Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 749683, 18 pages
doi:10.1155/2012/749683

Research Article

Semilinear Parabolic Equations on the Heisenberg Group with a Singular Potential

Houda Mokrani[1] and Fatimetou Mint Aghrabatt[2]

[1] *Laboratoire de Mathématiques Raphaël Salem, Université de Rouen, UMR 6085 CNRS, Avenue de l'Université, BP 12, 76801 Saint Etienne du Rouvray, France*
[2] *Laboratoire de Mathématiques Appliquées du Havre, Université du Havre, 25 rue Philippe Lebon, BP 540, 76058 Le Havre Cedex, France*

Correspondence should be addressed to Houda Mokrani, houdamokrani@yahoo.fr

Received 29 October 2011; Accepted 12 December 2011

Academic Editor: Shaher M. Momani

We discuss the asymptotic behavior of solutions for semilinear parabolic equations on the Heisenberg group with a singular potential. The singularity is controlled by Hardy's inequality, and the nonlinearity is controlled by Sobolev's inequality. We also establish the existence of a global branch of the corresponding steady states via the classical Rabinowitz theorem.

1. Introduction

In this paper, we study a class of parabolic equations on the Heisenberg group $\mathbb{H}^d$. Let us recall that the Heisenberg group is the space $\mathbb{R}^{2d+1}$ of the (noncommutative) law of product

$$(x, y, s) \cdot (x', y', s') = (x + x', y + y', s + s' + 2((y \mid x') - (y' \mid x))). \tag{1.1}$$

The left invariant vector fields are

$$X_j = \partial_{x_j} + 2y_j\partial_s, \qquad Y_j = \partial_{y_j} - 2x_j\partial_s, \quad j = 1, \dots, d \quad S = \partial_s = \frac{1}{4}[Y_j,\ X_j]. \tag{1.2}$$

In the sequel we will denote, we will denote $Z_j = X_j$ and $Z_{j+d} = Y_j$ for $j \in \{1, \dots, d\}$. We fix here some notations:

$$z = (x, y) \in \mathbb{R}^{2d}, \qquad w = (z, s) \in \mathbb{H}^d, \qquad \rho(z, s) = \left(|z|^4 + |s|^2\right)^{1/4}, \tag{1.3}$$

where ρ is the Heisenberg distance. Moreover, the Laplacian-Kohn operator on $\mathbb{H}^d$ and Heisenberg gradient is given by

$$\Delta_{\mathbb{H}^d} = \sum_{j=1}^{n} X_j^2 + Y_j^2; \qquad \nabla_{\mathbb{H}^d} = (Z_1, \dots, Z_{2d}). \tag{1.4}$$

Let Ω be an open and bounded domain of $\mathbb{H}^d$, we define thus the associated Sobolev space as follows

$$H^1\left(\Omega, \mathbb{H}^d\right) = \left\{ f \in L^2(\Omega);\ \nabla_{\mathbb{H}^d} f \in L^2(\Omega) \right\}, \tag{1.5}$$

and $H_0^1(\Omega, \mathbb{H}^d)$ is the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega, \mathbb{H}^d)$.

We are concerned in the following semilinear parabolic problem

$$\begin{gathered} \partial_t u - \Delta_{\mathbb{H}^d} u - \mu \frac{|z|^2}{\rho^4} u = \lambda u + |u|^{p-2} u, \quad w \in \Omega,\ t > 0, \\ u(w, 0) = u_0(w), \quad w \in \Omega, \\ u|_{\partial\Omega} = 0, \quad t > 0, \end{gathered} \tag{1.6}$$

where λ is a real constant and $2 < p < 2^*$; the index $2^* = 2 + 2/d$ is the critical index of Sobolev's inequality on the Heisenberg group [1, 2]:

$$\|u\|_{L^{2^*}(\Omega)} \le C_\Omega \|u\|_{H^1(\Omega,\ \mathbb{H}^d)}, \quad \forall u \in H_0^1\left(\Omega, \mathbb{H}^d\right). \tag{1.7}$$

D'Ambrosio in [3] has proved Hardy's inequality: let $u \in H_0^1(\Omega, \mathbb{H}^d)$, it holds that

$$\overline{\mu} \int_\Omega \frac{|z|^2}{\rho(w)^4} |u(w)|^2 dw \le \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2. \tag{1.8}$$

And by the work of Dou et al. [4], we have the following Hardy inequality with remainder terms for all $u \in C_0^\infty(\Omega \setminus \{0\})$:

$$\frac{1}{4} \int_\Omega \left(\ln\left(\frac{R}{\rho(w)} \right) \right)^{-2} \frac{|z|^2}{\rho(w)^4} |u(w)|^2 dw + \overline{\mu} \int_\Omega \frac{|z|^2}{\rho(w)^4} |u(w)|^2 dw \le \|\nabla_{\mathbb{H}^d} u\|_{L^2(\Omega)}^2, \tag{1.9}$$

for any $R \ge R_0$, where $R_0 = \sup_{w \in \Omega} \rho(w)$, and $\overline{\mu} = d^2$. Moreover $\overline{\mu}$ is optimal and it is not attained in $H_0^1(\Omega, \mathbb{H}^d)$.

Stimulated by the recent paper in the Euclidean space $\mathbb{R}^d$ of Karachalios and Zographopoulos [5] which studied the global bifurcation of nontrivial equilibrium solutions on the bounded domain case for a reaction term $f(s) = \lambda s - |s|^2 s$, where λ is a bifurcation parameter; our focus here is devoted to some results concerning the existence of a global attractor for the (1.6) and the existence of a global branch of the corresponding steady states

$$-\Delta_{\mathbb{H}^d} u - \mu \frac{|z|^2}{\rho(w)^4} u = \lambda u + |u|^{p-2} u \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.10}$$

with respect to λ. Let us recall some definitions on semiflows.

Definition 1.1. Let E be a complete metric space, a semiflow is a family of continuous maps $\mathcal{S}(t) : E \longrightarrow E$, $t \geq 0$, satisfying the semigroup identities

$$\mathcal{S}(0) = I, \qquad \mathcal{S}(t + t') = \mathcal{S}(t)\mathcal{S}(t'). \tag{1.11}$$

For $\mathcal{B} \subset E$ and $t \geq 0$,

$$\mathcal{S}(t)\mathcal{B} := \{u(t) = \mathcal{S}(t)u_0; u_0 \in \mathcal{B}\}. \tag{1.12}$$

The positive orbit of u through u_0 is the set

$$\gamma^+(u_0) = \{u(t) = \mathcal{S}(t)u_0, \ t \geq 0\}, \tag{1.13}$$

then the positive orbit of $\mathcal{B}$ is the set $\gamma^+(\mathcal{B}) = \cup_{t\geq 0}\mathcal{S}(t)\mathcal{B}$. The $\mathcal{W}$-limit set of u_0 is

$$\mathcal{W}(u_0) = \{\phi \in E : u(t_j) = \mathcal{S}(t_j)u_0 \longrightarrow \phi, \ t_j \longrightarrow +\infty\}. \tag{1.14}$$

The α-limit set of u_0 is

$$\alpha(u_0) = \{\phi \in E : u(t_j) \longrightarrow \phi, \ t_j \longrightarrow -\infty\}. \tag{1.15}$$

The subset $\mathcal{A}$ attracts a set $\mathcal{B}$ if $\operatorname{dist}(\mathcal{S}(t)\mathcal{B}, \mathcal{A}) \rightarrow 0$, $t \rightarrow +\infty$. $\mathcal{A}$ is invariant if $\mathcal{S}(t)\mathcal{A} = \mathcal{A}$ and for all $t \geq 0$.

