

Research Article

The (D) Property in Banach Spaces

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A Banach space E is said to have (D) property if every bounded linear operator $T : F \rightarrow E^*$ is weakly compact for every Banach space F whose dual does not contain an isomorphic copy of l_∞ . Studying this property in connection with other geometric properties, we show that every Banach space whose dual has (V^*) property of Pełczyński (and hence every Banach space with (V) property) has (D) property. We show that the space $L^1(\nu)$ of real functions, which are integrable with respect to a measure ν with values in a Banach space X , has (D) property. We give some other results concerning Banach spaces with (D) property.

1. Introduction

As it is well known, the properties or forms that remain invariant under a group of transformations are called “geometry.” The geometry of Banach spaces covers all of the properties that do not change under isomorphisms. We can collect these properties in two groups, namely, isometric and isomorphic ones. Isometric properties such as strict or smooth convexity are directly connected to the norms, whereas, isomorphic ones like Schur property or (V) property of Pełczyński depend on the topologies that the norms define, rather than the norms themselves. In the last half-century, defining new geometric properties of Banach spaces and studying them have gained great interest [1]. The reason for these developments is that examining the structure of Banach spaces with the help of these properties is easier than investigating them one by one. In the literature, there has been a plenty of geometric properties defined in Banach spaces so far. As very illuminative tools, Pełczyński’s fundamental paper [2] introducing the so-called (u) , (V) , and (V^*) properties and the second paper [3] that defined the (V) and (V^*) properties by the coincidence of (V) or (V^*) sets with the weakly relatively compact sets can be given as examples. Many important Banach spaces properties are (or can be) defined in the same way, that is, by the coincidence of two classes

of bounded sets. As an example, Phillips and weak Phillips properties were introduced by Freedman and Ülger in [4], and, then, further results on the weak Phillips property was given in a paper by Ülger [5].

In this paper, we introduce a new kind of geometric properties in Banach spaces as follows. Let E and F be two Banach spaces and E^* the dual space of E . A Banach space E is said to have (D) property if every bounded linear operator $T : F \rightarrow E^*$ is weakly compact for every Banach space F whose dual does not contain an isomorphic copy of l_∞ . We show that every Banach space whose dual has (V^*) property has (D) property. We show that (V) property implies (D) property; however, the converse implication does not hold. We also see that (D) property implies (W) property. We investigate some properties of such Banach spaces to some extent; for instance, we show that the James space J fails to have the (D) property.

Given a vector measure ν with values in a Banach space X , $L^1(\nu)$ denotes the space of (classes of) real functions that are integrable with respect to ν in the sense of Bartle et al. [6] and Lewis [7]. We show that L^1 -spaces have (D) property. This is a structural difference between $C(K)$ spaces (which enjoy (V) property) and L^1 -spaces (which fail to have (V) property).

2. Notations and Preliminaries

We will try to follow the standard notations in the Banach space theory. In order to prevent any doubt, we will fix some terminology. If E is a Banach space, B_E will be its closed unit ball and E^* its topological dual. The word operator will always mean linear bounded operator. A series $\sum x_n$ in E is said to be weakly unconditionally Cauchy (w.u.c. in short) if $\sum |x'(x_n)| < \infty$ for every $x' \in E^*$. An operator $T : E \rightarrow F$ is said to be unconditionally converging if T sends w.u.c. series $\sum x_n$ in E into unconditionally converging series $\sum Tx_n$ in F . An operator $T : E \rightarrow F$ is said to be weakly compact if $T(B_E)$ is relatively weakly compact. It is well known that every weakly compact operator is unconditionally converging [3].

A Banach space E is said to have (V) property if every unconditionally converging operator $T : E \rightarrow F$ is weakly compact for every Banach space F .

A Banach space E is said to have (V^*) property if, for every Banach space F , every operator $T : F \rightarrow E$ is weakly compact whenever its adjoint $T^* : E^* \rightarrow F^*$ is unconditionally converging.

The above definitions of (V) and (V^*) spaces were firstly introduced by Pełczyński, who showed that the space l_1 and abstract L -spaces enjoy the (V^*) property, whereas, the space c_0 and $C(K)$ spaces enjoy the (V) property [3]. Now, we introduce our following definition of (D) property in Banach spaces.

