Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 746823, 14 pages
doi:10.1155/2012/746823

Research Article

Characterizations of Irregular Multigenerator Gabor Frame on Periodic Subsets of $\mathbb{R}$

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Received 10 March 2012; Accepted 19 May 2012

Academic Editor: Sergey V. Zelik

We consider the multigenerator system $\{E_{mb_l}T_{na_l}\varphi_l, m,n \in \mathbb{Z}, l = 0,\ldots,r-1\}$ for $\varphi_0,\ldots,\varphi_{r-1} \in L^2(\mathbb{S})$ and $a_0, b_0,\ldots,a_{r-1}, b_{r-1} > 0$, where the parameters $b_0,\ldots,b_{r-1} > 0$ are not necessary the same. With the help of frame theory, we provide some sufficient or necessary conditions for the system to be a frame for $L^2(\mathbb{S})$. Moreover, we present some characterizations for this system to be a Parseval frame.

1. Introduction

For $a, b \in \mathbb{R}$, consider the translation operator $(T_a g)(x) = g(x - a)$ and the modulation operator $(E_b g)(x) = e^{2\pi ibx} g(x)$, both acting on $g \in L^2(\mathbb{R})$. We say that the system $\{E_{mb}T_{na}g, m, n \in \mathbb{Z}\}$ is a Gabor frame for $L^2(\mathbb{R})$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}g\rangle\right|^2 \leqslant B\|f\|^2 \tag{1.1}$$

holds for every $f \in L^2(\mathbb{R})$.

Gabor analysis is a pervasive signal processing method for decomposing and reconstructing signals from their time-frequency (TF) projections, and Gabor representation is used in many applications ranging from speech processing and texture segmentation to pattern and object recognition, among others. However, as it is widely recognized, a single-windowed Gabor expansion is not enough to analyze the dynamic TF contents of signals that contain a wide range of spatial and frequency components, the resolution of which is normally very poor. Therefore, if one could incorporate a set of multiple windows of various TF localizations in a frame system, the representation of signals of multiple and/or time-varying frequencies would have their corresponding windowing templates and resolutions to relate to. To this purpose, one of the best choices may be the multigenerator Gabor system.

Multigenerator Gabor system is firstly presented by Zibulski and Zeevi in [1]. Utilizing the piecewise Zak transform (PZT), they [2] discussed the frame operator associated with the multigenerator Gabor frame. They pointed out that the so-called Balian-Low theorem for multigenerator Gabor frame is generalized to consideration of a scheme of multigenerator which makes it possible to overcome in a way the constraint imposed by the original theorem in the case of a single window. Since then, researchers are interested in the study of both theory and application aspects of multigenerator Gabor frame; for detail, see [2–7]. Multigenerator Gabor systems may be both interesting and useful since they can increase the degree of freedom by incorporating windows of various types and widths.

Note that $a\mathbb{Z}$-periodic set in $\mathbb{R}$ can be used to model a signal that appears periodically but intermittently. Recently, some authors concerned Gabor analysis in $L^2(\mathbb{S})$, where $\mathbb{S}$ is an $a\mathbb{Z}$-periodic set in $\mathbb{R}$. Although classical Gabor analysis tools in $L^2(\mathbb{S})$ can be adjusted to treat such a scenario by padding with zeros outside the set $\mathbb{S}$, Gabor systems that fit exactly such a scenario might have been more efficient. Gabardo and Li [8] obtained density results for Gabor systems associated with periodic subsets of the real line. Lian and Li [9] studied the Gabor frame sets for subspaces. They pointed out that only periodic $\mathbb{S}$ in $\mathbb{R}$ is suitable for Gabor analysis.

Motivated by [7–9], we address the issue about the multigenerator Gabor frame in this paper. With the help of frame theory, we provide some sufficient or necessary conditions for the multigenerator Gabor frame system to be a frame for $L^2(\mathbb{S})$, and we obtain the characterization for the multigenerator Gabor system to be a Parseval frame.