The functional $\mathcal{J} : E \rightarrow \mathbb{R}$ is a Lyapunov functional for the semiflow $\mathcal{S}(t)$ if

(i) $\mathcal{J}$ is continuous.

(ii) $\mathcal{J}(\mathcal{S}(t)u_0) \leq \mathcal{J}(\mathcal{S}(t')u_0)$ for $0 \leq t' \leq t$.

(iii) $\mathcal{J}(\mathcal{S}(t))$ is constant for some orbit u and for all $t \in \mathbb{R}$.

We have the following theorem from Ball [6, 7].

Theorem 1.2. *Let $\mathcal{S}(t)$ be an asymptotically compact semiflow, and suppose that there exists a Lyapunov functional $\mathcal{J}$. Suppose further that the set $\mathcal{E}$ is bounded, then $\mathcal{S}(t)$ is dissipative, so there exists a global attractor $\mathcal{A}(t)$.*

For each complete orbit u containing u_0 lying in $\mathcal{A}(t)$, the limit sets $\alpha(u_0)$ and $\mathcal{W}(u_0)$ are connected subsets of $\mathcal{E}$ on which $\mathcal{J}$ is constant.

If $\mathcal{E}$ is totally disconnected (in particular countable), the limits

$$\phi_- = \lim_{t\to-\infty} u(t), \qquad \phi_+ = \lim_{t\to+\infty} u(t) \tag{1.16}$$

exist and are equilibrium points. furthermore, any solution $\mathcal{S}(t)u_0$ tends to an equilibrium point as $t \to \pm\infty$.

The outline of the paper is as follows. In Section 2, we study the existence of an unbounded connected branch of positive solutions of (1.10) with respect to the parameter λ by using global bifurcation theorem introduced by López-Gómez and Molina-Meyer in [8]. In Section 3, we describe the asymptotic behavior of solutions of (1.6) when u_0 has low energy smaller than the mountain pass level.

2. Existence of a Global Branch of the Corresponding Steady States

From the study of spectral decomposition of $H_0^1(\Omega,\mathbb{H}^d)$ with respect to the operator $-\Delta_{\mathbb{H}^d} - \mu(|z|^2/\rho(w)^4)$, where the singular potential V satisfies Hardy's inequality (1.8), we have the following.

Proposition 2.1. *Let $0<\mu\le\overline{\mu}$. Then there exist $0<\lambda_1<\lambda_2\le\lambda_3\le\cdots\le\lambda_k\le\cdots\to+\infty$, such that for each $k\ge1$, the following Dirichlet problem*

$$\begin{gathered} -\Delta_{\mathbb{H}^d}\phi_k - \mu\frac{|z|^2}{\rho(w)^4}\phi_k = \lambda_k\phi_k, \quad \text{in } \Omega \\ \phi_k\big|_{\partial\Omega} = 0 \end{gathered} \tag{2.1}$$

admits a nontrivial solution in $H_0^1(\Omega,\mathbb{H}^d)$. Moreover, $\{\phi_k\}_{k\ge1}$ constitutes an orthonormal basis of Hilbert space $H_0^1(\Omega,\mathbb{H}^d)$.

For the proof of this proposition, we refer to [9].

Remark that the first eigenvalue $\lambda_{1,\mu}$ characterized by

$$\lambda_{1,\mu} = \inf_{u\in H_0^1(\Omega,\mathbb{H}^d)\setminus\{0\}} \frac{\int_\Omega\left(|\nabla_{\mathbb{H}^d}u(w)|^2 - \mu\left(|z|^2/\rho(w)^4\right)|u(w)|^2\right)dw}{\|u\|_{L^2(\Omega)}^2}, \tag{2.2}$$

is simple with a positive associated eigenfunction $\phi_{1,\mu}$.

We discuss the behavior of $\lambda_{1,\mu}$ when $0<\mu<\overline{\mu}$ and $\mu\uparrow\overline{\mu}$.

Proposition 2.2. *Let $0<\mu<\overline{\mu}$ and $\mu\uparrow\overline{\mu}$. Then,*

(i) $(\lambda_{1,\mu})_\mu$ *is a decreasing sequence, and there exist $\lambda_*>0$ such that $\lambda_{1,\mu}\to\lambda_*$.*

(ii) *The corresponding normalized eigenfunction $\phi_{1,\mu}$ converging weakly to 0, in $H_0^1(\Omega,\mathbb{H}^d)$.*

Proof. (i) Let $\mu_1<\mu_2$. The characterization (2.2) of $\lambda_{1,\mu}$ implies that $\lambda_{1,\mu_1}>\lambda_{1,\mu_2}$.

The improved Hardy inequality (1.9) implies that $\lambda_{1,\mu}$ is bounded from below by $C_\Omega = 1/4\ \sup_\Omega(\ln(R/\rho(w)))^{-2}(|z|^2)/(\rho(w)^4)$. So, there exist $\lambda_*>0$ such that $\lambda_{1,\mu}\to\lambda_*$.

(ii) The eigenfunction $\phi_{1,\mu}$ satisfies, for any $v \in C_0^\infty(\Omega)$:

$$\int_\Omega \nabla_{\mathbb{H}^d}\phi_{1,\mu}\overline{\nabla_{\mathbb{H}^d} v}dw - \mu\int_\Omega \frac{|z|^2}{\rho(w)^4}\phi_{1,\mu}\,\overline{v}dw = \lambda_{1,\mu}\int_\Omega \phi_{1,\mu}\,\overline{v}dw. \tag{2.3}$$

We still denote by $\phi_{1,\mu}$ the sequence of normalized eigenfunction, forming a bounded sequence in $H_0^1(\Omega,\mathbb{H}^d)$. Then there exists $u \in H_0^1(\Omega,\mathbb{H}^d)$ such that

$$\begin{aligned} &\phi_{1,\mu} \rightharpoonup u \text{ in } H_0^1\left(\Omega,\mathbb{H}^d\right), \\ &\phi_{1,\mu} \rightarrow u \text{ in } L^q(\Omega), \quad \text{for any } 2 \le q < 2^*. \end{aligned} \tag{2.4}$$

For some fixed small enough $\varepsilon > 0$ and any $v \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega \frac{|z|^2}{\rho(w)^4}(\phi_{1,\mu} - u)\overline{v}dw &\le \|v\|_{L^\infty(\Omega)}\left(\int_\Omega |\phi_{1,\mu} - u|^{Q-\varepsilon/Q-2-\varepsilon}dw\right)^{Q-2-\varepsilon/Q-\varepsilon} \\ &\times \left(\int_\Omega\left(\frac{|z|}{\rho(w)^2}\right)^{Q-\varepsilon}dw\right)^{2/Q-\varepsilon}. \end{aligned} \tag{2.5}$$

