Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 752869, 14 pages
doi:10.1155/2012/752869

Research Article

Approximate Solutions of Fractional Nonlinear Equations Using Homotopy Perturbation Transformation Method

Yanqin Liu[1, 2]

[1] *Department of Mathematics, Dezhou University, Dezhou 253023, China*
[2] *Nonlinear Dynamics and Chaos Group, School of Management, Tianjin University, Tianjin 30072, China*

Correspondence should be addressed to Yanqin Liu, yqlin8801@yahoo.cn

Received 11 November 2011; Revised 25 December 2011; Accepted 30 January 2012

Academic Editor: Muhammad Aslam Noor

A homotopy perturbation transformation method (HPTM) which is based on homotopy perturbation method and Laplace transform is first applied to solve the approximate solution of the fractional nonlinear equations. The nonlinear terms can be easily handled by the use of He's polynomials. Illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.

1. Introduction

In recent years, system of fractional nonlinear partial differential equations [1–3] have attracted much attention in a variety of applied sciences. The importance of obtaining the exact and approximate solutions of fractional nonlinear equations in physics and mathematics is still a significant problem that needs new methods to discover exact and approximate solutions. But these nonlinear fractional differential equations are difficult to get their exact solutions [4–7]. So, numerical methods have been used to handle these equations [8–11], and some semianalytical techniques have also largely been used to solve these equations. Such as, Adomian decomposition method [12, 13], variational iteration method [14, 15], differential transform method [16, 17], Laplace decomposition method [18, 19], and homotopy perturbation method [20–25]. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work.

In this work, we will use homotopy perturbation transformation method introduced by Khan [26, 27] to solve fractional nonlinear partial differential equations. This new method basically illustrates how two powerful algorithms, homotopy perturbation method and Laplace transform method, can be combined and used to approximate the solutions of nonlinear equation. The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This paper considers the effectiveness of the homotopy perturbation transformation method in solving fractional nonlinear equations.

2. Description of the HPTM

To illustrate the basic idea of this method [26, 27], we consider a general fractional nonlinear nonhomogeneous partial differential equation with initial conditions of the form

$$D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{2.1}$$

$$u(x,0) = h(x), \qquad u_t(x,0) = f(x), \tag{2.2}$$

where $g(x,t)$ is the source term, N represents the general nonlinear differential operator and R is the linear differential operator, and $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of function $u(x,t)$ which is defined as

$${}_0D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x,\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1 < Re(\alpha)n,\ n \in N), \tag{2.3}$$

where $\Gamma(\cdot)$ denotes the Gamma function. The properties of fractional derivative can be found in [1, 2]. Laplace transform (denoted throughout this paper by L) of the Caputo operator is an important property will be used in this paper

$$L\left[{}_0D_t^{\alpha} u(x,t)\right] = s^{\alpha} u(x,s) - \sum_{k=0}^{n-1} u^k(x,0^+) s^{\alpha-1-k}, \quad (n-1 < \alpha \leqslant n). \tag{2.4}$$

Taking the Laplace transform on both sides of (2.1),

$$L\left[{}_0D_t^{\alpha} u(x,t)\right] + L[Ru(x,t)] + L[Nu(x,t)] = L\left[g(x,t)\right], \tag{2.5}$$

Using the property of the laplace transform, we have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{1}{s^{\alpha}} L[Ru(x,t)] - \frac{1}{s^{\alpha}} L[Nu(x,t)] + \frac{1}{s^{\alpha}} L\left[g(x,t)\right]. \tag{2.6}$$

Operating with the Laplace inverse on both sides of (2.6) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^{\alpha}} L[Ru(x,t)] + \frac{1}{s^{\alpha}} L[Nu(x,t)]\right], \tag{2.7}$$

where $G(x,t)$ represent the term arising from the source term and the prescribed initial conditions. Then, we apply the homotopy perturbation method; the basic assumption is that the solutions can be written as a power series in p

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n(x,t)=u_0+pu_1+p^2u_2+p^3u_3+\cdots, \tag{2.8}$$

and the nonlinear term can be decomposed as

$$Nu(x,t)=\sum_{n=0}^{\infty}p^n\mathcal{H}_n(u), \tag{2.9}$$

where $p\in[0,1]$ is an embedding parameter. $\mathcal{H}_n(u)$ is He's polynomials [28, 29] and can be generated by

$$\mathcal{H}_n(u_0,\ldots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0}, \quad n=0,1,2\ldots. \tag{2.10}$$