2. Notations

In this section, we present some notations and lemmas, which will be needed in the rest of the paper. Let $\mathbb{S}$ be $a\mathbb{Z}$-periodic subset of $\mathbb{R}$. Then, $\mathbb{S}$ is $aq\mathbb{Z}$-periodic subset of $\mathbb{R}$ for any given $q \in \mathbb{Z}$. Denote $\mathbb{S}_0 = [0, a) \bigcap \mathbb{S}$ and

$$L^i(\mathbb{S}) := \left\{ f \in L^i(\mathbb{R}) \mid \operatorname{supp}(f) \subset \mathbb{S} \right\}, \tag{2.1}$$

where $\operatorname{supp}(f) = \{t \in \mathbb{R} : f(t) \neq 0\}$ and $i = 1, 2$. Given a measurable set $\mathbb{F}$ in $\mathbb{R}$ and a constant $c > 0$. Define.

$$\mathbb{F}_{c,x} = \left\{ y \in \mathbb{F} : y = x + cj \text{ for some } j \in \mathbb{Z} \right\}, \quad x \in \mathbb{R}. \tag{2.2}$$

Consider the relation R between x, y in $\mathbb{F}$: xRy if and only if

$$\operatorname{card}(\mathbb{F}_{c,x}) = \operatorname{card}\left(\mathbb{F}_{c,y}\right), \tag{2.3}$$

where $\operatorname{card}(\mathbb{E})$ denotes the cardinality of $\mathbb{E}$ for a set $\mathbb{E}$. Then, it is easy to check that $\mathbb{F}$ has an equivalence relation R. Moreover, define

$$\mathbb{F}(c,k) := \{x \in \mathbb{F} : \operatorname{card}(\mathbb{F}_{c,x}) = k\}, \quad k \in \mathbb{N} \cup \{+\infty\}. \tag{2.4}$$

Note that $\{\mathbb{F}(c,k)\}_{k\in\mathbb{N}\cup\{+\infty\}}$ is an equivalence class under the relation R or a partition of $\mathbb{F}$. Thus

$$\mathbb{F}(c,k) \bigcap \mathbb{F}(c,k') = \emptyset \tag{2.5}$$

for $k, k' \in \mathbb{N} \cup \{+\infty\}$ and $k \neq k'$. Obviously, $\mathbb{F} \bigcap (\mathbb{F}(c,k') + cj) \subset \mathbb{F}(c,k')$ for given $j \in \mathbb{Z} \setminus \{0\}$. It follows that

$$\mathbb{F}(c,k) \bigcap \left(\mathbb{F}(c,k') + cj\right) = \emptyset \tag{2.6}$$

for $j \in \mathbb{Z} \setminus \{0\}$, $k, k' \in \mathbb{N} \cup \{+\infty\}$ and $k \neq k'$.

Example 2.1. Define $\mathbb{F}_n = (n - 1/2^{|n|+2}, n + 1/2^{|n|+2}) \subset \mathbb{R}$ for $n \in \mathbb{Z}$. Consider $\mathbb{F} = \bigcup_{n\in\mathbb{Z}} \mathbb{F}_n$. Then

$$\mu(\mathbb{F}) = \sum_{n\in\mathbb{Z}} \mu(\mathbb{F}_n) = 2\sum_{n\in\mathbb{N}} \mu(\mathbb{F}_n) - \frac{1}{2} = \sum_{n\in\mathbb{N}} \frac{1}{2^n} - \frac{1}{2} = \frac{3}{2} < \infty. \tag{2.7}$$

However, it is easy to check that $\mathbb{F}(c,\infty) = \mathbb{Z}$, which means $\mathbb{F}(c,\infty) \neq \emptyset$.

Remark 2.2. Note that $\mathbb{F}(c,k)$ is the subset as defined in (2.2) of [9]. We point out that Proposition 2.1 (v) in [9] is incorrect.

Definition 2.3. Let $g_n \in L^2(\mathbb{S})$ for $n \in \mathbb{Z}$. We say that the system $\{g_n, n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{S})$ if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{Z}} \left|\langle f, g_n\rangle\right|^2 \leqslant B\|f\|^2 \tag{2.8}$$

holds for every $f \in L^2(\mathbb{S})$; moreover, we say the frame $\{g_n, n \in \mathbb{Z}\}$ for $L^2(\mathbb{S})$ is tight if $A = B$. In particular, the frame $\{g_n, n \in \mathbb{Z}\}$ for $L^2(\mathbb{S})$ is Parseval if $A = B = 1$.