Thus,

$$\int_\Omega \frac{|z|^2}{\rho(w)^4}\phi_{1,\mu}\,\overline{v}dw \longrightarrow \int_\Omega \frac{|z|^2}{\rho(w)^4}u\overline{v}dw, \quad \text{as } \mu \uparrow \overline{\mu}. \tag{2.6}$$

We assume that $u \ne 0$, so passing to the limit in (2.3), we get that u is a nontrivial solution of the problem

$$-\Delta_{\mathbb{H}^d}u - \overline{\mu}\frac{|z|^2}{\rho(w)^4}u = \lambda_* u, \quad u \in H_0^1\left(\Omega,\mathbb{H}^d\right). \tag{2.7}$$

However, $\overline{\mu}$ is not achieved in $H_0^1(\Omega,\mathbb{H}^d)$, so $u = 0$. □

Thanks to Hardy's inequality (1.8) and Poincaré's inequality,

$$\|u\|_\mu = \left(\int_\Omega\left[|\nabla_{\mathbb{H}^d}u(w)|^2 - \mu\frac{|z|^2}{\rho(w)^4}|u(w)|^2\right]dw\right)^{1/2}, \tag{2.8}$$

is equivalent to the norm on $H_0^1(\Omega,\mathbb{H}^d)$ for all $0 \le \mu < \overline{\mu}$, so that we will use $\|\cdot\|_\mu$ as the norm of $H_0^1(\Omega,\mathbb{H}^d)$.

Theorem 2.3. *Let $\Omega \in \mathbb{H}^d$ a bounded domain and assume that $0 < \mu < \overline{\mu}$. Then, there exists an unbounded component $C^+_{\lambda_{1,\mu}} \subset \mathbb{R} \times H_0^1(\Omega,\mathbb{H}^d)$ of the set of positive solutions of* (1.10) *bifurcating from* $(\lambda_{1,\mu}, 0)$.

Proof. We introduce the Banach space $X = H_0^1(\Omega, \mathbb{H}^d)$, and the inner product in X is given by

$$\langle u, v\rangle_X \equiv \int_\Omega \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(w)^4} u\overline{v}\right] dw - \frac{\lambda_{1,\mu}}{2} \int_\Omega u\overline{v} dw. \tag{2.9}$$

Let

$$a(u, v) = \int_\Omega uv dw, \qquad \forall u, v \in X. \tag{2.10}$$

The bilinear form $a(u, v)$ is continuous in X, so the Riesz representation theorem implies that there exist a bounded linear operator L such that

$$a(u, v) = \langle Lu, v\rangle, \qquad \forall u, v \in X. \tag{2.11}$$

The operator L is self-adjoint and compact and its largest eigenvalue is characterized by

$$\nu_1 = \sup_{u\in X} \frac{\langle Lu, u\rangle}{\langle u, u\rangle_X} = \sup_{u\in X} \frac{\|u\|_{L^2(\Omega)}}{\int_\Omega \left[|\nabla_{\mathbb{H}^d} u|^2 - \mu\left(|z|^2/\rho(w)^4\right) |u|^2\right] dw} = \frac{1}{\lambda_{1,\mu}}. \tag{2.12}$$

We define the following energy functional on $H_0^1(\Omega, \mathbb{H}^d)$:

$$I_{\mu,\lambda}(u) = \frac{1}{2} \int_\Omega \left[|\nabla_{\mathbb{H}^d} u|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u|^{\,2}\right] dw - \frac{1}{p} \int_\Omega |u|^p dw - \frac{\lambda}{2} \int_\Omega |u|^2 dw. \tag{2.13}$$

Similar to the classical case, $I_{\mu,\lambda}(\cdot)$ is well defined on $H_0^1(\Omega, \mathbb{H}^d)$ and belongs to $C^1(H_0^1(\Omega, \mathbb{H}^d); \mathbb{R})$, and we have

$$\left\langle I'_{\mu,\lambda}(u), v\right\rangle = \int_\Omega \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(w)^4} u\overline{v} - |u|^{p-2} u\overline{v} - \lambda u\overline{v}\right] dw, \tag{2.14}$$

for any $v \in H_0^1(\Omega, \mathbb{H}^d)$. Let $N(\lambda, .) : \mathbb{R} \times X \to X^*$, X^* is the dual space of X, defined as by

$$\langle N(\lambda, u), v\rangle = \int_\Omega \left[\nabla_{\mathbb{H}^d} u \overline{\nabla_{\mathbb{H}^d} v} - \mu \frac{|z|^2}{\rho(w)^4} u\overline{v} - |u|^{p-2} u\overline{v} - \lambda u\overline{v}\right] dw, \tag{2.15}$$

for all $v \in X$. Since $I'_{\mu,\lambda}(u)$ is a bounded linear functional, $N(\lambda, \cdot)$ is well defined, and $N(\lambda, u) = u - G(\lambda, u)$, where $G(\lambda, u) = \lambda Lu + H(u)$,

$$\langle H(u), v\rangle = \int_\Omega |u|^{p-2} u\overline{v} dw, \qquad \forall v \in X. \tag{2.16}$$

So,

$$|\langle H(u), v\rangle| \leq \|u\|_{L^p(\Omega)}^{p-1} \|v\|_{L^p(\Omega)}. \tag{2.17}$$

By Sobolev embedding Sobolev theorem [10], we have

$$\frac{1}{\|u\|_X} |\langle H(u), v\rangle| \leq \|u\|_X^{p-2} \|v\|_X. \tag{2.18}$$

Then,

$$\lim_{\|u\|_X \to 0} \frac{\|H(u)\|_{X^*}}{\|u\|_X} = \lim_{\|u\|_X \to 0} \sup_{\|v\|_X \leq 1} \frac{1}{\|u\|_X} |\langle H(u), v\rangle| = 0. \tag{2.19}$$

Consequently, hypotheses (HL) and (HR) of [8] are hold. If $u \in H_0^1(\Omega, \mathbb{H}^d) \setminus \{0\}$ is a nonnegative solution of (1.10), then it follows from the strong maximum principle of J.-M. Bony [11] and the generalization of the Hopf boundary point lemma on the Heisenberg group [12], that u lies in the interior of the cone:

$$\operatorname{int}(\mathcal{P}) = \left\{ u \in H_0^1\left(\Omega, \mathbb{H}^d\right) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega \right\}. \tag{2.20}$$

Hence, the assumption (HP) of [8] is fulfilled. □

Remark 2.4. According to the theory of Rabinowitz [13], we can see that there is a continuum $C_{\lambda_{1,\mu}}$ of the set of nontrivial solutions of (1.10), and the continuum $C_{\lambda_{1,\mu}}$ consists of two subcontinua $C_{\lambda_{1,\mu}}^+$ and $C_{\lambda_{1,\mu}}^-$. However, this does not necessarily implies that the subcontinuum $C_{\lambda_{1,\mu}}^+$ satisfies the global alternative of Rabinowitz [13] by the reasons already explained by Dancer [14], López-Gómez and Molina-Meyer [8, 15]. Instead, the existence of a global subcontinuum $C_{\lambda_{1,\mu}}^+$ of the set of positive solutions follows by slightly adapting [8, Theorem 1.1].

