} C_k^{(n-\alpha)/2}\left(\frac{x\cdot y}{|x||y|}\right) & \text{if } |y|\geq 1, \end{cases} \tag{1.6}$$

where m is a nonnegative integer and $C_k^{\omega}(t)$ $(\omega = (n-\alpha)/2)$ is also the Gegenbauer polynomials. The expression arises from the generating function for Gegenbauer polynomials

$$\left(1-2tr+r^2\right)^{-\omega} = \sum_{k=0}^{\infty} C_k^{\omega}(t) r^k, \tag{1.7}$$

where $|r|<1$, $|t|\leq 1$ and $\omega>0$. Each coefficient $C_k^{\omega}(t)$ is called the Gegenbauer polynomial of degree k associated with ω, the function $C_k^{\omega}(t)$ is a polynomial of degree k in t.

Then we define the modified Poisson kernel $P_{\alpha,m}(x,y')$ and Green function $G_{\alpha,m}(x,y)$ respectively by

$$P_{\alpha,m}(x,y') = \begin{cases} P_\alpha(x,y') & \text{if } |y'| < 1, \\ P_\alpha(x,y') - \displaystyle\sum_{k=0}^{m-1} \frac{C_\alpha x_n |x|^k}{|y'|^{n-\alpha+2+k}} C_k^{(n-\alpha+2)/2}\left(\frac{x\cdot y'}{|x||y'|}\right) & \text{if } |y'| \geq 1; \end{cases} \tag{1.8}$$

$$G_{\alpha,m}(x,y) = \begin{cases} E_{n,m+1}(x-y) - E_{n,m+1}(x-y^*) & \text{if } \alpha = n = 2, \\ E_{\alpha,m+1}(x-y) - E_{\alpha,m+1}(x-y^*) & \text{if } 0 < \alpha < n, \end{cases} \tag{1.9}$$

where $x, y \in \overline{H}$ and $x \neq y$. We remark that the new kernel $P_{\alpha,m}(x,y')$ will be of order $|y'|^{-(n+m)}$ as $|y'| \to \infty$.

Write

$$\begin{aligned} U_{\alpha,m}(x,\nu) &= \int_{\partial H} P_{\alpha,m}(x,y')\,d\nu(y'), \\ G_{\alpha,m}(x,\mu) &= \int_H G_{\alpha,m}(x,y)\,d\mu(y), \end{aligned} \tag{1.10}$$

where ν (resp. μ) is a nonnegative measure on ∂H (resp. H). Here note that $U_{2,0}(x,\nu)$ (resp. $G_{2,0}(x,\mu)$) is nothing but the general Poisson integral (resp. Green potential).

Let k be a nonnegative Borel measurable function on $\mathbf{R}^n \times \mathbf{R}^n$, and set

$$k(y,\mu) = \int_E k(y,x)\,d\mu(x), \qquad k(\mu,x) = \int_E k(y,x)\,d\mu(y), \tag{1.11}$$

for a nonnegative measure μ on a Borel set $E \subset \mathbf{R}^n$. We define a capacity C_k by

$$C_k(E) = \sup \mu(\mathbf{R}^n), \quad E \subset H, \tag{1.12}$$

where the supremum is taken over all nonnegative measures μ such that S_μ (the support of μ) is contained in E and $k(y,\mu) \leq 1$ for every $y \in H$.

For $\beta \leq 1$ and $\delta \leq 1$, we consider the function $k_{\alpha,\beta,\delta}$ defined by

$$k_{\alpha,\beta,\delta}(y,x) = x_n^{-\beta} y_n^{-\delta} G_\alpha(x,y) \quad \text{for } x, y \in H. \tag{1.13}$$

If $\beta = \delta = 1$, then $k_\alpha = k_{\alpha,1,1}$ is extended to be continuous on $\overline{H} \times \overline{H}$ in the extended sense, where $\overline{H} = H \cup \partial H$.

Recently, Siegel-Talvila [2] proved the following result.