Given a frame $\{g_n \in L^2(\mathbb{S}), n \in \mathbb{Z}\}$ for $L^2(\mathbb{S})$, a dual frame of $\{g_n \in L^2(\mathbb{S}), n \in \mathbb{Z}\}$ for $L^2(\mathbb{S})$ is a frame $\{h_n \in L^2(\mathbb{S}), n \in \mathbb{Z}\}$ for $L^2(\mathbb{S})$ which satisfies the reconstruction property

$$f = \sum_{n\in\mathbb{Z}} \langle f, g_n\rangle h_n, \quad \forall f \in L^2(\mathbb{S}). \tag{2.9}$$

For fixed positive integer r, let $\varphi_0, \psi_0, \ldots, \varphi_{r-1}, \psi_{r-1} \in L^2(\mathbb{S})$. For given $a_0, b_0, \ldots, a_{r-1}, b_{r-1} \in \mathbb{R}$, we say that the system $\{E_{mb_l} T_{na_l} \varphi_l, m,n \in \mathbb{Z},\ l = 0, \ldots, r-1\}$ is a multigenerator Gabor frame for $L^2(\mathbb{S})$ if it is a frame for $L^2(\mathbb{S})$. Given a multigenerator Gabor frame

$\{E_{mb_l}T_{na_l}\varphi_l, m, n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ for $L^2(\mathbb{S})$, a dual multigenerator Gabor frame $\{E_{mb_l}T_{na_l}\psi_l, m, n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ for $L^2(\mathbb{S})$ is a multigenerator Gabor frame

$$f = \sum_{l=0}^{r-1} \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb_l}T_{na_l}\varphi_l \rangle E_{mb_l}T_{na_l}\psi_l \tag{2.10}$$

for any $f \in L^2(\mathbb{S})$.

The following lemma follows from general characterizations of shift-invariant frames, see [10, Corollary 1.6.2]. Alternatively, it can be proved similarly to [11, Theorem 8.4.4].

Lemma 2.4. *Let $g_n \in L^2(\mathbb{R})$, $n \in \mathbb{Z}$, $b > 0$ and suppose that*

$$B := \frac{1}{b} \sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \left| \sum_{n\in\mathbb{Z}} g_n(x) \overline{g_n\left(x - \frac{k}{b}\right)} \right| < \infty. \tag{2.11}$$

Then $\{E_{mb}g_n, m, n \in \mathbb{Z}\}$ is a Bessel sequences with upper frame bound B for $L^2(\mathbb{R})$. If also

$$A := \frac{1}{b} \inf_{x\in\mathbb{R}} \left(\sum_{n\in\mathbb{Z}} |g_n(x)|^2 - \sum_{k\neq 0} \left| \sum_{n\in\mathbb{Z}} g_n(x) \overline{g_n\left(x - \frac{k}{b}\right)} \right| \right) > 0, \tag{2.12}$$

then $\{E_{mb}g_n, m, n \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$ with bounds A and B.

3. Sufficient and Necessary Conditions

In this section, we provide some sufficient and necessary conditions for a class of the multigenerator Gabor frame system to be a frame for $L^2(\mathbb{S})$.

Firstly, we obtain the following theorem for the multigenerator Gabor system with the parameters a and b, which discloses the relationship between the Gabor system $L^2(\mathbb{R})$ and its subspace $L^2(\mathbb{S})$.

Theorem 3.1. *Let $\varphi_0, \dots, \varphi_{r-1} \in L^2(\mathbb{S})$ and $a, b > 0$. Then the following results hold.*

(I) *If the Gabor system $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ is a frame for $L^2(\mathbb{R})$, then it is a frame for $L^2(\mathbb{S})$.*

(II) *If the Gabor system $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ is a Bessel sequence for $L^2(\mathbb{S})$ with upper bound B, then it is a Bessel sequence for $L^2(\mathbb{R})$ with the same upper bound.*

Proof. The part (I) follows from the fact that $L^2(\mathbb{S}) \subset L^2(\mathbb{R})$.

Next, we prove the second part. Suppose that the Gabor system $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ is a Bessel sequence for $L^2(\mathbb{S})$. Then there exists a constant $B > 0$ such that

$$\sum_{l=0}^{r-1} \sum_{n\in\mathbb{Z}} \left| \langle f, E_{mb}T_{na}\varphi_l \rangle \right|^2 \leqslant B\|f\|^2, \quad \forall f \in L^2(\mathbb{S}). \tag{3.1}$$

Observe that

$$\langle f, E_{mb}T_{na}\varphi_l\rangle = \int_{\mathbb{R}} f(x)\overline{\varphi_l(x-na)}e^{-2\pi mbx}dx = \int_{\mathbb{S}} f(x)\overline{\varphi_l(x-na)}e^{-2\pi mbx}dx, \tag{3.2}$$

since $f\overline{\varphi_l} \in L^2(\mathbb{S})$, $l = 0, \dots, r-1$, for all $f \in L^2(\mathbb{R})$. It follows that

$$\langle f, E_{mb}T_{na}\varphi_l\rangle = \langle f\chi_{\mathbb{S}}, E_{mb}T_{na}\varphi_l\rangle. \tag{3.3}$$