3. Asymptotic Behavior of Solutions for Problem (1.6)

Similar to [16, 17], we are interested here in the description of the behavior of solutions of (1.6) when u_0 has low energy smaller than the mountain pass level

$$\begin{aligned} c_{\mu,\lambda} &= \inf_{h\in\Gamma} \max_{t\in[0,1]} I_{\mu,\lambda}(h(t)), \text{ where} \\ \Gamma &= \left\{ h \in C\left([0,1]; H_0^1\left(\Omega, \mathbb{H}^d\right)\right); h(0) = 0 \text{ and } h(1) = e \right\}. \end{aligned} \tag{3.1}$$

In view of [9], since $2 < p < 2^*$, the functional $I_{\mu,\lambda}$ satisfies the Palais-Smale condition and admits at least a positive solution (called mountain pass solution).

Proposition 3.1. *Let* $u_0 \in H_0^1(\Omega, \mathbb{H}^d)$, $\lambda > 0$, *and* $0 < \mu < \overline{\mu}$, *the problem* (1.6) *has a unique weak solution* u *such that*

$$u \in C\Big([0,T); H_0^1\Big(\Omega, \mathbb{H}^d\Big)\Big) \cap C^1\Big([0,T); H^{-1}\Big(\Omega, \mathbb{H}^d\Big)\Big), \tag{3.2}$$

and we have

$$\frac{d}{dt} I_{\mu,\lambda}(u(t)) = -\|\partial_t u\|_{L^2(\Omega)}^2. \tag{3.3}$$

Proof. By means of the Hill-Yosida theorem, $\mathcal{T}(t) = \{e^{-tL_\mu}\}_{t\geq 0}$ is the semigroup generated by the operator $L_\mu = -\Delta_{\mathbb{H}^d} - \mu|z|^2/\rho(z,s)^4$. Let f the function defined by $f(t) = \lambda t + |t|^{p-2}t$, for $t \in \mathbb{R}$. Since $f : H_0^1(\Omega, \mathbb{H}^d) \to H^{-1}(\Omega)$ is locally Lipschitz, so by Pazy [18, Theorem 1.4] or Cazenave and Haraux [19, Theorem 6.2.2], there exists a unique solution of (1.6) defined on a maximal interval $[0, T_{\max})$, where $0 < T_{\max} \leq +\infty$ and

$$u \in C\Big([0,T); H_0^1\Big(\Omega, \mathbb{H}^d\Big)\Big) \cap C^1\Big([0,T); H^{-1}(\Omega)\Big), \tag{3.4}$$

satisfying the variation of constants formula

$$u(t) = \mathcal{T}(t)u_0 + \int_0^t \mathcal{T}(t-\tau) f(u(\tau))\ d\tau. \tag{3.5}$$

Moreover, if $T_{\max} < +\infty$, we say that $T_{\max}$ is blow-up time, whereas if $T_{\max} = +\infty$, we say that u is global solution.

We will show that u satisfies (3.3): Let $u \in D(L_\mu)$, ($D(L_\mu)$ is the domain of definition of L_μ), and $t \in [0,T), T < T_{\max}$. Since $I_{\mu,\lambda} \in C^1(H_0^1(\Omega, \mathbb{H}^d); \mathbb{R})$, we have

$$\begin{aligned} \left\langle I'_{\mu,\lambda}(u), \Delta_{\mathbb{H}^d} u + \mu \frac{|z|^2}{\rho(w)^4} u + f(u) \right\rangle &= -\int_\Omega \left| \Delta_{\mathbb{H}^d} u + \mu \frac{|z|^2}{\rho(w)^4} u + f(u) \right|^2 dw \\ &= -\int_\Omega |\partial_t u|^2 dw. \end{aligned} \tag{3.6}$$

Set $g(t) = f(u(t))$, and let $g_n \in C^1([0,T]; H_0^1(\Omega, \mathbb{H}^d))$, $u_{0n} \in D(L_\mu)$ such that

$$\begin{gathered} g_n \longrightarrow g \text{ in } C^1\Big([0,T]; H_0^1\Big(\Omega, \mathbb{H}^d\Big)\Big), \\ u_{0n} \longrightarrow u_0 \text{ in } H_0^1\Big(\Omega, \mathbb{H}^d\Big). \end{gathered} \tag{3.7}$$

Define $u_n(t) = \mathcal{T}(t)u_{0n} + \int_0^t \mathcal{T}(t-\tau) g_n(\tau) d\tau$, then, $u_n \in C^1([0,T]; H_0^1(\Omega, \mathbb{H}^d))$ and satisfies

$$\partial_t u_n - \Delta_{\mathbb{H}^d} u_n - \mu V u_n = g_n \qquad u_n \longrightarrow u \text{ in } H_0^1\Big(\Omega, \mathbb{H}^d\Big). \tag{3.8}$$

Thus, from (3.6),

$$I_{\mu,\lambda}(u_n(t)) - I_{\mu,\lambda}(u_{0n}) = \int_0^t \left\langle I'_{\mu,\lambda}(u_n(\tau)), \Delta_{\mathbb{H}^d} u_n + \mu \frac{|z|^2}{\rho(w)^4} u_n + g_n(\tau) \right\rangle d\tau$$
$$= -\int_0^t \|\partial_\tau u_n(\tau)\|^2_{L^2(\Omega)} d\tau + \int_0^t \left\langle I'_{\mu,\lambda}(u_n(\tau)), g_n(\tau) - f(u_n(\tau)) \right\rangle d\tau. \tag{3.9}$$

Passing to the limit, we deduce (3.3). □

Next, we introduce the following sets:

$$\mathcal{O}^+ \equiv \left\{ u \in H_0^1\left(\Omega, \mathbb{H}^d\right) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \left\langle I'_{\mu,\lambda}(u), u \right\rangle > 0 \right\},$$
$$\mathcal{O}^- \equiv \left\{ u \in H_0^1\left(\Omega, \mathbb{H}^d\right) : I_{\mu,\lambda}(u) < c_{\mu,\lambda}; \left\langle I'_{\mu,\lambda}(u), u \right\rangle < 0 \right\}, \tag{3.10}$$
$$\mathcal{N} \equiv \left\{ u \in H_0^1\left(\Omega, \mathbb{H}^d\right) : \left\langle I'_{\mu,\lambda}(u), u \right\rangle = 0 \right\}.$$

$\mathcal{N}$ is named the Nehari manifold relative to $I_{\mu,\lambda}$. The mountain-pass level $c_{\mu,\lambda}$ defined in (3.1) may also be characterized as

$$c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u). \tag{3.11}$$

Theorem 3.2. *If there exist $t_0 \geq 0$ such that $I_{\mu,\lambda}(u(t_0)) \leq 0$, then $u(t)$ blows up in finite time.*

Proof. Let $t_0 \geq 0$ such that $I_{\mu,\lambda}(u(t_0)) \leq 0$, and we suppose that $u(t)$ is a global solution for the problem (1.6). Since $u(t)$ satisfy (3.3), we have

$$I_{\mu,\lambda}(u(t_0)) = I_{\mu,\lambda}(u(t)) + \int_{t_0}^t \|\partial_\tau u(\tau)\|^2_{L^2(\Omega)} d\tau. \tag{3.12}$$

Set $g(t) \equiv \int_\Omega |u(t)|^2 dw$, then

$$\frac{d}{dt} g(t) = \int_\Omega u(t) \partial_t u(t) dw$$
$$= -2 \int_\Omega \left[|\nabla_{\mathbb{H}^d} u(t)|^2 - \mu \frac{|z|^2}{\rho(w)^4} |u(t)|^2 \right] dw$$
$$+ 2\lambda \int_\Omega |u(t)|^2 dw + 2 \int_\Omega |u(t)|^p dw \tag{3.13}$$
$$= 4 \int_{t_0}^t \|\partial_\tau u(\tau)\|^2_{L^2(\Omega)} d\tau - 4 I_{\mu,\lambda}(u(t_0)) + 2\left(1 - \frac{2}{p}\right) \int_\Omega |u(t)|^p dw$$
$$\geq 2\left(1 - \frac{2}{p}\right) \int_\Omega |u(t)|^p dw > 0.$$

Hence, we get for $t \geq t_0$, $g(t) \geq g(t_0) = \int_\Omega |u(t_0)|^2 dw$.