Substituting (2.8) and (2.9) in (2.7) we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left(L^{-1}\left[\frac{1}{s^\alpha}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)\right]+\frac{1}{s^\alpha}L\left[\sum_{n=0}^{\infty}p^n\mathcal{H}_n(u)\right]\right]\right). \tag{2.11}$$

Equating the terms with identical powers in p, we obtain the following approximations:

$$\begin{aligned}
p^0 &: u_0(x,t)=G(x,t),\\
p^1 &: u_1(x,t)=-L^{-1}\left[\frac{1}{s^\alpha}L[Ru_0(x,t)]+\frac{1}{s^\alpha}L[\mathcal{H}_0(u)]\right],\\
p^2 &: u_2(x,t)=-L^{-1}\left[\frac{1}{s^\alpha}L[Ru_1(x,t)]+\frac{1}{s^\alpha}L[\mathcal{H}_1(u)]\right],\\
&\vdots
\end{aligned} \tag{2.12}$$

The best approximations for the solution are

$$u(x,t)=u_0+u_1+u_2+u_3+\cdots. \tag{2.13}$$

This method does not resort to linearization or assumptions of weak nonlinearity, the solution generated in the form of general solution, and it is more realistic compared to the method of simplifying the physical problems.

3. Approximate Solutions of Fractional Equations

In order to assess the advantages and the accuracy of the homotopy perturbation transform method for fractional nonlinear equations, we have applied it to the following several problems.

Case 1. Consider the following time fractional advection nonhomogeneous equation [27]:

$$D_t^{\alpha} u(x,t) + uu_x = 2t + x + t^3 + xt^2, \tag{3.1}$$

$$u(x,0) = 0, \tag{3.2}$$

where $0 < \alpha \leqslant 1$, taking the Laplace transform on both sides of (3.1)-(3.2)

$$L\left[D_t^{\alpha} u(x,t)\right] + L[uu_x] = L\left[2t + x + t^3 + xt^2\right]. \tag{3.3}$$

Using the property of the Laplace transform, we have

$$L[u(x,t)] = \frac{1}{s^{\alpha}}\left(\frac{2}{s^2} + \frac{x}{s} + \frac{6}{s^4} + \frac{2x}{s^3}\right) - \frac{1}{s^{\alpha}} L[uu_x]. \tag{3.4}$$

Operating with the Laplace inverse on both sides of (3.4) gives

$$u(x,t) = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - L^{-1}\left[\frac{1}{s^{\alpha}} L[uu_x]\right]. \tag{3.5}$$

Then, we apply the homotopy perturbation method, and substituting (2.8) and (2.9) in (3.5) we get

$$\sum_{n=0}^{\infty} p^n u_n = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u)\right]\right]\right), \tag{3.6}$$

where $\mathscr{H}_n(u)$ is He's polynomials that represents nonlinear term uu_x; we have a few terms of the He's polynomials for uu_x which are given by

$$\begin{aligned} \mathscr{H}_0(u) &= u_0 u_{0x}, \\ \mathscr{H}_1(u) &= u_0 u_{1x} + u_1 u_{0x}, \\ \mathscr{H}_2(u) &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x}, \\ &\vdots \end{aligned} \tag{3.7}$$

Comparing the coefficient of like powers of p, we have

$$
\begin{aligned}
u_0(x,t) &= \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)}, \\
u_1(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}}L[u_0 u_{0x}]\right] \\
&= -\frac{x\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)} - \frac{2\Gamma(2+2\alpha)t^{1+3\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)\Gamma(2+3\alpha)} - \frac{4x\Gamma(3+2\alpha)t^{2+3\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)} \\
&\quad - \frac{4\Gamma(3+3\alpha)t^{4+2\alpha}}{\Gamma(2+\alpha)\Gamma(3+\alpha)\Gamma(4+3\alpha)} - \frac{6\Gamma(4+2\alpha)t^{3+3\alpha}}{\Gamma(1+\alpha)\Gamma(4+\alpha)\Gamma(4+3\alpha)} \\
&\quad - \frac{4x\Gamma(5+2\alpha)t^{4+3\alpha}}{\Gamma^2(3+\alpha)\Gamma(5+3\alpha)} - \frac{12\Gamma(6+2\alpha)t^{5+3\alpha}}{\Gamma(3+\alpha)\Gamma(4+\alpha)\Gamma(6+3\alpha)}, \\
u_2(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}}L[u_0 u_{1x} + u_1 u_{0x}]\right] \\
&= \frac{2x\Gamma(1+2\alpha)\Gamma(1+4\alpha)t^{5\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)} + \frac{2\Gamma(1+2\alpha)\Gamma(2+4\alpha)t^{1+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(2+\alpha)\Gamma(1+3\alpha)\Gamma(2+5\alpha)} \\
&\quad + \frac{2\Gamma(2+2\alpha)\Gamma(2+4\alpha)t^{1+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(2+\alpha)\Gamma(2+3\alpha)\Gamma(2+5\alpha)} + \frac{4x\Gamma(1+2\alpha)\Gamma(3+4\alpha)t^{2+5\alpha}}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(1+3\alpha)\Gamma(3+5\alpha)} + \cdots \\
&\ \vdots
\end{aligned}
\tag{3.8}
$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.1) in series form is given by