Theorem A. *Let f be a measurable function on $\mathbf{R}^{n-1}$ satisfying $\int_{\mathbf{R}^{n-1}}(|f(y')|/(1+|y'|)^{(n+m)})\, dy' < \infty$. Then the function $v(x) = \int_{\mathbf{R}^{n-1}} P_{2,m}(x,y')f(y')dy'$ satisfies*

$$\begin{gathered}
v \in C^2(H) \cap C^0\left(\overline{H}\right), \\
\Delta v = 0, \quad x \in H, \\
\lim_{x \to x'} v(x) = f(x') \quad \text{nontangentially } a \cdot e\ x' \in \partial H, \\
v(x) = o\left(x_n^{1-n}|x|^{m+n}\right) \quad \text{as } |x| \longrightarrow \infty,\ x \in H.
\end{gathered} \tag{1.14}$$

Our first aim is to establish the following theorem.

Theorem 1.1. *Let $\gamma > 0$ and $\alpha + \gamma - n - 2 \le m < \alpha + \gamma - n - 1$. If ν is a nonnegative measure on ∂H satisfying*

$$\int_{\partial H} \frac{1}{(1+|y'|)^{\gamma}} d\nu(y') < \infty, \tag{1.15}$$

then there exists a Borel set $E \subset H$ with properties:

(1) $\lim_{|x| \to \infty, x \in H-E} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+1} U_{\alpha,m}(x,\nu) = 0$;

(2) $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta,1}}(E_i) < \infty$,

where $E_i = \{x \in E : 2^i \le |x| < 2^{i+1}\}$.

Theorem 1.2. *Let $\gamma > \delta - 1$ and $\alpha + \gamma - n - \delta - 1 \le m < \alpha + \gamma - n - \delta$. If μ is a nonnegative measure on H satisfying*

$$\int_{H} \frac{y_n^{\delta}}{(1+|y|)^{\gamma}} d\mu(y) < \infty, \tag{1.16}$$

then there exists a Borel set $F \subset H$ with properties:

(1) $\lim_{|x| \to \infty, x \in H-F} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+\delta} G_{\alpha,m}(x,\mu) = 0$;

(2) $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+\delta)} C_{k_{\alpha,\beta,\delta}}(F_i) < \infty$,

where $F_i = \{x \in F : 2^i \le |x| < 2^{i+1}\}$.

Remark 1.3. If $\delta = 1$, then $F = E$.

Next we generalize Theorem A to the modified α-potentials on H, which is defined by

$$R_{\alpha}(x) = U_{\alpha,m}(x,\nu) + G_{\alpha,m}(x,\mu), \tag{1.17}$$

where $0 < \alpha \le n$ and ν (resp. μ) is a nonnegative measure on ∂H (resp. H) satisfying (1.15) (resp. (1.16) ($\delta = 1$)). Clearly, $R_2(x)$ is a superharmonic function.

The following theorem follows readily from Theorems 1.1 and 1.2.

Theorem 1.4. *Let $R_\alpha(x)$ be defined by* (1.17)*. Then there exists a Borel set $E \subset H$ satisfying Theorem 1.1*(2) *such that*

$$\lim_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+1}R_\alpha(x) = 0. \tag{1.18}$$

Remark 1.5. In the case $0 \le \beta \le 1$, by using Lemma 2.5 below, we can easily show that $C_{k_{2,\beta,1}} = \lambda^\beta$ in the notation of [3]. Thus, Theorem 1.1(2) with $\alpha = 2$ means that E is β-rarefied at infinity in the sense of [3]. In particular, this condition with $\alpha = 2, \beta = 1$(resp. $\alpha = 2, \beta = 0$) means that E is minimally thin at infinity (resp. rarefied at infinity) in the sense of [4].

Finally we are concerned with the best possibility of Theorem 1.4 as to the size of the exceptional set.

Theorem 1.6. *Let $\gamma > 0$, $E \subset H$ be a Borel set satisfying Theorem 1.1*(2) *and let $R_\alpha(x)$ be defined by* (1.17)*. Then one can find a nonnegative measure λ defined on $\overline{H}$ satisfying*

$$\int_{\overline{H}} \frac{1}{(1+|y|)^\gamma} d\lambda(y) < \infty, \tag{1.19}$$

such that

$$\limsup_{|x|\to\infty, x\in E} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+1}R_\alpha(x) = \infty, \tag{1.20}$$

where $d\lambda(y') = d\nu(y')(y' \in \partial H)$ and $d\lambda(y) = y_n d\mu(y)(y \in H)$.

