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*Research Article*

# Periodic Solutions for Duffing Type $p$-Laplacian Equation with Multiple Constant Delays

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Using inequality techniques and coincidence degree theory, new results are provided concerning the existence and uniqueness of periodic solutions for the Duffing type $p$-Laplacian equation with multiple constant delays of the form $(\varphi_p(x'(t)))' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) = e(t)$. Moreover, an example is provided to illustrate the effectiveness of the results in this paper.

## 1. Introduction

Referring to the work of Esmailzadeh and Nakhaie-Jazar [1], Duffing type equation is the simplest case of a vibrating system with nonlinear restoring force generator element. This is equivalent to a mechanical vibrating system with either a hard or soft spring. Thus, this equation and its modifications have been extensively and intensively studied. In particular, the existence of periodic solutions for Duffing type equations with and without delays have been discussed by various researchers (see, e.g., [2–8] and the references given therein). However, to the best of our knowledge, the existence and uniqueness of periodic solutions of Duffing type $p$-Laplacian equation whose delays more than two have not been sufficiently researched. Motivated by this, we shall consider the Duffing type $p$-Laplacian equations with multiple constant delays of the form

$$\left(\varphi_p\left(x'(t)\right)\right)' + Cx'(t) + g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) = e(t), \tag{1.1}$$

where $p > 1$ and $\varphi_p : R \to R$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi_p(0) = 0$, $C$ and $\tau_k$ are constants, $e : R \to R$ and $g_0$, $g_k : R \times R \to R$ are continuous functions, $\tau_k$ and $e$ are $T$-periodic, $g_0$ and $g_k$ are $T$-periodic in the first argument, $T > 0$ and $k = 1, 2, \ldots, n$. The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of $T$-periodic solutions of (1.1). The results of this paper are new and complement previously known results. Moreover, we give an example to illustrate the results.

## 2. Preliminary Results

Throughout this paper, we will denote

$$\begin{gathered} C_T^1 := \left\{ x \in C^1(R) : x \text{ is } T\text{-periodic} \right\}, \\ |x|_k = \left( \int_0^T |x(t)|^k dt \right)^{1/k} \quad (k > 0), \qquad |x|_\infty = \max_{t \in [0,T]} |x(t)|. \end{gathered} \tag{2.1}$$

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = \tilde{f}(t, x, x'), \qquad x(0) = x(T), \qquad x'(0) = x'(T), \tag{2.2}$$

where $\tilde{f} \in C(R^3, R)$ is $T$-periodic in the first variable, we have the following lemma.

**Lemma 2.1** (see [9]). *Let $\Omega$ be an open bounded set in $C_T^1$, if the following conditions hold.*

(i) *For each $\lambda \in (0, 1)$ the problem*

$$(\varphi_p(x'(t)))' = \lambda \tilde{f}(t, x, x'), \qquad x(0) = x(T), \qquad x'(0) = x'(T), \tag{2.3}$$

*has no solution on $\partial\Omega$.*

(ii) *The equation*

$$F(a) := \frac{1}{T} \int_0^T \tilde{f}(t, a, 0) dt = 0 \tag{2.4}$$

*has no solution on $\partial\Omega \bigcap R$.*

(iii) *The Brouwer degree of $F$*

$$\deg\left(F, \Omega \bigcap R, 0\right) \neq 0. \tag{2.5}$$

*Then, the periodic boundary value problem* (2.2) *has at least one $T$-periodic solution on $\overline{\Omega}$.*

We can easily obtain the homotopic equation of (1.1) as follows:

$$(\varphi_p(x'(t)))' + \lambda C x'(t) + \lambda \left[ g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k)) \right] = \lambda e(t), \quad \lambda \in (0, 1). \tag{2.6}$$

**Lemma 2.2.** *Assume that the following conditions are satisfied.*
*$(A_1)$ There exists a constant $d > 0$ such that*

(1) $\sum_{k=0}^{n} g_k(t, x_k) - e(t) < 0$ for $x_k > d,\ t \in R,\ k = 0, 1, 2, \dots, n,$

(2) $\sum_{k=0}^{n} g_k(t, x_k)\ - e(t) > 0$ for $x_k < -d,\ t \in R,\ k = 0, 1, 2, \dots, n.$

*Moveover, if $x(t)$ is a $T$-periodic solution of* (2.6)*, then*

$$|x|_\infty \le d + \frac{1}{2}\sqrt{T}\left|x'\right|_2. \tag{2.7}$$

*Proof.* Let $x(t)$ be a $T$-periodic solution of (2.6). Then, integrating (2.6) over $[0, T]$, we have

$$\int_0^T \left[ g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t - \tau_k)) - e(t) \right] dt = 0. \tag{2.8}$$