Therefore,

$$\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\left|\langle f, E_{mb}T_{na}\varphi_l\rangle\right|^2 = \sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\left|\langle f\chi_{\mathbb{S}}, E_{mb}T_{na}\varphi_l\rangle\right|^2 \leqslant B\|f\chi_{\mathbb{S}}\|^2 \leqslant B\|f\|^2 \tag{3.4}$$

for all $f \in L^2(\mathbb{R})$. This implies that $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0,\dots,r-1\}$ is a Bessel sequence for $L^2(\mathbb{R})$ with the same upper bound B. □

Moreover, we have the following sufficient condition for the multigenerator Gabor system with the parameters a and b.

Theorem 3.2. *Let $\varphi_0,\dots,\varphi_{r-1} \in L^2(\mathbb{S})$ and $a, b > 0$. Moreover, suppose that*

$$B := \frac{1}{b}\sup_{x\in[0,1/b)}\sum_{k\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right| < \infty. \tag{3.5}$$

Then $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0,\dots,r-1\}$ is a Bessel sequence for $L^2(\mathbb{S})$ with upper frame bound B. If also

$$A := \frac{1}{b}\inf_{x\in[0,1/b)}\left(\sum_{n\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\right|^2 - \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}}\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right|\right) > 0, \tag{3.6}$$

then $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0,\dots,r-1\}$ is a frame for $L^2(\mathbb{S})$ with bounds A and B. That means $\{E_{mb}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0,\dots,r-1\}$ is a multigenerator Gabor frame for $L^2(\mathbb{S})$.

Proof. Define

$$\begin{aligned} H_1(x) &:= \sum_{k\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right|, \\ H_2(x) &:= \sum_{n\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\right|^2 - \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}}\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right|, \end{aligned} \tag{3.7}$$

then H_1 and H_2 are $1/b$-periodic functions. Thus

$$B = \frac{1}{b}\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right| < \infty,$$
$$A = \frac{1}{b}\inf_{x\in\mathbb{R}}\left(\sum_{n\in\mathbb{Z}}\left|\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\right|^2 - \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}}\sum_{l=0}^{r-1}T_{na}\varphi_l(x)\overline{T_{na}\varphi_l\left(x+\frac{k}{b}\right)}\right|\right) > 0. \tag{3.8}$$

Define

$$g_n(x) := (T_{ka}\varphi_l)(x), \tag{3.9}$$

where $n = l + rk$ and $l = 0,\ldots,r-1$. Then, one obtains from (3.8) that

$$B = \frac{1}{b}\sup_{x\in\mathbb{R}}\sum_{k\in\mathbb{Z}}\left|\sum_{n\in\mathbb{Z}}g_n(x)\overline{g_n\left(x-\frac{k}{b}\right)}\right| < \infty,$$
$$A = \frac{1}{b}\inf_{x\in\mathbb{R}}\left(\sum_{n\in\mathbb{Z}}|g_n(x)|^2 - \sum_{k\neq 0}\left|\sum_{n\in\mathbb{Z}}g_n(x)\overline{g_n\left(x-\frac{k}{b}\right)}\right|\right) > 0, \tag{3.10}$$

respectively. Note that $L^2(\mathbb{S}) \subset L^2(\mathbb{R})$. By Lemma 2.4, one obtains the results. □

Remark 3.3. Theorem 3.2 is similar to [11, Theorem 8.4.4]. Note that our result extends [11, Theorem 8.4.4] to the multigenerator and the periodic subset cases.

The following theorem gives necessary condition for the system $\{E_{mb_l}T_{na}\varphi_l, m, n \in \mathbb{Z},\ l = 0,\ldots,r-1\}$ to be a multigenerator Gabor frame for $L^2(\mathbb{S})$. It depends on the interplay among the function $\varphi_0,\ldots,\varphi_{r-1}$, the corresponding translation parameters a, b_0, …, b_{r-1}, and the subset $\mathbb{S}$.