2. Some Lemmas

Throughout this paper, let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 2.1 (see [5]). *Let m be a nonnegative integer and $M > 0$.*

(1) *If $1 \le |y'| \le |x|/2$, then $|P_{\alpha,m}(x, y')| \le M((x_n|x|^{m-1})/|y'|^{n+m-\alpha+1})$.*

(2) *If $|y'| \ge 2|x|$ and $|y'| \ge 1$, then $|P_{\alpha,m}(x, y')| \le M((x_n\ |x|^m)/|y'|^{n+m-\alpha+2})$.*

Lemma 2.2. *There exists a positive constant M such that $G_\alpha(x, y) \le M((x_n\ y_n)/|x - y|^{n-\alpha+2})$, where $0 < \alpha \le n$, $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in H.*

This can be proved by simple calculation.

Lemma 2.3. *Gegenbauer polynomials have the following properties:*

(1) $|C_k^\omega(t)| \le C_k^\omega(1) = \Gamma(2\omega + k)/((\Gamma(2\omega)\Gamma(k + 1)),\ |t| \le 1;$

(2) $(d/dt)C_k^\omega(t) = 2\omega C_{k-1}^{\omega+1}(t),\ k \ge 1;$

(3) $\sum_{k=0}^{\infty} C_k^{\omega}(1) r^k = (1-r)^{-2\omega}$;

(4) $|C_k^{(n-\alpha)/2}(t) - C_k^{(n-\alpha)/2}(t^*)| \leq (n-\alpha) C_{k-1}^{(n-\alpha+2)/2}(1)|t-t^*|, \ |t| \leq 1, \ \ |t^*| \leq 1.$

Proof. (1) and (2) can be derived from [1]. (3) follows by taking $t = 1$ in (1.7); (4) follows by (1), (2) and the Mean Value Theorem for Derivatives. □

Lemma 2.4. *For $x, y \in \mathbf{R}^n (\alpha = n = 2)$, one has the following properties:*

(1) $|\Im \sum_{k=0}^{m} (x^k / y^{k+1}) \ | \leq \sum_{k=0}^{m-1} ((2^k \ x_n \ |x|^k \ )/|y|^{k+2})$;

(2) $|\Im \sum_{k=0}^{\infty} (x^{k+m+1}/y^k) \ | \leq 2^{m+1} x_n |x|^m$;

(3) $|G_{n,m}(x,y) - G_n(x,y)| \leq M \sum_{k=1}^{m} ((k x_n y_n |x|^{k-1})/|y|^{(k+1)})$;

(4) $|G_{n,m}(x,y)| \leq M \sum_{k=m+1}^{\infty} ((k x_n y_n x^{k-1})/|y|^{(k+1)})$.

The following lemma can be proved by using Fuglede (see [6], Théorèm 7.8).

Lemma 2.5. *For any Borel set E in H, one has $C_{k_{\alpha,\beta,1}}(E) = \widehat{C}_{k_{\alpha,\beta,1}}(E)$ and*

$$\widehat{C}_{k_{\alpha,\beta,\delta}}(E) = \inf \lambda(H) \ \left(\textit{resp.} \ \inf \lambda\left(\overline{H}\right)\right) \quad \textit{if } \delta < 1 \ (\textit{resp. } \delta = 1), \tag{2.1}$$

where the infimum is taken over all nonnegative measures λ on H (resp. $\overline{H}$) such that $k_{\alpha,\beta,\delta}(\lambda, x) \geq 1$ for every $x \in E$.