$$
u(x,t) = \frac{2t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{6t^{\alpha+3}}{\Gamma(4+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)} - \frac{x\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)} + \cdots. \tag{3.9}
$$

If we take $\alpha = 1$, the first few components the solution of (3.1) are as follows:

$$
\begin{aligned}
u_0(x,t) &= t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3}, \\
u_1(x,t) &= -\frac{t^4}{4} - \frac{xt^3}{3} - \frac{2xt^5}{15} - \frac{7t^6}{72} - \frac{xt^7}{63} - \frac{t^8}{96}, \\
u_2(x,t) &= \frac{5t^12}{8064} + \frac{2xt^11}{2079} + \frac{2783t^10}{302400} + \frac{38xt^9}{2835} + \frac{143t^8}{2880} + \frac{22xt^7}{315} + \frac{7t^6}{12} + \frac{2xt^5}{15}, \\
&\ \vdots
\end{aligned}
\tag{3.10}
$$

The noise terms $-t^4/4 - xt^3/3$ between the components u_0 and u_1 can be canceled and the remaining term of u_0 still satisfies the equation. For this special case, the exact solution is therefore

$$u(x,t) = t^2 + xt. \tag{3.11}$$

which was given in [27].

Case 2. Consider the following time-space fractional nonlinear Fokker-Planck equation [30]:

$$D_t^{\alpha} u(x,t) = -D_x^{\beta}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) + D_x^{2\beta}\left(u^2\right), \tag{3.12}$$

$$u(x,0) = x^2, \tag{3.13}$$

where $0 < \alpha, \beta \leqslant 1$, and α and β are parameters describing the order of the time- and space-fractional derivatives, respectively. D_x^{β} is also the Caputo fractional derivative with respect to x and is defined as

$$D_x^{\beta} u(x,t) = \frac{1}{\Gamma(m-\beta)} \int_0^x \frac{u^{(m)}(\xi,t)d\xi}{(x-\xi)^{\beta+1-m}}, \quad (m-1 < Re(\beta) \leqslant m,\ m \in N) \tag{3.14}$$

taking the Laplace transform on both sides of (3.12)-(3.13)

$$s^{\alpha} L[u(x,t)] - x^2 s^{\alpha-1} = L\left[-D_x^{\beta}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) + D_x^{2\beta}\left(u^2\right)\right]. \tag{3.15}$$

We have

$$L[u(x,t)] = \frac{x^2}{s} + \frac{1}{s^{\alpha}} L\left[-D_x^{\beta}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) + D_x^{2\beta}\left(u^2\right)\right]. \tag{3.16}$$

Operating with the Laplace inverse on both sides of (3.16) gives

$$u(x,t) = x^2 + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[-D_x^{\beta}\left(\frac{4u^2}{x} - \frac{xu}{3}\right) + D_x^{2\beta}\left(u^2\right)\right]\right]. \tag{3.17}$$

Then, we apply the homotopy perturbation method, and substituting (2.8) and (2.9) in (3.17) we get

$$\sum_{n=0}^{\infty} p^n u_n = x^2 + p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[D_x^{2\beta}\left(\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u)\right) - D_x^{\beta}\left(\frac{4\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u)}{x} - \frac{x\sum_{n=0}^{\infty} p^n u_n}{3}\right)\right]\right]\right), \tag{3.18}$$

where $\mathscr{H}_n(u)$ is He's polynomials that represents nonlinear term u^2; we have a few terms of the He's polynomials for u^2 which are given by

$$\mathscr{H}_0(u) = u_0^2, \tag{3.19}$$

$$\mathscr{H}_1(u) = 2u_0u_1, \tag{3.20}$$

$$\begin{gathered} \mathscr{H}_2(u) = u_1^2 + 2u_0u_2, \\ \vdots \end{gathered} \tag{3.21}$$