Using the integral mean-value theorem, it follows that there exists $t_1 \in [0, T]$ such that

$$g_0(t_1, x(t_1)) + \sum_{k=1}^{n} g_k(t_1, x(t_1 - \tau_k)) - e(t_1) = 0. \tag{2.9}$$

We now prove that there exists a constant $t_2 \in R$ such that

$$|x(t_2)| \le d. \tag{2.10}$$

Indeed, suppose otherwise. Then,

$$|x(t)| > d \quad \forall t \in R. \tag{2.11}$$

Let $\tau_0 = 0$. From $(A_1)$, (2.9), and (2.11), we see that there exist $0 \le i,\ j \le n$ such that

$$x(t_1 - \tau_i) = \max_{0 \le k \le n} x(t_1 - \tau_k) \ge \min_{0 \le k \le n} x(t_1 - \tau_k) = x\left(t_1 - \tau_j\right), \tag{2.12}$$

which, together with (2.11), implies

$$-d > x(t_1 - \tau_i) = \max_{0 \le k \le n} x(t_1 - \tau_k) \quad \text{or} \quad x\left(t_1 - \tau_j\right) = \min_{0 \le k \le n} x(t_1 - \tau_k) > d. \tag{2.13}$$

Without loss of generality, we may assume that $x(t_1 - \tau_j) > d$ (the situation is analogous for $-d > x(t_1 - \tau_i)$). Then, we have

$$x(t_1 - \tau_i) \ge x(t_1 - \tau_k) \ge x\left(t_1 - \tau_j\right) > d, \quad k = 0, 1, 2, \dots, n. \tag{2.14}$$

According to (2.14) and $(A_1)$, we obtain

$$0 > \sum_{k=0}^{n} g_k(t_1, x(t_1 - \tau_k)) - e(t_1), \tag{2.15}$$

this contradicts the fact (2.9); thus, (2.10) is true.

Let $t_2 = mT + t_0$ where $t_0 \in [0, T]$ and $m$ is an integer. Then, by the same approach used in the proof of inequality (3.3) of [7], we have

$$|x|_\infty = \max_{t\in[t_0,t_0+T]} |x(t)| \le \max_{t\in[t_0,t_0+T]} \left\{ d + \frac{1}{2}\left( \int_{t_0}^{t} |x'(s)|ds + \int_{t-T}^{t_0} |x'(s)|ds \right) \right\} \le d + \frac{1}{2}\sqrt{T}|x'|_2. \tag{2.16}$$

This completes the proof of Lemma 2.2. □

**Lemma 2.3.** *Let $(A_1)$ holds. Assume that the following condition is satisfied.*

*$(A_2)$ There exist nonnegative constants $\overline{b_0}, b_0, b_1, b_2, \ldots, b_n$ such that*

$$\begin{gathered}\overline{b_0}|x_1 - x_2|^2 \le -(g_0(t, x_1) - g_0(t, x_2))(x_1 - x_2), \\ \overline{b_0} > b_1 + b_2 + \cdots + b_n, \ |g_k(t, x_1) - g_k(t, x_2)| \le b_k|x_1 - x_2|,\end{gathered} \tag{2.17}$$

*for all $t, x_1, x_2 \in R$, $k = 0, 1, 2, \ldots, n$.*

*Then,* (1.1) *has at most one $T$-periodic solution.*

*Proof.* Suppose that $x_1(t)$ and $x_2(t)$ are two $T$-periodic solutions of (1.1). Set $Z(t) = x_1(t) - x_2(t)$. Then, we obtain

$$\begin{aligned}&(\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))' + C(x_1'(t) - x_2'(t)) + [g_0(t, x_1(t)) - g_0(t, x_2(t))] \\ &\quad + \sum_{k=1}^{n} [g_k(t, x_1(t - \tau_k)) - g_k(t, x_2(t - \tau_k))] = 0.\end{aligned} \tag{2.18}$$

Multiplying $Z(t)$ and (2.18) and then integrating it from 0 to $T$, from $(A_2)$ and Schwarz inequality, we get

$$\begin{aligned}\overline{b_0}|Z|_2^2 &= \overline{b_0}\int_0^T |Z(t)|^2 dt \\ &\le -\int_0^T (x_1(t) - x_2(t))[g_0(t, x_1(t)) - g_0(t, x_2(t))]dt \\ &= -\int_0^T (\varphi_p(x_1'(t)) - \varphi_p(x_2'(t)))(x_1'(t) - x_2'(t))dt \\ &\quad + \sum_{k=1}^{n}\int_0^T [g_k(t, x_1(t - \tau_k)) - g_k(t, x_2(t - \tau_k))]Z(t)dt\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{n} b_k \int_0^T |x_1(t-\tau_k) - x_2(t-\tau_k)||Z(t)|dt \\
&\leq \sum_{k=1}^{n} b_k \left( \int_0^T |x_1(t-\tau_k) - x_2(t-\tau_k)|^2 dt \right)^{1/2} |Z|_2 \\
&= \sum_{k=1}^{n} b_k |Z|_2^2.
\end{aligned} \tag{2.19}$$