A Banach space E is said to have (D) property if every bounded linear operator $T : F \rightarrow E^*$ is weakly compact for every Banach space F whose dual does not contain an isomorphic copy of l_∞ .

Let (Ω, Σ) be a measurable space, X a Banach space with unit ball B_X and dual space X^* , and $\nu : \Sigma \rightarrow X$ a countably additive vector measure. The semivariation of ν is the set function $\|\nu\|(A) = \sup\{|x^*\nu|(A) : x^* \in B_{X^*}\}$, where $|x^*\nu|$ is the variation of the scalar measure $x^*\nu$. A Rybakov control measure for ν is a measure $\lambda = |x^*\nu|$ such that $\lambda(A) = 0$ if and only if $\|\nu\|(A) = 0$ (see [8]).

Following Lewis [7], we will say that a measurable function $f : \Omega \rightarrow IR$ is *integrable* with respect to ν if

- (1) f is x^*v integrable for every $x^* \in X^*$,
 (2) for each $A \in \Sigma$ there exists an element of X denoted by $\int_A f d\nu$, such that

$$x^* \int_A f d\nu = \int_A f dx^*v \quad \text{for every } x^* \in X^*. \quad (2.1)$$

Identifying two functions if the set where they differ has null semivariation, we obtain a linear space of classes of functions that, when endowed with the norm

$$\|f\|_v = \sup \left\{ \int_A |f| d|x^*v| : x^* \in B_{X^*} \right\}, \quad (2.2)$$

becomes a Banach space. We will denote it by $L^1(v)$. It is a Banach lattice for the $\|v\|$ -almost everywhere order. Simple functions are dense in $L^1(v)$ and the identity is a continuous injection of the space of $\|v\|$ -essentially bounded functions into $L^1(v)$.

3. Main Results on the (D) Property

We start with Lemma 3.1., which is adapted from a result given by Godefroy and Saab in [9].

Lemma 3.1. *Let E be a Banach space with the (V^*) property and (x_n) a bounded but not relatively compact sequence in E . Then, there is a subsequence (x_{n_k}) equivalent to the l_1 unit vector basis so that the closed linear space $[(x_{n_k})]$ is complemented in E .*

Now, we give the following lemma, which we need for the same operations done in Ülger's paper [5].

Lemma 3.2. *Let E be a Banach space and F a dual Banach space not containing an isomorphic copy of l_∞ . Then, $T : E^* \rightarrow F$ is unconditionally converging.*

Proof. For a contradiction, suppose that we have an operator $T : E^* \rightarrow F$ that is unconditionally converging for a Banach space E and a dual Banach space F not containing an isomorphic copy of l_∞ . Then, there exists a subspace M of E^* such that M is isomorphic to c_0 and the restriction $T|_M$ of T is an isomorphism on M by [10]. Since $M \cong c_0$, we have $M^{**} \cong l_\infty$. Considering the natural injection $i : M \rightarrow E^*$ and the natural projection $p : E^{***} \rightarrow E^*$, take the composition $T \circ p \circ i^{**} : M^{**} \rightarrow F$. Let us denote it as $S = T \circ p \circ i^{**}$. Since $M^{**} \cong l_\infty$ is an injective space [11] and every operator from an injective space to a space not containing an isomorphic copy of l_∞ is weakly compact from Corollary 1.4 of Rosenthal [12], S is weakly compact. Then the restriction $S|_M = (T \circ p \circ i^{**})|_M$ is weakly compact. Since $S|_M = (T \circ p \circ i^{**})|_M = T|_M$ holds, $T|_M$ is weakly compact, which means $(T|_M)^{**}(M^{**}) \subset c_0$ by Goldstein theorem. Since the restriction $T|_M$ is an isomorphism, a contradiction occurs that means $l_\infty \subset c_0$. Then, $T : E^* \rightarrow F$ is unconditionally converging. \square

Theorem 3.3. *Let E be a Banach space whose dual has the (V^*) property. Then, E has the (D) property.*

Proof. Let E be a Banach space whose dual has the (V^*) property and F a Banach space whose dual does not contain an isomorphic copy of l_∞ . Let an operator $T : F \rightarrow E^*$ be given.

Consider the adjoint operator $T^* : E^{**} \rightarrow F^*$. Then, by Lemma 3.2, the operator $T^* : E^{**} \rightarrow F^*$ is unconditionally converging. By the definition of (V^*) property, the operator $T : F \rightarrow E^*$ is weakly compact.