Comparing the coefficient of like powers of p, we have (3.22)

$$\begin{aligned}
u_0(x,t) &= x^2, \\
u_1(x,t) &= L^{-1}\left[\frac{1}{s^\alpha}L\left[D_x^{2\beta}\left(u_0^2\right) - D_x^\beta\left(\frac{4u_0^2}{x} - \frac{xu_0}{3}\right)\right]\right], \\
&= \frac{24t^\alpha x^{4-2\beta}}{\Gamma(1+\alpha)\Gamma(5-2\beta)} - \frac{22t^\alpha x^{3-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)}, \\
u_2(x,t) &= L^{-1}\left[\frac{1}{s^\alpha}L\left[D_x^{2\beta}(2u_0u_1) - D_x^\beta\left(\frac{8u_0u_1}{x} - \frac{xu_1}{3}\right)\right]\right] \\
&= -\frac{184t^{2\alpha}x^{5-3\beta}\Gamma(6-2\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(5-2\beta)} + \frac{48t^{2\alpha}x^{6-4\beta}\Gamma(7-2\beta)}{\Gamma(1+2\alpha)\Gamma(7-4\beta)\Gamma(5-2\beta)} \\
&\quad + \frac{506t^{2\alpha}x^{4-2\beta}\Gamma(5-\beta)}{3\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(4-\beta)} - \frac{44t^{2\alpha}x^{5-3\beta}\Gamma(6-\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(4-\beta)}, \\
&\ \vdots
\end{aligned} \tag{3.22}$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.12) in series form is given by

$$u(x,t) = x^2 + \frac{24t^\alpha x^{4-2\beta}}{\Gamma(1+\alpha)\Gamma(5-2\beta)} - \frac{22t^\alpha x^{3-\beta}}{\Gamma(1+\alpha)\Gamma(4-\beta)} - \frac{184t^{2\alpha}x^{5-3\beta}\Gamma(6-2\beta)}{\Gamma(1+2\alpha)\Gamma(6-3\beta)\Gamma(5-2\beta)} + \cdots. \tag{3.23}$$

If we take $\alpha = \beta = 1$, the first few components the solution of (3.12) are as follows:

$$u_0(x,t) = x^2,$$
$$u_1(x,t) = x^2 t,$$
$$u_2(x,t) = x^2 \frac{t^2}{2!}, \tag{3.24}$$
$$\vdots$$

and so on. Hence, for this special case, we have

$$u(x,t) = x^2\left(1 + t + \frac{t^2}{2!} + \cdots\right) = x^2 e^t \tag{3.25}$$

which was given in [30].

Figure 1 shows the approximate solution for (3.12)-(3.13) by using the homotopy perturbation transformation method when choosing $x = 0.8$, $\alpha = 1$. From the figure, it is clear to see the time evolution of fractional Fokker-Planck equation and we also know the approximate solution of the model is continuous with the fractional parameter β. Figure 2 shows the approximate solution for (3.12)-(3.13) when $t = 0.8$, $\alpha = 1$, and the approximate solution of the model is continuous with the fractional parameter β. Figures 3 and 4 show the approximate solution for (3.12)-(3.13) when the parameter $\beta = 1$, and from the figures, we also know that the solution of the fractional nonlinear equation changes with the parameters α and x, t.

Table 1 shows the approximate solutions for (3.12) by using the homotopy perturbation transformation method, Adomian decomposition method, variational iteration method and the exact solution $u(x,t) = x^2 e^t$ when $\alpha = \beta = 1$. It is noted that only the third-order of the homotopy perturbation transformation solution is used in evaluating the approximate solutions in Table 1, and it is evident that the method used in this paper and the Adomian decomposition method have high accuracy compare with the variational iteration method, and we take 15 terms of the VIM solution. And for nonlinear equations, Adomian's polynomials are very difficult to calculate. In brief, the homotopy perturbation transformation method is an effectiveness tool to solve fractional nonlinear equation only using Mathematica symbol computing software.