Let $\epsilon \in (1, p/2)$, so we deduce by (3.13), that for any $t \geq t_0$:

$$\begin{aligned}
-\frac{1}{\epsilon - 1}\frac{d}{dt}g^{1-\epsilon}(t) &= g^{-\epsilon}(t)\frac{d}{dt}g(t) \\
&\geq 2\left(1 - \frac{2}{p}\right)g^{-\epsilon}(t)\int_\Omega |u(t)|^p dw \\
&\geq Cg^{-\epsilon}(t)\left(\int_\Omega |u(t)|^2 dw\right)^{p/2} \\
&\geq C\left(\int_\Omega |u(t_0)|^2 dw\right)^{(p/2)-\epsilon}.
\end{aligned} \tag{3.14}$$

Hence, for any $t \geq t_0$ sufficiently large, we have

$$\begin{aligned}
0 < \left(\int_\Omega |u(t)|^2 dw\right)^{1-\epsilon} &= g^{1-\epsilon}(t) \\
&\leq g^{1-\epsilon}(t_0) + C(\epsilon - 1)g^{(p/2)-\epsilon}(t_0)(t_0 - t).
\end{aligned} \tag{3.15}$$

Then

$$-1 < C(\epsilon - 1)g^{(p/2)-1}(t_0)(t_0 - t), \tag{3.16}$$

and so $t < t_0 + [C(\epsilon - 1)g^{p/2-1}(t_0)]^{-1}$, which is a contradiction. □

Theorem 3.3. *Assume that $u_0 \in \mathcal{O}^+$ and $\lambda < \lambda_{1,\mu}$, then the problem* (1.6) *admits a global solution $u(t)$. Moreover, there exists a positive number α such that*

$$\|u(t)\| = O\left(e^{-\alpha t}\right), \quad \text{as } t \longrightarrow +\infty. \tag{3.17}$$

Proof. Let $u_0 \in \mathcal{O}^+$, and let $u(t) = u(w, t, u_0)$ be the unique solution established in Proposition 3.1. From inequality (3.3), we have that $t \mapsto I_{\mu,\lambda}(u(t))$ is strictly decreasing, so

$$I_{\mu,\lambda}(u(t)) \leq I_{\mu,\lambda}(u_0) \leq c_{\mu,\lambda}. \tag{3.18}$$

Suppose there exists $t^* \in (0, T_{\max})$ such that $u(t^*) \notin \mathcal{O}^+$. Then,

$$\left\langle I'_{\mu,\lambda}(u(t^*)), u(t^*)\right\rangle \leq 0. \tag{3.19}$$

Moreover, since the application $t \mapsto \langle I'_{\mu,\lambda}(u(t)), u(t)\rangle$ is continuous, there exists $t_0 \in (0, t^*]$ such that

$$\left\langle I'_{\mu,\lambda}(u(t_0)), u(t_0)\right\rangle = 0. \tag{3.20}$$

Hence, $u(t_0) = 0$ in Ω or $u(t_0) \in \mathcal{N}$. If $u(t_0) = 0$ in Ω, then by the uniquess of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [t_0, T_{\max})$. Thus, $u(t)$ is global by extending to 0 for all $t \geq T_{\max}$, and so $I_{\mu,\lambda}(u(t)) > 0$ for any $t \geq 0$ by Theorem 3.2. But $I_{\mu,\lambda}(u(t_0)) = 0$, which is a contradiction. So, we conclude that $u(t) \in \mathcal{O}^+$ for all $t \in [t_0, T_{\max})$.

On other hand, we can write

$$\begin{aligned} I_{\mu,\lambda}(u(t)) = & \frac{1}{p}\left\langle I'_{\mu,\lambda}(u(t)), u(t)\right\rangle \\ & + \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega}\left[|\nabla_{\mathbb{H}^d}u(t)|^2 - \mu\frac{|z|^2}{\rho(w)^4}|u(t)|^2\right]dw - \left(\frac{1}{2}-\frac{1}{p}\right)\lambda\int_{\Omega}|u(t)|^2 dw \\ & > \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega}\left[|\nabla_{\mathbb{H}^d}u(t)|^2 - \mu\frac{|z|^2}{\rho(w)^4}|u(t)|^2\right]dw - \left(\frac{1}{2}-\frac{1}{p}\right)\lambda\int_{\Omega}|u(t)|^2 dw \\ & \geq \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\frac{\lambda}{\lambda_{1,\mu}}\right)\|u(t)\|_{\mu}^2 > 0. \end{aligned} \tag{3.21}$$

Since $u(t)$ satisfy (3.3), we have

$$\int_{t_0}^{t}\|\partial_\tau u(\tau)\|_{L^2(\Omega)}^2 d\tau + \left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\frac{\lambda}{\lambda_{1,\mu}}\right)\|u(t)\|_{\mu}^2 \leq I_{\mu,\lambda}(u(t_0)) < c_{\mu,\lambda}. \tag{3.22}$$

Then we have

$$\int_{t_0}^{t}\|\partial_\tau u(\tau)\|_{L^2(\Omega)}^2 d\tau < c_{\mu,\lambda}, \qquad \|u(t)\|_{\mu}^2 < \left[\left(\frac{1}{2}-\frac{1}{p}\right)\left(1-\frac{\lambda}{\lambda_{1,\mu}}\right)\right]^{-1} c_{\mu,\lambda}, \tag{3.23}$$

which implies that $u(t)$ is a global solution of the problem (1.6), and $\mathcal{O}^+$ is invariant set. Letting $t \to +\infty$ in (3.23), the integral $\int_{t_0}^{t}\|\partial_\tau u(\tau)\|_{L^2(\Omega)}^2 d\tau$ is finitely determined. Therefore, there exists a sequence (t_n) $n \geqslant 0$ with $t_n \to +\infty$ as $n \to +\infty$ such that

$$\int_{\Omega}|\partial_\tau u(t_n)|^2 dw \longrightarrow 0, \qquad u(t_n) \rightharpoonup v \text{ in } H_0^1\left(\Omega, \mathbb{H}^d\right). \tag{3.24}$$