If any Banach space X enjoys the (V) property, then dual space X^* has the (V^*) property; therefore, we immediately have the following result. \square

Corollary 3.4. *If a Banach space X has the (V) property, then it has the (D) property.*

Corollary 3.4 shows that the space c_0 and $C(K)$ spaces have the (D) property. But, we give the following example to show that the converse of Corollary 3.4 does not hold.

Example 3.5. Let Y be the space constructed by Bourgain and Delbaen in [13]. Since $Y^* \cong l_1$ and the space l_1 has the Schur property, the space Y has (D) property. However, since the space Y is not reflexive and does not contain a copy of c_0 , it fails to have the (V) property.

As a commonly known example, the space l_1 does not have (D) property because the injection $I : c_0 \rightarrow l_\infty$ is not weakly compact. Recall that the dual space of c_0 , that is, l_1 , does not contain a copy of l_∞ . By the definition of (D) property it is clear that any Banach space with (D) property does not contain a complemented copy of l_1 .

Any Banach space with the (V^*) property is weakly sequentially complete (w.s.c.) and every closed subspace of a w.s.c. space is also w.s.c.; so such space does not contain an isomorphic copy of l_∞ . Recall that the space l_∞ is not w.s.c.. If a Banach space has the (V) property, then its dual has the (V^*) property [3]. Hence, Corollary 3.4 extends a previous result of E. Saab and P. Saab [14], which we give as a corollary below.

Corollary 3.6. *Let E be a Banach space with the (V) property. Then, every operator $T : E \rightarrow E^*$ is weakly compact [14].*

E. Saab and P. Saab, in [14], introduced a property called the (W) property. A Banach space X has the (W) property if every operator $T : E \rightarrow E^*$ is weakly compact.

Corollary 3.7. *Let E be a Banach space having the (D) property. Then, E has (W) property.*

Proof. Let E be a Banach space having the (D) property. Then, E cannot contain a complemented subspace that is isomorphic to l_1 , and E^* cannot contain a subspace isomorphic to l_∞ . It follows that every operator $T : E \rightarrow E^*$ is weakly compact, that is, E has the (W) property.

Taking into consideration Theorem 3.3 and Corollary 3.7, we have the following well-known result. \square

Corollary 3.8. *Let the dual space E^* has the (V^*) property. Then, E has (W) property.*

Theorem 3.9. *Let E be a Banach space whose dual has the (V^*) property and F any Banach space. If any bounded linear operator $T : F \rightarrow E^*$ is not weakly compact, then T fixes an isomorphic copy of l_1 .*

Proof. Let E^* have the (V^*) property. Suppose that $T : F \rightarrow E^*$ is not weakly compact. Then, there is a bounded sequence (y_n) in F so that $(T(y_n))$ is not relatively weakly compact. Hence, from Lemma 3.1, a subsequence generates a complemented subspace $[(T(y_{n_k}))]$ isomorphic to l_1 . It is easy to see that $[(y_{n_k})]$ must be a complemented subspace of F isomorphic to l_1 .

The example we gave after Corollary 3.4 shows that (D) property does not imply (V) property is not, of course, the unique counterexample. For example, we want to give

Theorem 3.10, which indicates another structural difference between $C(K)$ spaces (which enjoy (V) property) and L^1 -spaces (which fail to have (V) property). \square

Theorem 3.10. *The space $L^1(v)$ has (D) property.*

Proof. Let F be a Banach space whose dual does not contain an isomorphic copy of l_∞ , and, a bounded operator $T : F \rightarrow L^1(v)^*$ be given. Let λ be a Rybakov control measure for v . Then, $L^1(v)$ is an order continuous Banach function space with weak unit over the finite measure space $(\Omega, \Sigma, \lambda)$ (see [15]). Thus, it can be regarded as a lattice ideal in $L^1(\lambda)$, and the dual space $L^1(v)^*$ can be identified with the space of functions g in $L^1(\lambda)$ such that $fg \in L^1(\lambda)$, for all f in $L^1(v)$, where the action of g over $L^1(v)$ is given by integration with respect to λ [16]. Suppose that (f_n) is a bounded sequence in F ; then $T(f_n)$ can be considered as a bounded set in $L^1(\lambda)$. Then, by Dunford's theorem in [8], we have a weakly convergence subsequence $(T(f_{n_k}))$, which means that the operator T is weakly compact. Hence, $L^1(v)$ has the (D) property.