Case 3. Consider the following time fractional nonhomogeneous nonlinear system [31]:

$$D_t^{\alpha} u - u_x v - u = 1, \tag{3.26}$$

$$D_t^{\beta} v + u v_x + v = 1, \tag{3.27}$$

with the initial conditions

$$u(x,0) = e^{-x}, \qquad v(x,0) = e^{x}, \tag{3.28}$$

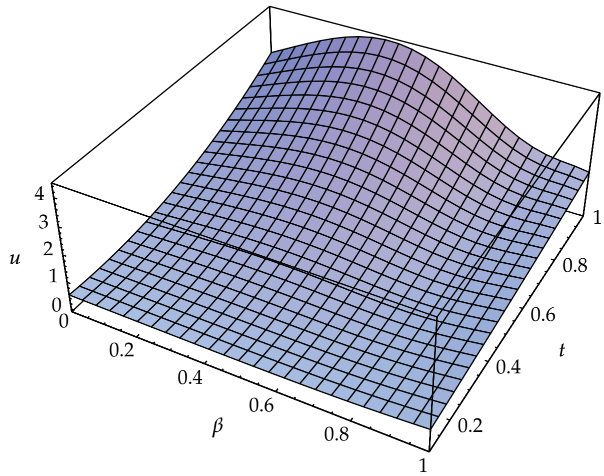

Figure 1: The surface of second-order approximate solution of (3.12) when $x = 0.8$, $\alpha = 1$.

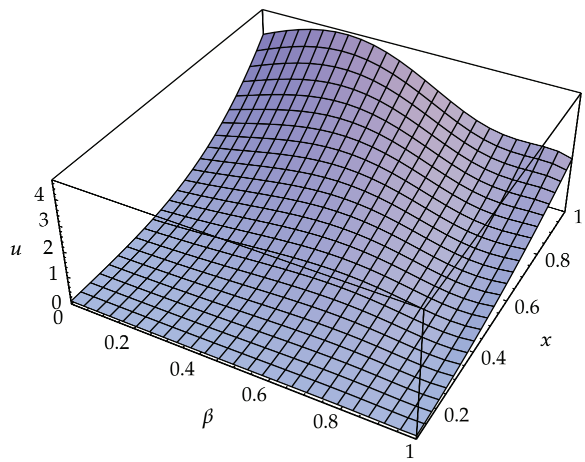

Figure 2: The surface of second-order approximate solution of (3.12) when $t = 0.8$, $\alpha = 1$.

Table 1: Approximate values and exact solutions when $\alpha = 1$, $\beta = 1$ for (3.12).

t	x	$\text{solution}_{\text{HPTM}}$	$\text{solution}_{\text{ADM}}$	$\text{solution}_{\text{VIM}}$	exact solution
0.06	0.25	0.066367	0.066367	0.066363	0.066365
	0.50	0.265468	0.265468	0.265450	0.265459
	0.75	0.597303	0.597303	0.597262	0.597283
	1.0	1.061970	1.061970	1.061800	1.061840
0.2	0.25	0.076417	0.076416	0.076250	0.076338
	0.50	0.305667	0.305667	0.305000	0.305351
	0.75	0.687750	0.687750	0.686250	0.687039
	1.0	1.222670	1.222670	1.220000	1.221400
0.4	0.25	0.093833	0.093833	0.092500	0.093239
	0.50	0.375330	0.375330	0.370000	0.372956
	0.75	0.844500	0.844500	0.832500	0.839151
	1.0	1.501330	1.501330	1.480000	1.491820

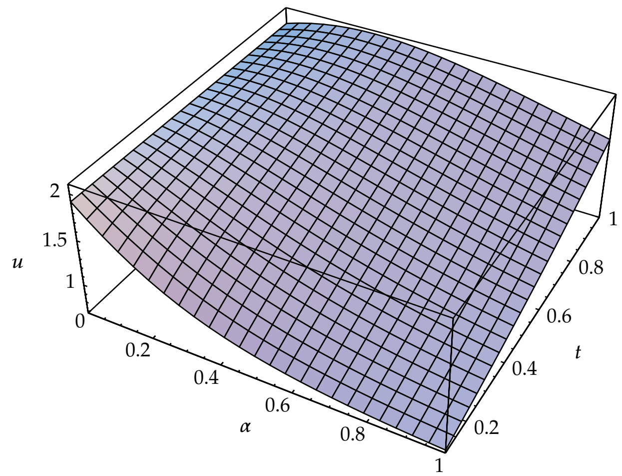

Figure 3: The surface of second-order approximate solution of (3.12) when $x = 0.8$, $\beta = 1$.