3. Proof of Theorems

Proof of Theorem 1.1. For any $\epsilon_1 > 0$, there exists $R_{\epsilon_1} > 2$ such that

$$\int_{\{y' \in \partial H, |y'| \geq R_{\epsilon_1}\}} \frac{1}{(1+|y'|)^{\gamma}} d\nu(y') < \epsilon_1. \tag{3.1}$$

For fixed $x \in H$ and $|x| \geq 2R_{\epsilon_1}$, we write

$$\begin{aligned} U_{\alpha,m}(x,\nu) &= \int_{G_1} P_{\alpha,m}(x,y') d\nu(y') + \int_{G_2} P_{\alpha,m}(x,y') d\nu(y') \\ &\quad + \int_{G_3} [P_{\alpha,m}(x,y') - P_{\alpha}(x,y')] d\nu(y') + \int_{G_4} P_{\alpha}(x,y') d\nu(y') \\ &\quad + \int_{G_5} P_{\alpha,m}(x,y') d\nu(y') \\ &= U_1(x) + U_2(x) + U_3(x) + U_4(x) + U_5(x), \end{aligned} \tag{3.2}$$

where

$$G_1 = \{y' \in \partial H : |y'| < 1\}, \qquad G_2 = \left\{y' \in \partial H : 1 \le |y'| < \frac{|x|}{2}\right\},$$
$$G_3 = G_4 = \left\{y' \in \partial H : \frac{|x|}{2} \le |y'| < 2|x|\right\}, \qquad G_5 = \{y' \in \partial H : |y'| \ge 2|x|\}. \tag{3.3}$$

First note that

$$|U_1(x)| \le x_n \left(\frac{|x|}{2}\right)^{\alpha-n-2} \int_{G_1} d\nu(y') \le M x_n |x|^{\alpha-n-2}. \tag{3.4}$$

By Lemma 2.1(1), we have

$$\begin{aligned} |U_2(x)| &\le M x_n |x|^{m-1} \int_{G_2} \frac{1}{|y'|^{n+m-\alpha+1}} d\nu(y') \\ &\le M x_n |x|^{m-1} \int_{G_2} |y'|^{\alpha+\gamma-n-m-1} \frac{1}{|y'|^{\gamma}} d\nu(y') \\ &\le M x_n |x|^{\alpha+\gamma-n-2} \int_{G_2} \frac{1}{|y'|^{\gamma}} d\nu(y'). \end{aligned} \tag{3.5}$$

Write

$$U_2(x) = U_{21}(x) + U_{22}(x), \tag{3.6}$$

where

$$\begin{aligned} U_{21}(x) &= \int_{G_2 \cap B_{n-1}(0, R_{\epsilon_1})} P_{\alpha,m}(x, y') d\nu(y'), \\ U_{22}(x) &= \int_{G_2 - B_{n-1}(0, R_{\epsilon_1})} P_{\alpha,m}(x, y') d\nu(y'). \end{aligned} \tag{3.7}$$

If $|x| > 2R_{\epsilon_1}$, then we have

$$\begin{aligned} |U_{21}(x)| &\le M R_{\epsilon_1}^{\alpha+\gamma-n-m-1} x_n |x|^{m-1}, \\ |U_{22}(x)| &\le M \epsilon_1 x_n |x|^{\alpha+\gamma-n-2} \end{aligned} \tag{3.8}$$

from (3.5).

So

$$|U_2(x)| \le M \epsilon_1 x_n^{\beta} |x|^{\alpha+\gamma-n-\beta-1}. \tag{3.9}$$

We have by Lemma 2.3(3)

$$
\begin{aligned}
|U_3(x)| &\leq M \int_{G_3} \sum_{k=0}^{m-1} \frac{x_n |x|^k}{|y'|^{n-\alpha+2+k}} C_k^{(n-\alpha+2)/2}(1) d\nu(y') \\
&\leq M x_n |x|^m \sum_{k=0}^{m-1} \frac{1}{2^k} C_k^{(n-\alpha+2)/2}(1) \int_{G_3} |y'|^{\alpha+\gamma-n-m-2} \frac{1}{|y'|^{\gamma}} d\nu(y') \\
&\leq M \varepsilon_1 x_n |x|^{\alpha+\gamma-n-2}.
\end{aligned}
\tag{3.10}
$$