Letting $t_n \to +\infty$, we obtain that $v \in H_0^1(\Omega, \mathbb{H}^d)$ is a solution of problem (1.10). So

$$\left\langle I'_{\mu,\lambda}(v), v\right\rangle = 0. \tag{3.25}$$

If $v \neq 0$, then $v \in \mathcal{N}$, and so

$$I_{\mu,\lambda}(v) \geq c_{\mu,\lambda}. \tag{3.26}$$

Since $u(t_n)$ satisfies (3.3), it follows by Hölder inequality and from (3.24), that

$$\begin{aligned}
\left|\left\langle I'_{\mu,\lambda}(u(t_n,.)),u(t_n,.)\right\rangle\right| &\le \left|\int_\Omega u(t_n,w)\partial_t u(tn,w)dw\right| \\
&\le \|u(t_n,.)\|_{L^2(\Omega)}\|\partial_t u(t_n,.)\|_{L^2(\Omega)} \\
&\le \sqrt{\lambda_{1,\mu}}\|u(t_n,.)\|_\mu\|\partial_t u(t_n,.)\|_{L^2(\Omega)} \\
&\le C\|\partial_t u(t_n,.)\|_{L^2(\Omega)}.
\end{aligned} \tag{3.27}$$

Therefore,

$$\lim_{n\to+\infty}\left\langle I'_{\mu,\lambda}(u(t_n)),u(t_n)\right\rangle = 0. \tag{3.28}$$

We deduce by (3.22), (3.25), and (3.28) that

$$\begin{aligned}
I_{\mu,\lambda}(v) &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left[|\nabla_{\mathbb{H}^d}v(w)|^2-\mu\frac{|z|^2}{\rho^4(w)}|v(w)|^2\right]dw-\left(\frac{1}{2}-\frac{1}{p}\right)\lambda\int_\Omega|v(w)|^2dw \\
&\le \lim_{n\to+\infty}\left(\frac{1}{2}-\frac{1}{p}\right)\int_\Omega\left[|\nabla_{\mathbb{H}^d}u(t_n,w)|^2-\mu\frac{|z|^2}{\rho^4(w)}|v(t_n,w)|^2\right]dw \\
&\quad -\lim_{n\to+\infty}\left(\frac{1}{2}-\frac{1}{p}\right)\lambda\int_\Omega|v(t_n,w)|^2dw+\lim_{n\to+\infty}\left\langle I'_{\mu,\lambda}(u(t_n)),u(t_n)\right\rangle \\
&\le \lim_{n\to+\infty}I_{\mu,\lambda}(u(t_n)) \\
&\le I_{\mu,\lambda}(u_0)<c_{\mu,\lambda},
\end{aligned} \tag{3.29}$$

which contradicts (3.26), and so $v = 0$ in Ω. Hence, by (3.24), we have

$$u(t_n,.)\longrightarrow 0 \text{ in } L^q(\Omega), \quad 2\le q<2^*. \tag{3.30}$$

Since

$$\|u(t_n,.)\|_\mu^2=\left\langle I'_{\mu,\lambda}(u(t_n)),u(t_n)\right\rangle+\lambda\int_\Omega|u(t_n,w)|^2dw+\int_\Omega|u(t_n,w)|^pdw\to 0, \quad \text{as } n\longrightarrow+\infty, \tag{3.31}$$

we have

$$u(t_n,.)\longrightarrow 0 \text{ in } H_0^1\left(\Omega,\mathbb{H}^d\right), \quad \text{as } n\to+\infty. \tag{3.32}$$

For simplicity, let us denote by t the divergent sequence and by $u(t) = u(t_n, w)$. We have from (3.29) that

$$\begin{aligned} I_{\mu,\lambda}(u(t)) &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} \left[|\nabla_{\mathbb{H}^d} u(t,w)|^2 - \mu \frac{|z|^2}{\rho^4(w)} |u(t,w)|^2 \right] dw \\ &\quad - \left(\frac{1}{2} - \frac{1}{p}\right) \lambda \int_{\Omega} |u(t,w)|^2 dw \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u(t)\|_{\mu}^2 - \lambda \left(\frac{1}{2} - \frac{1}{p}\right) \|u(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.33}$$

So, due to (3.3) we have

$$\begin{aligned} \|u(t)\|_{\mu}^2 &= \frac{2p}{p-2} I_{\mu,\lambda}(u(t)) + \lambda \|u(t)\|_{L^2(\Omega)}^2 \\ &\leq \frac{2p}{p-2} I_{\mu,\lambda}(u_0) + \lambda \|u(t)\|_{L^2(\Omega)}^2 \\ &< \frac{2p}{p-2} c_{\mu,\lambda} + o(1). \end{aligned} \tag{3.34}$$

Therefore, there exists t_0 such that for all $t \geq t_0$,

$$\|u(t)\|_{\mu}^2 \leq \frac{2p}{p-2} c_{\mu,\lambda}. \tag{3.35}$$

On the other hand,

$$\begin{aligned} \int_{\Omega} |u(t,w)|^p dw &\leq C_{\Omega}^p \left(\frac{\overline{\mu}}{\overline{\mu} - \mu} \right)^{p/2} \|u(t)\|_{\mu}^p \\ &\leq C_{\Omega}^p \left(\frac{\overline{\mu}}{\overline{\mu} - \mu} \right)^{p/2} \left[\frac{2p}{p-2} c_{\mu,\lambda} \right]^{p-2/2} \|u(t)\|_{\mu}^2. \end{aligned} \tag{3.36}$$

Let $C_1 = C_{\Omega}^p (\overline{\mu}/\overline{\mu} - \mu)^{p/2} [(2p/p-2) c_{\mu,\lambda}]^{(p-2)/2}$, we have

$$\begin{aligned} (1 - C_1) \|u(t)\|_{\mu}^2 &\leq \|u(t)\|_{\mu}^2 - \|u(t)\|_{L^p(\Omega)}^p \\ &\leq \left\langle I'_{\mu,\lambda}(u(t)), u(t) \right\rangle + \lambda \|u(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.37}$$

Let us recall that if we set $g(t) \equiv \int_\Omega |u(t,w)|^2 dw$, then

$$\begin{aligned}\frac{d}{dt}g(t) &= 2\int_\Omega u(t,w)\partial_t u(t,w)dw \\ &= -2\int_\Omega\left[|\nabla_{\mathbb{H}^d}u(t,w)|^2 - \mu\frac{|z|^2}{\rho^4(w)}|u(t,w)|^2\right]dw + 2\lambda\int_\Omega |u(t,w)|^2 dw \\ &\quad + \lambda\int_\Omega |v(t_n,w)|^p dw \\ &= -2\left\langle I'_{\mu,\lambda}(u(t)), u(t)\right\rangle.\end{aligned} \tag{3.38}$$

So we get from (3.22) that for any $t \geq t_0$, we have

$$\begin{aligned}\int_t^{+\infty}\left\langle I'_{\mu,\lambda}(u(\tau)), u(\tau)\right\rangle \mathrm{d}\tau &= \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{2\lambda_{\mu,\lambda}}\|u(t)\|^2_\mu \leq \frac{p}{(p-2)\lambda_{\mu,\lambda}}I_{\mu,\lambda}(u(t)).\end{aligned} \tag{3.39}$$