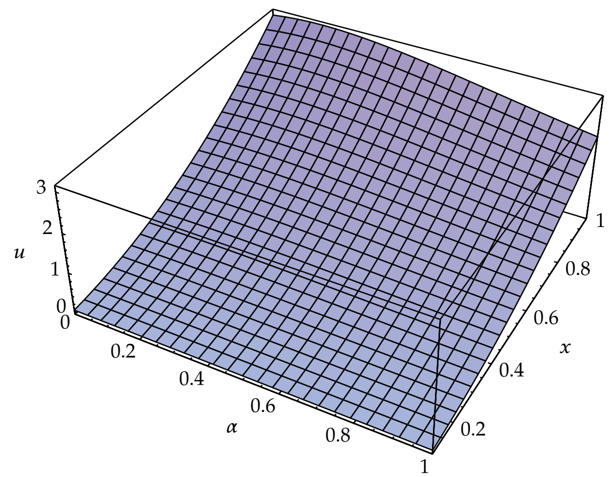

Figure 4: The surface of second-order approximate solution of (3.12) when $t = 0.8$, $\beta = 1$.

where $0 < \alpha, \beta \leqslant 1$; in a similar way as above we have

$$\sum_{n=0}^{\infty} p^n u_n = e^{-x} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{1n}(u,v) + \sum_{n=0}^{\infty} p^n u_n\right]\right]\right),$$
$$\sum_{n=0}^{\infty} p^n v_n = e^{x} + \frac{t^{\beta}}{\Gamma(\beta+1)} - p\left(L^{-1}\left[\frac{1}{s^{\alpha}} L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{2n}(u,v) + \sum_{n=0}^{\infty} p^n v_n\right]\right]\right), \tag{3.29}$$

where $\mathscr{H}_{1n}(u,v)$ and $\mathscr{H}_{2n}(u,v)$ are He's polynomials that represent nonlinear term vu_x and uv_x respectively, and we have a few terms of the He's polynomials for vu_x and uv_x which are given by

$$\begin{aligned}
\mathscr{H}_{10}(u,v) &= v_0u_{0x},\\
\mathscr{H}_{11}(u,v) &= v_1u_{0x} + v_0u_{1x},\\
\mathscr{H}_{12}(u,v) &= v_2u_{0x} + v_1u_{1x} + v_0u_{2x},\\
&\ \vdots\\
\mathscr{H}_{20}(u,v) &= u_0v_{0x},\\
\mathscr{H}_{21}(u,v) &= u_1u_{0x} + u_0v_{1x},\\
\mathscr{H}_{22}(u,v) &= u_2v_{0x} + u_1v_{1x} + u_0v_{2x},\\
&\ \vdots
\end{aligned} \tag{3.30}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned}
u_0(x,t) &= e^{-x} + \frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
v_0(x,t) &= e^{x} + \frac{t^{\beta}}{\Gamma(1+\beta)},\\
u_1(x,t) &= L^{-1}\left[\frac{1}{s^{\alpha}}L[v_0u_{0x} + u_0]\right],\\
&= -\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{e^{-x}t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{e^{-x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},\\
v_1(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}}L[u_0v_{0x} + v_0]\right],\\
&= -\frac{t^{\beta}}{\Gamma(1+\beta)} - \frac{e^{x}t^{\beta}}{\Gamma(1+\beta)} - \frac{t^{2\beta}}{\Gamma(1+2\beta)} - \frac{e^{x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)},\\
u_2(x,t) &= L^{-1}\left[\frac{1}{s^{\alpha}}L[v_1u_{0x} + v_0u_{1x} + u_1]\right]\\
&= -\frac{2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{e^{-x}t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)}\\
&\quad + \frac{e^{-x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{2t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} - \frac{e^{-x}t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} + \frac{e^{-x}t^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)}\\
&\quad - \frac{e^{-x}t^{2\alpha+\beta}\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+2\alpha+\beta)} + \frac{e^{-x}t^{2\alpha+2\beta}\Gamma(1+\alpha+2\beta)}{\Gamma(1+\beta)\Gamma(1+\alpha+\beta)\Gamma(1+2\alpha+2\beta)},
\end{aligned}$$