By Lemma 2.1(2), we obtain

$$
|U_5(x)| \leq M x_n |x|^m \int_{G_5} |y'|^{\alpha+\gamma-n-m-2} \frac{1}{|y'|^{\gamma}} d\nu(y') \leq M \epsilon_1 x_n |x|^{\alpha+\gamma-n-2}. \tag{3.11}
$$

Note that $U_4(x) = x_n^{\beta} \int_{G_4} k_{\alpha,\beta,1}(y', x) d\nu(y')$. In view of (1.15), we can find a sequence $\{a_i\}$ of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ and $\sum_{i=1}^{\infty} a_i b_i < \infty$, where

$$
b_i = \int_{\{y' \in \partial H : 2^{i-1} < |y'| < 2^{i+2}\}} \frac{1}{|y'|^{\gamma}} d\nu(y'). \tag{3.12}
$$

Consider the sets

$$
E_i = \left\{ x \in H : 2^i \leq |x| < 2^{i+1},\ x_n^{-\beta} U_4(x) \geq a_i^{-1} 2^{-i(n-\alpha+\beta-\gamma+1)} \right\}, \tag{3.13}
$$

for $i = 1, 2, \ldots$. If μ is a nonnegative measure on H such that $S_\mu \subset E_i$ and $k_{\alpha,\beta,1}(y', \mu) \leq 1$ for $y' \in \partial H$, then we have

$$
\begin{aligned}
\int_H d\mu &\leq a_i 2^{i(n-\alpha+\beta-\gamma+1)} \int x_n^{-\beta} U_4(x) d\mu(X) \\
&\leq M a_i 2^{i(n-\alpha+\beta-\gamma+1)} \int_{\{y' \in \partial H : 2^{i-1} < |y'| < 2^{i+2}\}} k_{\alpha,\beta,1}(y', \mu) d\nu(y') \\
&\leq M a_i 2^{i(n-\alpha+\beta-\gamma+1)} \int_{\{y' \in \partial H : 2^{i-1} < |y'| < 2^{i+2}\}} d\nu(y') \\
&\leq M 2^{i(n-\alpha+\beta+1)} a_i b_i
\end{aligned}
\tag{3.14}
$$

so that

$$
C_{k_{\alpha,\beta,1}}(E_i) \leq M 2^{i(n-\alpha+\beta+1)} a_i b_i \tag{3.15}
$$

which yields

$$\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+1)} C_{k_{\alpha,\beta,1}}(E_i) < \infty. \tag{3.16}$$

Setting $E = \bigcup_{i=1}^{\infty} E_i$, we see that Theorem 1.1(2) is satisfied and

$$\limsup_{|x|\to\infty, x\in H-E} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+1} U_4(x) \leq \limsup_{i\to\infty} 2^{|n-\alpha+\beta-\gamma+1|} a_i^{-1} = 0. \tag{3.17}$$

Combining (3.4) and (3.9)–(3.17), Theorem 1.1(1) holds.

Then we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. For any $\epsilon_2 > 0$, there exists $R_{\epsilon_2} > 2$ such that

$$\int_{\{y\in H, |y|\geq R_{\epsilon_2}\}} \frac{y_n^{\delta}}{(1+|y|)^{\gamma}} d\mu(y) < \epsilon_2. \tag{3.18}$$