Since $\overline{b_0} > b_1 + b_2 + \cdots + b_n$, we have

$$Z(t) \equiv 0 \quad \forall t \in R. \tag{2.20}$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in R$. Therefore, (1.1) has at most one $T$-periodic solution. The proof of Lemma 2.3 is now complete. □

## 3. Main Results

**Theorem 3.1.** *Let $(A_1)$ and $(A_2)$ hold. Then,* (1.1) *has a unique $T$-periodic solution in $C_T^1$.*

*Proof.* By Lemma 2.3, it is easy to see that (1.1) has at most one $T$-periodic solution in $C_T^1$. Thus, in order to prove Theorem 3.1, it suffices to show that (1.1) has at least one $T$-periodic solution in $C_T^1$. To do this, we are going to apply Lemma 2.1. Firstly, we claim that the set of all possible $T$-periodic solutions of (2.6) in $C_T^1$ is bounded.

Let $x(t) \in C_T^1$ be a $T$-periodic solution of (2.6). Multiplying $x(t)$ and (2.6) and then integrating it from 0 to $T$, we have

$$-\int_0^T \varphi_p(x'(t))x'(t)dt + \lambda \int_0^T x(t) \left[ g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) - e(t) \right] dt = 0. \tag{3.1}$$

Since $x(0) = x(T)$, then there exists $t_0 \in [0, T]$ such that $x'(t_0) = 0$. And since $\varphi_p(0) = 0$, we have

$$|\varphi_p(x'(t))| = \left| \int_{t_0}^{t} (\varphi_p(x'(s)))' ds \right| \leq \lambda \int_{t_0}^{t_0+T} \left| g_0(t, x(t)) + \sum_{k=1}^{n} g_k(t, x(t-\tau_k)) - e(t) \right| dt, \tag{3.2}$$

where $t \in [t_0, t_0 + T]$.

In view of (3.1), $(A_2)$, and Schwarz inequality, we get

$$\begin{aligned}
\overline{b_0}|x|_2^2 &= b_0 \int_0^T |x(t)|^2 dt \\
&\leq -\int_0^T (x(t) - 0)(g_0(t, x(t)) - g_0(t, 0))dt
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\lambda}\int_0^T \varphi_p(x'(t))x'(t)dt + \sum_{k=1}^n \int_0^T \left[g_k(t, x(t-\tau_k)) - g_k(t,0)\right]x(t)dt \\
&\quad + \sum_{k=0}^n \int_0^T g_k(t,0)x(t)dt - \int_0^T x(t)e(t)dt \\
&\le \sum_{k=1}^n b_k \int_0^T |x(t-\tau_k)||x(t)|dt + \sum_{k=0}^n \int_0^T \left|g_k(t,0)\right||x(t)|dt + \sqrt{T}|e|_\infty |x|_2 \\
&\le \sum_{k=1}^n b_k \left(\int_0^T |x(t-\tau_k)|^2 dt\right)^{1/2} |x|_2 + \sqrt{T}\sum_{k=0}^n \left|g_k(t,0)\right|_\infty |x|_2 + \sqrt{T}|e|_\infty |x|_2 \\
&= \sum_{k=1}^n b_k |x|_2^2 + \sqrt{T}\sum_{k=0}^n \left|g_k(t,0)\right|_\infty |x|_2 + \sqrt{T}|e|_\infty |x|_2.
\end{aligned}
\tag{3.3}
$$

It follows that

$$
|x|_2 \le \frac{\sqrt{T}\sum_{k=0}^n \left|g_k(t,0)\right|_\infty + \sqrt{T}|e|_\infty}{\overline{b_0} - \sum_{k=1}^n b_k} := \theta. \tag{3.4}
$$