$$
\begin{aligned}
v_2(x,t) &= -L^{-1}\left[\frac{1}{s^{\alpha}}L[u_1 v_{0x} + u_0 v_{1x} + v_1]\right] \\
&= -\frac{t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^{x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} - \frac{e^{x}t^{2\alpha+\beta}}{\Gamma(1+2\alpha+\beta)} + \frac{2t^{2\beta}}{\Gamma(1+2\beta)} \\
&\quad + \frac{e^{x}t^{2\beta}}{\Gamma(1+2\beta)} + \frac{2t^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)} + \frac{e^{x}t^{\alpha+2\beta}}{\Gamma(1+\alpha+2\beta)} + \frac{t^{3\beta}}{\Gamma(3\beta)} \\
&\quad + \frac{e^{x}t^{\alpha+2\beta}\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)} + \frac{e^{x}t^{2\alpha+2\beta}\Gamma(1+2\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\alpha+\beta)\Gamma(1+2\alpha+2\beta)}, \\
&\ \vdots
\end{aligned}
\tag{3.31}
$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (3.26) and (3.27) in series form is given by

$$
\begin{aligned}
u(x,t) &= e^{-x} + \frac{e^{-x}t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{e^{-x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^{-x}t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots, \\
v(x,t) &= e^{x} - \frac{e^{x}t^{\beta}}{\Gamma(1+\beta)} + \frac{t^{2\beta}}{\Gamma(1+2\beta)} - \frac{e^{x}t^{\alpha+\beta}}{\Gamma(1+\alpha+\beta)} + \frac{e^{x}t^{2\beta}}{\Gamma(1+2\beta)} + \cdots.
\end{aligned}
\tag{3.32}
$$

If we take $\alpha = \beta = 1$, the first few components the solution of (3.26) and (3.27) are as follows:

$$
\begin{aligned}
u_0(x,t) &= e^{-x} + t, \\
v_0(x,t) &= e^{x} + t, \\
u_1(x,t) &= -t + e^{-x}t + \frac{t^2}{2} - \frac{e^{-x}t^2}{2}, \\
v_1(x,t) &= -t - e^{x}t - \frac{t^2}{2} - \frac{e^{x}t^2}{2}, \\
u_2(x,t) &= -\frac{t^2}{2} + e^{-x}t^2 + \frac{t^3}{2} - \frac{e^{-x}t^3}{3} + \frac{e^{-x}t^4}{8}, \\
v_2(x,t) &= \frac{t^2}{2} + e^{x}t^2 + \frac{t^3}{2} + \frac{e^{x}t^3}{3} + \frac{e^{x}t^4}{8}, \\
&\ \vdots
\end{aligned}
\tag{3.33}
$$

and so on. Hence, for this special case, we have

$$u(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{-x+t},$$
$$v(x,t) = e^{x}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^{x-t}, \tag{3.34}$$

which was given in [31].

4. Conclusion

In this work, a homotopy perturbation transformation method which is based on homotopy perturbation method and Laplace transform is used to solve fractional nonlinear partial equations. The nonlinear terms can be easily handled by the use of He's polynomials. It is worth mentioning that the method is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the result, and the size reduction amounts to an improvement of the performance of the approach. The HPTM can be applied for some other engineering system with less computational work.

Acknowledgments

The authors express our thanks to the referees for their fruitful advices and comments. This paper is supported by the National Science Foundation of Shandong Province (Grant no. Y2007A06 and ZR2010Al019) and the China Postdoctoral Science Foundation (Grant no. 20100470783.)

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, Singapore, 2000.

[3] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, p. 77, 2000.

[4] S. Wang and M. Xu, "Axial Couette flow of two kinds of fractional viscoelastic fluids in an annulus," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 2, pp. 1087–1096, 2009.

[5] X. J. Xiaoyun and X. M. Yu, "Analysis of fractional anomalous diffusion caused by an instantaneous point source in disordered fractal media," *International Journal of Non-Linear Mechanics*, vol. 41, no. 1, pp. 156–165, 2006.

[6] J. H. Ma and M. Y. Liu, "Exact solutions for a generalized nonlinear fractional Fokker-Planck equation," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 1, pp. 515–521, 2010.

[7] Y.-Q. Liu and J.-H. Ma, "Exact solutions of a generalized multi-fractional nonlinear diffusion equation in radical symmetry," *Communications in Theoretical Physics*, vol. 52, no. 5, pp. 857–861, 2009.

[8] T. E. Simos, "Closed Newton-Cotes trigonometrically-fitted formulae of high order for long-time integration of orbital problems," *Applied Mathematics Letters*, vol. 22, no. 10, pp. 1616–1621, 2009.