For fixed $x \in H$ and $|x| \geq 2R_{\epsilon_2}$, we write

$$\begin{aligned} G_{\alpha,m}(x,\mu) = &\int_{H_1} G_{\alpha}(x,y)d\mu(y) + \int_{H_2} G_{\alpha}(x,y)d\mu(y) \\ &+ \int_{H_3} [G_{\alpha,m}(x,y) - G_{\alpha}(x,y)]d\mu(y) + \int_{H_4} G_{\alpha,m}(x,y)d\mu(y) \\ &+ \int_{H_5} G_{\alpha}(x,y)d\mu(y) + \int_{H_6} [G_{\alpha,m}(x,y) - G_{\alpha}(x,y)]d\mu(y) \\ &+ \int_{H_7} G_{\alpha,m}(x,y)d\mu(y) \\ = &V_1(x) + V_2(x) + V_3(x) + V_4(x) + V_5(x) + V_6(x) + V_7(x), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} H_1 &= \left\{y \in H : |y| \geq R_{\epsilon_2},\ |x-y| \leq \frac{|x|}{2}\right\}, \\ H_2 &= \left\{y \in H : |y| \geq R_{\epsilon_2},\ \frac{|x|}{2} < |x-y| \leq 3|x|\right\}, \\ H_3 &= \{y \in H : |y| \geq R_{\epsilon_2},\ |x-y| \leq 3|x|\}, \\ H_4 &= \{y \in H : |y| \geq R_{\epsilon_2},\ |x-y| > 3|x|\}, \\ H_5 &= H_6\{y \in H : 1 \leq |y| < R_{\epsilon_2}\}, \\ H_7 &= \{y \in H : |y| < 1\}. \end{aligned} \tag{3.20}$$

We distinguish the following two cases.

Case 1. $0 < \alpha < n$.

Note that

$$V_1(x) = x_n^{\beta} \int_{H_1} k_{\alpha,\beta,\delta}(y,x)d\lambda(y), \tag{3.21}$$

where $d\lambda(y) = y_n^{\delta} d\mu(y)$.

By the lower semicontinuity of $k_{\alpha,\beta,\delta}(y,x)$, then we can prove the following fact in the same way as $U_4(x)$ in the proof of Theorem 1.1:

$$\lim_{|x|\to\infty, x\in H-F} x_n^{-\beta}|x|^{n-\alpha+\beta-\gamma+\delta}V_1(x) = 0, \tag{3.22}$$

where $F = \bigcup_{i=1}^{\infty} F_i$, $F_i = \{x \in F : 2^i \le |x| < 2^{i+1}\}$ and $\sum_{i=1}^{\infty} 2^{-i(n-\alpha+\beta+\delta)} C_{k_{\alpha,\beta,\delta}}(F_i) < \infty$.

Moreover, by Lemma 2.2

$$\begin{aligned} |V_2(x)| &\le M x_n \int_{H_2} \frac{y_n}{|x-y|^{n-\alpha+2}} d\mu(y) \\ &\le M x_n |x|^{\alpha-n-2} \int_{H_2} |y|^{\gamma-\delta+1} \frac{y_n^{\delta}}{|y|^{\gamma}} d\mu(y) \\ &\le M\epsilon_2 x_n |x|^{\alpha+\gamma-n-\delta-1}. \end{aligned} \tag{3.23}$$

Note that $C_0^{\omega}(t) \equiv 1$. By (3) and (4) in Lemma 2.3, we take $t = x \cdot y/|x||y|$, $t^* = x \cdot y^*/|x||y^*|$ in Lemma 2.3(4) and obtain

$$\begin{aligned} V_3(x)| &\le \int_{H_3} \sum_{k=1}^{m} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{(n-\alpha+2)/2}(1) \frac{x_n y_n}{|x||y|} d\mu(y) \\ &\le M x_n |x|^{m-1} \sum_{k=1}^{m} \frac{1}{4^{k-1}} C_{k-1}^{(n-\alpha+2)/2}(1) \int_{H_3} |y|^{\alpha+\gamma-n-m-\delta} \frac{y_n^{\delta}}{|y|^{\gamma}} d\mu(y) \\ &\le M\varepsilon_2 x_n |x|^{\alpha+\gamma-n-\delta-1}. \end{aligned} \tag{3.24}$$

Similarly, we have by (3) and (4) in Lemma 2.3

$$\begin{aligned} |V_4(x)| &\le \int_{H_4} \sum_{k=m+1}^{\infty} \frac{|x|^k}{|y|^{n-\alpha+k}} 2(n-\alpha) C_{k-1}^{(n-\alpha+2)/2}(1) \frac{x_n y_n}{|x||y|} d\mu(y) \\ &\le M x_n |x|^{m} \sum_{k=m+1}^{\infty} \frac{1}{2^{k-1}} C_{k-1}^{(n-\alpha+2)/2}(1) \int_{H_4} |y|^{\alpha+\gamma-n-m-\delta-1} \frac{y_n^{\delta}}{|y|^{\gamma}} d\mu(y) \\ &\le M\varepsilon_2 x_n |x|^{\alpha+\gamma-n-\delta-1}. \end{aligned} \tag{3.25}$$