Again from $(A_2)$ and Schwarz inequality, (3.2) and (3.4) yield

$$
\begin{aligned}
\left|x'\right|_\infty^{p-1} &= \max_{t\in[t_0,\, t_0+T]} \left\{\left|\varphi_p(x'(t))\right|\right\} = \max_{t\in[t_0, t_0+T]} \left\{\left|\int_{t_0}^t (\varphi_p(x'(s)))' ds\right|\right\} \\
&\le \int_{t_0}^{t_0+T} \left|g_0(t, x(t)) + \sum_{k=1}^n g_k(t, x(t-\tau_k)) - e(t)\right| dt \\
&= \int_0^T \left|g_0(t, x(t)) + \sum_{k=1}^n g_k(t, x(t-\tau_k)) - e(t)\right| dt \\
&\le \int_0^T \left|g_0(t,x(t)) - g_0(t,0)\right| dt + \sum_{k=1}^n \int_0^T \left|g_k(t, x(t-\tau_k)) - g_k(t,0)\right| dt \\
&\quad + \sum_{k=0}^n \int_0^T \left|g_k(t,0)\right| dt + T|e|_\infty \\
&\le b_0 \int_0^T |x(t)|dt + \sum_{k=1}^n \int_0^T b_k |x(t-\tau_k)|dt + \sum_{k=0}^n T\left|g_k(t,0)\right|_\infty + T|e|_\infty \\
&\le b_0\sqrt{T}|x|_2 + \sum_{k=1}^n b_k\sqrt{T}|x|_2 + \sum_{k=0}^n T\left|g_k(t,0)\right|_\infty + T|e|_\infty
\end{aligned}
$$

$$
\begin{aligned}
&\leq b_0\sqrt{T}\theta + \sum_{k=1}^{n} b_k\sqrt{T}\theta + \sum_{k=0}^{n} T\left|g_k(t,0)\right|_\infty + T|e|_\infty \\
&:= \overline{\eta},
\end{aligned}
\tag{3.5}
$$

which, together with (2.7), implies that there exists a positive constant $M > 1 + (\overline{\eta})^{1/(p-1)}$ such that, for all $t \in R$,

$$
\left|x'\right|_\infty < M, \qquad |x|_\infty \leq d + \frac{1}{2}\sqrt{T}\left|x'\right|_2 \leq d + \frac{1}{2}T\left|x'\right|_\infty < M. \tag{3.6}
$$

Set

$$
\Omega = \left\{x \in C_T^1 : |x|_\infty \leq M, \left|x'\right|_\infty \leq M\right\}, \tag{3.7}
$$

then we know that (2.6) has no $T$-periodic solution on $\partial\Omega$ as $\lambda \in (0,1)$ and when $x(t) \in \partial\Omega \bigcap R$, $x(t) = M$ or $x(t) = -M$, from $(A_2)$, we can see that

$$
\begin{gathered}
\frac{1}{T}\int_0^T \left\{-g_0(t,M) - \sum_{k=1}^{n} g_k(t,M) + e(t)\right\} dt > 0, \\
\frac{1}{T}\int_0^T \left\{-g_0(t,-M) - \sum_{k=1}^{n} g_k(t,-M) + e(t)\right\} dt < 0,
\end{gathered}
\tag{3.8}
$$

so condition (ii) of Lemma 2.1 is also satisfied. Set

$$
H(x,\mu) = \mu x - (1-\mu)\frac{1}{T}\int_0^T \left[g_0(t,x) + \sum_{k=1}^{n} g_k(t,x) - e(t)\right] dt, \tag{3.9}
$$

and when $x \in \partial\Omega \bigcap R$, $\mu \in [0,1]$, we have

$$
xH(x,\mu) = \mu x^2 - (1-\mu)x\frac{1}{T}\int_0^T \left[g_0(t,x) + \sum_{k=1}^{n} g_k(t,x) - e(t)\right] dt > 0. \tag{3.10}
$$

Thus, $H(x,\mu)$ is a homotopic transformation and

$$
\begin{aligned}
\deg\left\{F, \Omega \bigcap R, 0\right\} &= \deg\left\{-\frac{1}{T}\int_0^T \left[g_0(t,x) + \sum_{k=1}^{n} g_k(t,x) - e(t)\right] dt, \Omega \bigcap R, 0\right\} \\
&= \deg\left\{x, \Omega \bigcap R, 0\right\} \neq 0,
\end{aligned}
\tag{3.11}
$$

so condition (iii) of Lemma 2.1 is satisfied. In view of the previous Lemma 2.1, (1.1) has at least one solution with period $T$. This completes the proof. □

## 4. Example and Remark

*Example 4.1.* Let $p = 4$, $g_0(t,x) = -10e^{20+\sin t}x$, $g_1(t,x) = -2e^{2+\sin t}\sin x$, and $g_2(t,x) = -3e^{3+\cos t}\cos x$ for all $t,x \in R$. Then, the following Liénard type $p$-Laplacian equation with two constant delays

$$(\varphi_p x'(t))' + 55x'(t) + g_0(t,x(t)) + g_1(t,x(t-1)) + g_2(t,x(t-2)) = \cos t \tag{4.1}$$

has a unique $2\pi$-periodic solution.

*Proof.* From (4.1), it is straight forward to check that all the conditions needed in Theorem 3.1 are satisfied. Therefore, (4.1) has at least one $2\pi$-periodic solution. □

*Remark 4.2.* Obviously, the results in [2–5] obtained on Duffing type $p$-Laplacian equation with single delay and without multiple delays cannot be applicable to (4.1). This implies that the results of this paper are essentially new.

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