So from (3.37) and (3.39), we have for any $t \geq t_0$ that

$$\begin{aligned}\int_t^{+\infty} I_{\mu,\lambda}(u(\tau))d\tau &\leq \frac{1}{2\lambda_{\mu,\lambda}}\int_t^{+\infty}\|u(\tau)\|^2_\mu d\tau \\ &\leq (1-C_1)^{-1}\frac{1}{2\lambda_{\mu,\lambda}}\left[\int_t^{+\infty}\left\langle I'_{\mu,\lambda}(u(\tau)), u(\tau)\right\rangle d\tau + \lambda\int_t^{+\infty}\|u(\tau)\|^2_{L^2(\Omega)}d\tau\right] \\ &\leq \frac{(1-C_1)^{-1}}{2\lambda^2_{\mu,\lambda}}\frac{p}{p-2}I_{\mu,\lambda}(u(t)) + (1-C_1)^{-1}\frac{\lambda}{2\lambda_{\mu,\lambda}}\int_t^{+\infty}\|u(\tau)\|^2_{L^2(\Omega)}d\tau.\end{aligned} \tag{3.40}$$

Since $\lim_{t\to+\infty}\|u(t)\|^2_{L^2(\Omega)} = 0$, there exists $t_1 > t_0$ such that for any $t \geq t_1$, we have

$$\int_t^{+\infty} I_{\mu,\lambda}(u(\tau))d\tau \leq \frac{(1-C_1)^{-1}}{2\lambda^2_{\mu,\lambda}}\frac{p}{p-2}I_{\mu,\lambda}(u(t)). \tag{3.41}$$

Thus,

$$\int_t^{+\infty} I_{\mu,\lambda}(u(\tau))d\tau \leq C(t_1)e^{-\alpha t}, \tag{3.42}$$

with $\alpha = ((1-C_1)^{-1}/2\lambda^2_{\mu,\lambda})(p/p-2)$. But we remark that

$$I_{\mu,\lambda}(u(t+1)) \leq \int_t^{t+1} I_{\mu,\lambda}(u(\tau))d\tau < \int_t^{+\infty} I_{\mu,\lambda}(u(\tau))d\tau, \tag{3.43}$$

hence, we deduce that for any $t \geq t_1$, we have

$$I_{\mu,\lambda}(u(t+1)) < C(t_1)e^{-\alpha t}, \tag{3.44}$$

and we can conclude that for any $t \geq t_1$, we have

$$\|u(t)\|_{\mu} = O\left(e^{-\alpha t}\right). \tag{3.45}$$

□

Remark 3.4. for small u_0, Theorem 3.3 is an immediate consequence from the fact that, according to the linearized stability principle, the trivial solution is linearly asymptotically stable. In other words, from the fact that the principle eigenvalue of the linearization at $u = 0$ is positive.

Questions of stability for nonlinear systems are frequently resolved via linearized stability or Lyapunov-type methods. Here, we proved the asymptotic stability under Lyapunov function to obtain estimates in $L^p(\Omega)$.

Corollary 3.5. *Assume that $u_0 \in \mathcal{O}^+$ and $\lambda < \lambda_{1,\mu}$. Then any solution $u(t)$ of* (1.6) *tends to the trivial equilibrium point, as $t \to +\infty$.*

Proof. It follows from (3.45) that the semiflow $\mathfrak{T}(t)$ is eventually bounded, see [7]. Since the resolvent of the operator L_μ is compact, $\mathfrak{T}(t)$ is compact for $t > 0$ (see [20, Theorem 3.3], thus by [18, Corollary 3.2.2], $\mathfrak{T}(t)$ is asymptotically smooth and so by [7, Proposition 2.3] is asymptotically compact. It remains to shows that $\mathcal{E}$, the set of equilibrium points of $\mathfrak{T}(t)$, is bounded: $u(t) \in \mathcal{E}$, so $u(t) \in \mathcal{N}$. Then from (3.3) and Poincaré's inequality, we have

$$\begin{aligned}\left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_{\mu}^{2} &= I_{\mu,\lambda}(u(t)) + \lambda\left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_{L^2(\Omega)}^{2} \\ &\leq I_{\mu,\lambda}(u_0) + \lambda\left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_{L^2(\Omega)}^{2} \\ &\leq c_{\mu,\lambda} + \frac{\lambda}{\lambda_{1,\mu}}\left(\frac{1}{2}-\frac{1}{p}\right)\|u(t)\|_{\mu}^{2},\end{aligned} \tag{3.46}$$

which implies that the set $\mathcal{E}$ is bounded. Then, by Theorem 1.2, $\mathfrak{T}(t)$ is dissipative and by (3.45), we have $\operatorname{dist}(\mathfrak{T}(t)\mathcal{B}, 0) \to 0$ as $t \to +\infty$, for every bounded set $\mathcal{B} \subset H_0^1(\Omega, \mathbb{H}^d)$. So, we conclude that the global attractor $\mathcal{A} = 0$, and that any solution $u(t) = \mathcal{S}(t)u_0$ tends to the trivial equilibrium point as $t \to +\infty$, when $u_0 \in \mathcal{O}^+$. □

Theorem 3.6. *Assume that $u_0 \in \mathcal{O}^-$. Then the solution $u(t)$ of the problem* (1.6) *blows up in finite time.*

Proof. Let $u_0 \in \mathcal{O}^-$, and let $u(t) = u(w, t, u_0)$ be the unique solution, the existence of which has been proved in Proposition 3.1. From the inequality (3.3), we have that $t \mapsto I_{\mu,\lambda}(u(t))$ is strictly decreasing, so

$$I_{\mu,\lambda}(u(t)) = I_{\mu,\lambda}(u_0) = c_{\mu,\lambda}. \tag{3.47}$$

Suppose there exists $\tilde{t} \in (0, T_{\max})$ such that $u(\tilde{t}) \notin \mathcal{O}^-$. Then

$$\left\langle I'_{\mu,\lambda}(u(t)), u(t) \right\rangle \geq 0. \tag{3.48}$$

And since the application $t \mapsto \langle I'_{\mu,\lambda}(u(t)), u(t) \rangle$ is continuous, there exists $\tilde{t}_0 \in (0, \tilde{t}]$ such that

$$\left\langle I'_{\mu,\lambda}(u(t_0)), u(t_0) \right\rangle = 0. \tag{3.49}$$

Hence, $u(\tilde{t}_0) = 0$ in Ω or $u(\tilde{t}_0) \in \mathcal{N}$. If $u(\tilde{t}_0) = 0$ in Ω, then by the uniquess of $u(t)$, we conclude that $u(t) = 0$ for any $t \in [\tilde{t}_0, T_{\max})$. Thus, $u(t)$ is global by extending to 0 for all $t \geq T_{\max}$, and thanks to Theorem 3.2, $I_{\mu,\lambda}(u(t)) > 0$ for any $t \geq 0$. But $I_{\mu,\lambda}(u(\tilde{t}_0)) = 0$, which is a contradiction, and so $u(\tilde{t}_0) \in \mathcal{N}$. But by [21],

$$c_{\mu,\lambda} = \inf_{u \in \mathcal{N}} I_{\mu,\lambda}(u), \tag{3.50}$$

then $c_{\mu,\lambda} \leq I_{\mu,\lambda}(u(\tilde{t}_0))$, which contradicts (3.47). So, we conclude that $u(t) \in \mathcal{O}^-$ for all $t \in [\tilde{t}_0, T_{\max})$. We suppose by contradiction that $T_{\max} = +\infty$, that is, $u(t) = u(t, \cdot)$ exists for all $t \geq 0$. For $u \in \mathcal{O}^-$, we have

$$\frac{d}{dt} \|u(t, \cdot)\|^2_{L^2(\Omega)} = -2 \left\langle I'_{\mu,\lambda}(u(t_0)), u(t_0) \right\rangle > 0. \tag{3.51}$$