[9] A. A. Kosti, Z. A. Anastassi, and T. E. Simos, "Construction of an optimized explicit Runge-Kutta-Nyström method for the numerical solution of oscillatory initial value problems," *Computers & Mathematics with Applications*, vol. 61, no. 11, pp. 3381–3390, 2011.

[10] Z. A. Anastassi and T. E. Simos, "New trigonometrically fitted six-step symmetric methods for the efficient solution of the Schrödinger equation," *Communications in Mathematical and in Computer Chemistry*, vol. 60, no. 3, pp. 733–752, 2008.

[11] Z. A. Anastassi and T. E. Simos, “Numerical multistep methods for the efficient solution of quantum mechanics and related problems,” *Physics Reports*, vol. 482/483, pp. 1–240, 2009.
[12] G. Adomian, “A review of the decomposition method in applied mathematics,” *Journal of Mathematical Analysis and Applications*, vol. 135, no. 2, pp. 501–544, 1988.
[13] A.-M. Wazwaz and S. M. El-Sayed, “A new modification of the Adomian decomposition method for linear and nonlinear operators,” *Applied Mathematics and Computation*, vol. 122, no. 3, pp. 393–405, 2001.
[14] J. H. He, “Variational iteration method- a kind of non-linear analytical technique: some examples,” *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.
[15] A.-M. Wazwaz, “The variational iteration method for analytic treatment of linear and nonlinear ODEs,” *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 120–134, 2009.
[16] V. S. Ertürk, S. Momani, and Z. Odibat, “Application of generalized differential transform method to multi-order fractional differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.
[17] A. Al-rabtah, V. S. Ertürk, and S. Momani, “Solutions of a fractional oscillator by using differential transform method,” *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1356–1362, 2010.
[18] E. Yusufoglu, “Numerical solution of Duffing equation by the Laplace decomposition algorithm,” *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 572–580, 2006.
[19] Y. Khan, “An effective modification of the laplace decomposition method for nonlinear equations,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 11-12, pp. 1373–1376, 2009.
[20] J. H. He, “Application of homotopy perturbation method to nonlinear wave equations,” *Chaos, Solitons and Fractals*, vol. 26, no. 3, pp. 695–700, 2005.
[21] X. C. Li, M. Y. Xu, and X. Y. Jiang, “Homotopy perturbation method to time-fractional diffusion equation with a moving boundary condition,” *Applied Mathematics and Computation*, vol. 208, no. 2, pp. 434–439, 2009.
[22] S. Momani and Z. Odibat, “Homotopy perturbation method for nonlinear partial differential equations of fractional order,” *Physics Letters A*, vol. 365, no. 5-6, pp. 345–350, 2007.
[23] J.-H. He, “Recent development of the homotopy perturbation method,” *Topological Methods in Nonlinear Analysis*, vol. 31, no. 2, pp. 205–209, 2008.
[24] M. A. Noor, “Iterative methods for nonlinear equations using homotopy perturbation technique,” *Applied Mathematics & Information Sciences*, vol. 4, no. 2, pp. 227–235, 2010.
[25] M. A. Noor, “Some iterative methods for solving nonlinear equations using homotopy perturbation method,” *International Journal of Computer Mathematics*, vol. 87, no. 1–3, pp. 141–149, 2010.
[26] M. Madani, M. Fathizadeh, Y. Khan, and A. Yildirim, “On the coupling of the homotopy perturbation method and Laplace transformation,” *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1937–1945, 2011.
[27] Y. Khan and Q. Wu, “Homotopy perturbation transform method for nonlinear equations using He’s polynomials,” *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 1963–1967, 2011.
[28] A. Ghorbani, “Beyond Adomian polynomials: He polynomials,” *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1486–1492, 2009.
[29] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, “Traveling wave solutions of seventh-order generalized KdV equations using he’s polynomials,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 2, pp. 227–233, 2009.
[30] A. Yildirim, “Application of the homotopy perturbation method for the Fokker-Planck equation,” *International Journal for Numerical Methods in Biomedical Engineering*, vol. 26, no. 9, pp. 1144–1154, 2010.
[31] M. Khan, M. A. Gondal, and S. Kumar, “A novel homotopy perturbation tranform algorithm for linear and nonlinear system of partial differential equations,” *World Applied Sciences Journal*, vol. 12, no. 12, pp. 2352–2357, 2011.