By Lemma 2.2, we get

$$\begin{aligned} |V_5(x)| &\le Mx_n|x|^{m-1}\int_{H_5}\frac{y_n}{|x-y|^{n-\alpha+2}}d\mu(y) \\ &\le Mx_n|x|^{\alpha-n-2}\int_{H_2}|y|^{\gamma-\delta+1}\frac{y_n^{\delta}}{|y|^{\gamma}}d\mu(y) \\ &\le MR_{\epsilon_2}^{\gamma-\delta+1}x_n|x|^{\alpha-n-2}, \\ |V_7(x)| &\le Mx_n|x|^{\alpha-n-2}. \end{aligned} \tag{3.26}$$

Similar to the estimate of $V_3(x)$, we obtain

$$\begin{aligned} |V_6(x)| &\le \int_{H_6}\sum_{k=1}^{m}\frac{|x|^k}{|y|^{n-\alpha+k}}2(n-\alpha)C_{k-1}^{(n-\alpha+2)/2}(1)\frac{x_n y_n}{|x||y|}d\mu(y) \\ &\le Mx_n\sum_{k=1}^{m}C_{k-1}^{(n-\alpha+2)/2}(1)x^{k-1}R_{\varepsilon}^{m-k}\int_{H_6}|y|^{\alpha+\gamma-n-m-\delta}\frac{y_n^{\delta}}{|y|^{\gamma}}d\mu(y) \\ &\le MR_{\epsilon_2}^{\alpha+\gamma-n-\delta-1}x_n|x|^{m-1}. \end{aligned} \tag{3.27}$$

Combining (3.22)–(3.27), we see that Theorem 1.2(1) holds. Then we prove Case 1.

Case 2. $\alpha = n = 2$.

Since the estimates of $V_1(x)$, $V_2(x)$, $V_5(x)$ and $V_7(x)$ are similar to those of Case 1, we omit them. (3.22), (3.23), and (3.26) still hold in Case 2.

Moreover, by Lemma 2.4(3), we find

$$\begin{aligned} |V_3(x)| &\le M\int_{H_3}\sum_{k=1}^{m}\frac{kx_n y_n|x|^{k-1}}{|y|^{k+1}}d\mu(y) \\ &\le Mx_n|x|^{m-1}\sum_{k=1}^{m}\frac{k}{4^{k-1}}\int_{H_3}|y|^{\gamma-m-\delta}\frac{y_n^{\delta}}{|y|^{\gamma}}d\mu(y) \\ &\le M\varepsilon_2 x_n|x|^{\gamma-\delta-1}. \end{aligned} \tag{3.28}$$

By Lemma 2.4(4), we have

$$\begin{aligned} |V_4(x)| &\le M\int_{H_4}\sum_{k=m+1}^{\infty}\frac{kx_n y_n|x|^{k-1}}{|y|^{k+1}}d\mu(y) \\ &\le Mx_n|x|^{m}\sum_{k=m+1}^{\infty}\frac{k}{2^{k-1}}\int_{H_4}|y|^{\gamma-m-\delta-1}\frac{y_n^{\delta}}{|y|^{\gamma}}d\mu(y) \\ &\le M\varepsilon_2 x_n|x|^{\gamma-\delta-1}. \end{aligned} \tag{3.29}$$

Similar to the estimate of $V_3(x)$, we have

$$|V_6(x)| \leq MR_{\epsilon_2}^{\gamma-\delta-1} x_n |x|^{m-1}. \tag{3.30}$$

Combining (3.22), (3.23), (3.26) and (3.28)–(3.30), we see that Theorem 1.2(1) holds. Then we prove Case 2.