It is known that dual spaces of the Banach spaces having the geometric properties such as (V), weak Phillips, or Grothendieck properties are weakly sequentially complete; see [3–5]. At this point the following question arises: does the dual of any Banach space with (D) property have to be weakly sequentially complete, or not? We could not give an answer to this question yet. Regarding (D) property, we now just give the following theorem that reveals a result opposite to our expectation about the James space J , which is not weakly sequentially complete as shown in [17]. \square

Proposition 3.11. *The James space J fails to have the (D) property.*

Proof. By recalling the construction of the James space as in [18], we will give a proof using contradiction method as follows: suppose the James space J has the (D) property. Take F as J^* and any bounded sequence (y_n) in F . Since F does not contain any isomorphic copy of l_1 , by Rosenthal's l_1 theorem, (y_n) has a weakly Cauchy subsequence (y_{n_j}) . Then, considering the adjoint operator $S^* : J^{**} \rightarrow F^*$, for each $x'' \in J^{**}$, we see that $S^*(x'') \in F^*$. By the equality $\langle y_{n_j}, S^*(x'') \rangle = \langle S(y_{n_j}), x'' \rangle$ for $x'' \in J^{**}$, the subsequence $(S(y_{n_j}))$ becomes a weakly Cauchy subsequence in J^* . For yielding a contradiction, using the same technique in [19], suppose that $(S(y_{n_j}))$ does not converge weakly in J^* ; then by using [19, Lemma 6] we have a subsequence $(S(x_{n_j}))$ of $(S(y_{n_j}))$ and a constant $M < \infty$ such that

$$\left\| \sum c_j v'_j \right\| \leq M \left\| \sum c_j S(x_{n_j}) \right\| \dots \tag{3.1}$$

for every finitely supported real sequence (c_j) . According to [19, Proposition 5] we may assume that there is a bounded linear operator $R : F \rightarrow F$ such that $R(u_{n_j}) = x_{n_j}$ for all j . Therefore, by [19]

$$\left\| \sum c_j v'_j \right\| \leq M \left\| \sum c_j S(x_{n_j}) \right\| \leq M \|S\| \left\| \sum c_j x_{n_j} \right\| \leq M \|S\| \|R\| \left\| \sum c_j x_{n_j} \right\| \tag{3.2}$$

for all finitely supported real sequence (c_j) . But this means that the operator T from [19, Proposition 3] is bounded, yielding a contradiction. Hence, $(S(y_{n_j}))$ converges weakly in J^* , which means that the identity operator $S : J^* \rightarrow J^*$ is weakly compact. However, this case is impossible. Therefore, the James space J fails to have (D) property.

The James space J fails to have (D) property by Proposition 3.7, and J is not weakly sequentially complete because it is not reflexive and does not contain any isomorphic copy of l_1 . \square

Theorem 3.12. *Any nonreflexive space E with E^{**} not containing l_∞ does not have the (D) property.*

Proof. Let E be a nonreflexive space E with E^{**} not containing l_∞ . For the contradiction, suppose that E enjoys the (D) property and G is a subspace of E^* that does not contain a complemented copy of l_1 . Since the condition that F^* does not contain a copy of l_∞ is equivalent to the condition that F does not contain a complemented copy of l_1 [17], the embedding $i : G \rightarrow E^*$ is weakly compact. Hence, G is reflexive. That is, the identity operator $I : E^* \rightarrow E^*$ is weakly compact, which is a contradiction.

Before raising some questions, we want to recall some geometric properties in the literature we think can be related with the property (D) . A Banach space E is said to have the (weak) Phillips property if the natural projection $p : E^{***} \rightarrow E^*$ is sequentially weak*-to-norm (weak*-to-weak) continuous [4, 5]. A Banach space E is said to have the Grothendieck property if, for every separable Banach space F , every operator $T : E \rightarrow F$ is weakly compact [8]. \square

Open Questions

We do not know whether (D) property implies or is implied by either the weak Phillips property or the Grothendieck property. We do not know whether weak sequential completeness of any dual Banach space E^* implies that the space E itself has (D) property. Also, we do not know whether (D) property of any Banach space implies (V^*) property of its dual space.

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