Then $t \mapsto \|u(t, \cdot)\|_{L^2(\Omega)}$ is strictly increasing and so

$$\lim_{t \to +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = c \in (0, +\infty]. \tag{3.52}$$

We suppose that $c < +\infty$. Following the same reasoning as in the proof of Theorem 3.3, we deduce that we can select a divergent subsequence, still denoted by t, such that when $t \to +\infty$,

$$u(t, \cdot) \longrightarrow 0 \qquad \text{in } H^1_0\left(\Omega, \mathbb{H}^d\right). \tag{3.53}$$

Letting $t \to +\infty$ in the inequality

$$\sqrt{\lambda_{1,\mu}} \|u(t, \cdot)\|_{L^2(\Omega)} \leq \|u(t, \cdot)\|_{\mu}, \tag{3.54}$$

we get that $0 < c \leq 0$, which is a contradiction. So we conclude that

$$\lim_{t \to +\infty} \|u(t, \cdot)\|_{L^2(\Omega)} = +\infty. \tag{3.55}$$

Set $g(t) = \|u(t,\cdot)\|^2_{L^2(\Omega)}$, so

$$\begin{aligned}
-\frac{2}{p-2}\frac{d}{dt}g^{1-p/2}(t) &= g'(t)g^{-p/2}(t) \\
&= -2\|u(t,\cdot)\|^{-p}_{L^2(\Omega)}\left(\|u(t,\cdot)\|^2_{\mu} - \lambda\|u(t,\cdot)\|^2_{L^2(\Omega)}\|u(t,\cdot)\|^2_{L^p(\Omega)}\right) \\
&\geq -2\|u(t,\cdot)\|^{-p}_{L^2(\Omega)}\|u(t,\cdot)\|^2_{\mu} + 2\|u(t,\cdot)\|^{-p}_{L^2(\Omega)}\|u(t,\cdot)\|^2_{L^p(\Omega)}.
\end{aligned} \tag{3.56}$$

By Hölder inequality, we have

$$\|u(t,\cdot)\|^p_{L^p(\Omega)} \geq |\Omega|^{1-p/2}\|u(t,\cdot)\|^p_{L^2(\Omega)}, \tag{3.57}$$

and by (3.55), there exist $t_1 > 0$ and a constant $C_1 > 0$ such that for $t \geq t_1$, we have

$$\|u(t,\cdot)\|_{L^2(\Omega)} \geq C_1. \tag{3.58}$$

Then, there exist $t_1 > 0$ and a constant $C_2 > 0$ such that for $t \geq t_1$, we have

$$-\frac{2}{p-2}\frac{d}{dt}g^{1-p/2}(t) \geq -2\lambda_{1,\mu}C_1^{2-p} + 2|\Omega|^{1-p/2} \geq C_2. \tag{3.59}$$

Hence, we have from (3.59), that for any $t \geq t_1$,

$$0 < g(t) \leq g(t_1) + \frac{p-2}{2}C_2(t-t_1), \tag{3.60}$$

which is a contradiction if t is sufficiently large. So we conclude that $T_{\max} < +\infty$. □

Acknowledgment

The author is glad to thank the referee for a careful and very constructive reading of the paper and making many good suggestions.

References

[1] G. B. Folland, "Subelliptic estimates and function spaces on nilpotent Lie groups," *Arkiv för Matematik*, vol. 13, no. 2, pp. 161–207, 1975.

[2] D. Jerison and J. M. Lee, "Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem," *Journal of the American Mathematical Society*, vol. 1, no. 1, pp. 1–13, 1988.

[3] L. D'Ambrosio, "Some Hardy inequalities on the Heisenberg group," *Differential Equations*, vol. 40, no. 4, pp. 552–564, 2004.

[4] J. Dou, P. Niu, and Z. Yuan, "A Hardy inequality with remainder terms in the Heisenberg group and the weighted eigenvalue problem," *Journal of Inequalities and Applications*, vol. 2007, Article ID 32585, 24 pages, 2007.

[5] N. I. Karachalios and N. B. Zographopoulos, "The semiflow of a reaction diffusion equation with a singular potential," *Manuscripta Mathematica*, vol. 130, no. 1, pp. 63–91, 2009.

[6] J. M. Ball, "On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations," *Journal of Differential Equations*, vol. 27, no. 2, pp. 224–265, 1978.

[7] J. M. Ball, "Global attractors for damped semilinear wave equations," *Discrete and Continuous Dynamical Systems. Series A*, vol. 10, no. 1-2, pp. 31–52, 2004.

[8] J. López-Gómez and M. Molina-Meyer, "Bounded components of positive solutions of abstract fixed point equations: mushrooms, loops and isolas," *Journal of Differential Equations*, vol. 209, no. 2, pp. 416–441, 2005.

[9] H. Mokrani, "Semi-linear sub-elliptic equations on the Heisenberg group with a singular potential," *Communications on Pure and Applied Analysis*, vol. 8, no. 5, pp. 1619–1636, 2009.

[10] L. Capogna, D. Danielli, and N. Garofalo, "An embedding theorem and the Harnack inequality for nonlinear subelliptic equations," *Communications in Partial Differential Equations*, vol. 18, no. 9-10, pp. 1765–1794, 1993.

[11] J.-M. Bony, "Principe du maximum, inégalite de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés," *Annales de l'Institut Fourier*, vol. 19, pp. 277–304, 1969.

[12] I. Birindelli and A. Cutrì, "A semi-linear problem for the Heisenberg Laplacian," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 94, pp. 137–153, 1995.

[13] P. H. Rabinowitz, "Some global results for nonlinear eigenvalue problems," vol. 7, pp. 487–513, 1971.

[14] E. N. Dancer, "Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one," *The Bulletin of the London Mathematical Society*, vol. 34, no. 5, pp. 533–538, 2002.

[15] J. López-Gómez, *Spectral Theory and Nonlinear Functional Analysis*, vol. 426 of *Chapman & Hall/CRC Research Notes in Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2001.

[16] R. Ikehata and T. Suzuki, "Stable and unstable sets for evolution equations of parabolic and hyperbolic type," *Hiroshima Mathematical Journal*, vol. 26, no. 3, pp. 475–491, 1996.

[17] L. E. Payne and D. H. Sattinger, "Saddle points and instability of nonlinrar parabolic equations," *Hiroshima Mathematical Journal*, vol. 30, pp. 117–127, 2000.

[18] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[19] T. Cazenave and A. Haraux, *Introduction aux Problèmes D'évolution Semi-Linéaires*, vol. 1 of *Mathématiques & Applications*, Ellipses, Paris, France, 1990.

[20] J. K. Hale, *Asymptotic behavior of dissipative systems*, vol. 25 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 1988.

[21] M. Willem, *Minimax Theorems*, vol. 24 of *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser, Boston, Mass, USA, 1996.