Hence we complete the proof of Theorem 1.2. □

Proof of Theorem 1.6. We prove the case $0 < \alpha < n$, because the case $\alpha = n = 2$ can be proved similarly. Further, we only need prove

$$\limsup_{|x|\to\infty, x\in E} x_n^{1-\beta} |x|^{n-\alpha+\beta-\gamma+1} k_\alpha(\lambda, x) = \infty. \tag{3.31}$$

By Lemma 2.5, for each i we can find λ_i on $\overline{H}$ such that $\lambda_i(\overline{H}) < C_{k_{\alpha,\beta,1}}(E_i)+1$ and $k_{\alpha,\beta,1}(\lambda_i, x) \geq 1$ on E_i. Denote by λ_i' the restriction of λ_i to the set $\{y \in \overline{H} : 2^{i-1} < |y| < 2^{i+2}\}$.

Set $\lambda = \sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta-\gamma+1)} \lambda_i'$, where $\{a_i\}$ is a sequence of positive numbers such that $\lim_{i\to\infty} a_i = \infty$ but $\sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta+1)} \{C_{k_{\alpha,\beta,1}}(E_i) + 1\} < \infty$. Then

$$\begin{aligned} \int_{\overline{H}} \frac{1}{(1+|y|)^\gamma} d\lambda(y) &= \sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta-\gamma+1)} \int_{\overline{H}} \frac{1}{(1+|y|)^\gamma} d\lambda_i'(y) \\ &\leq M \sum_{i=1}^{\infty} a_i 2^{-i(n-\alpha+\beta-\gamma+1)} \left\{ C_{k_{\alpha,\beta,1}}(E_i) + 1 \right\} < \infty. \end{aligned} \tag{3.32}$$

If $x \in E_i$, then

$$\begin{aligned} k_{\alpha,\beta,1}(\lambda_i', x) &\geq 1 - \int_{\{y\in\overline{H}:|y|\leq 2^{i-1}\}\cap\{y\in\overline{H}:|y|\geq 2^{i+2}\}} k_{\alpha,\beta,1}(y,x) d\lambda_i(y) \\ &\geq 1 - M2^{-i(n-\alpha+\beta+1)} \left\{ C_{k_{\alpha,\beta,1}}(E_i) + 1 \right\}. \end{aligned} \tag{3.33}$$

We also have

$$x_n^{1-\beta} |x|^{n-\alpha+\beta-\gamma+1} k_\alpha(\lambda, x) = |x|^{n-\alpha+\beta-\gamma+1} k_{\alpha,\beta,1}(\lambda, x) \geq a_i k_{\alpha,\beta,1}(\lambda_i', x), \tag{3.34}$$

which implies that

$$\limsup_{|x|\to\infty, x\in E} x_n^{1-\beta} |x|^{n-\alpha+\beta-\gamma+1} k_\alpha(\lambda, x) = \infty. \tag{3.35}$$

Thus λ satisfies all the conditions in the Theorem 1.6. □

Acknowledgments

The authors are extremely grateful to the referee for useful suggestions that improved the contents of the paper. This work is supported by the National Natural Science Foundation of China under Grant 11071020 and Specialized Research Fund for the Doctoral Program of Higher Education under Grant 20100003110004.

References

[1] G. Szegő, *Orthogonal Polynomials*, vol. 23, American Mathematical Society Colloquium Publications, Providence, RI, USA, 4th edition, 1975.
[2] D. Siegel and E. Talvila, "Sharp growth estimates for modified Poisson integrals in a half space," *Potential Analysis*, vol. 15, no. 4, pp. 333–360, 2001.
[3] H. Aikawa, "On the behavior at infinity of nonnegative superharmonic functions in a half space," *Hiroshima Mathematical Journal*, vol. 11, no. 2, pp. 425–441, 1981.
[4] M. Essén and H. L. Jackson, "On the covering properties of certain exceptional sets in a half-space," *Hiroshima Mathematical Journal*, vol. 10, no. 2, pp. 233–262, 1980.
[5] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, vol. 1, Academic Press, London, UK, 1976.
[6] B. Fuglede, "Le théorème du minimax et la théorie fine du potentiel," *Annales de l'Institut Fourier*, vol. 15, pp. 65–88, 1965.