Theorem 3.4. *Let $\varphi_0,\ldots,\varphi_{r-1} \in L^2(\mathbb{S})$ and $a, b_0,\ldots,b_{r-1} > 0$. Assume that $\{E_{mb_l}T_{na}\varphi_l, m, n \in \mathbb{Z},\ l = 0,\ldots,r-1\}$ is a multigenerator Gabor frame for $L^2(\mathbb{S})$ with bounds A and B. Then,*

$$A\chi_{\mathbb{S}}(x) \leqslant \sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) \leqslant B\chi_{\mathbb{S}}(x), \quad \text{a.e. } \mathbb{R}. \tag{3.11}$$

Proof. Firstly, note that $\mathbb{S}$ is a $a\mathbb{Z}$-periodic subset of $\mathbb{R}$. Therefore, $\varphi_l(\cdot - na) \in L^2(\mathbb{S})$ for all $n \in \mathbb{Z}$ and $l = 0,\ldots,r-1$. Thus

$$\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) = 0, \quad \text{a.e. } \mathbb{R}\setminus\mathbb{S}. \tag{3.12}$$

The rest part of the proof is by contradiction. Assume that the upper condition in (3.11) is violated on $\mathbb{S}$. Then there exists a measurable set $\Delta \subset \mathbb{S}$ with measure $\mu(\Delta) > 0$ such that

$$\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}\left|\varphi_l(x-na)\right|^2\right) > B, \quad \text{a.e. } \Delta. \tag{3.13}$$

Similar to the discussion in [11, Proposition 8.3.2], we can assume that

$$\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}\left|\varphi_l(x-na)\right|^2\right) > B+\epsilon, \quad \text{a.e. } \Delta \tag{3.14}$$

for small $\epsilon > 0$. Note that $\mathbb{S}$ is a $a\mathbb{Z}$-periodic subset of $\mathbb{R}$. We can assume further that $\Delta \subset \mathbb{S}_0$. Define

$$\begin{aligned}\frac{1}{b^0} &:= \min\left\{\frac{1}{b_l}, l = 0,\cdots,r-1\right\},\\ \Delta_k &= \Delta\bigcap\left[\frac{k-1}{b^0},\frac{k}{b^0}\right], \quad k\in\mathbb{Z}.\end{aligned} \tag{3.15}$$

Then, there exists $k_0 \in \mathbb{Z}$ such that $\mu(\Delta_{k_0}) > 0$ and

$$\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}\left|\varphi_l(x-na)\right|^2\right) > B+\epsilon, \quad \text{a.e. } \Delta_{k_0}. \tag{3.16}$$

If not, that is $\mu(\Delta_k) = 0$ for all $k \in \mathbb{Z}$, then

$$\mu(\Delta) = \mu\left(\bigcup_{k\in\mathbb{Z}}\Delta_k\right) = \sum_{k\in\mathbb{Z}}\mu(\Delta_k) = 0. \tag{3.17}$$

This contradicts to $\mu(\Delta) > 0$. Therefore, we can also assume that Δ is contained in an interval of length $1/b^0$ and that Δ is a subset of $\mathbb{S}_0$.

Now consider the function $f = \chi_\Delta$ and note that $\|f\|^2 = \mu(\Delta)$. Then for any $n \in \mathbb{Z}$, the function $f\overline{T_{na}\varphi_l}$ has support in Δ. Since the functions $\{\sqrt{b_l}E_{mb_l}, m \in \mathbb{Z}\}$ constitute an orthonormal basis for $L^2(I)$ for every interval I of length $1/b_l$ for fixed $l = 0,\ldots,r-1$, we have

$$\sum_{m\in\mathbb{Z}}\left|\langle f, E_{mb_l}T_{na}\varphi_l\rangle\right|^2 = \sum_{m\in\mathbb{Z}}\left|\left\langle f\overline{T_{na}\varphi_l}, E_{mb_l}\right\rangle\right|^2 = \frac{1}{b_l}\int_{-\infty}^{\infty}\left|f(x)\right|^2\left|\varphi_l(x-na)\right|^2dx. \tag{3.18}$$

Thus,

$$\sum_{m,n\in\mathbb{Z}}\left|\langle f, E_{mb_l}T_{na}\varphi_l\rangle\right|^2 = \int_{-\infty}^{\infty}\left|f(x)\right|^2\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}\left|\varphi_l(x-na)\right|^2\right)dx. \tag{3.19}$$

Therefore,

$$\begin{aligned}\sum_{l=0}^{r-1}\sum_{m,n\in\mathbb{Z}}\left|\left\langle f,E_{mb_l}T_{na}\varphi_l\right\rangle\right|^2 &= \int_{-\infty}^{\infty}|f(x)|^2\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right)dx\\ &> \int_{\Delta}(B+\epsilon)dx = (B+\epsilon)\mu(\Delta)\\ &= (B+\epsilon)\|f\|^2 > B\|f\|^2.\end{aligned}\tag{3.20}$$

This contradicts to the assumption that B is an upper frame bound for$\{E_{mb_l}T_{na}\varphi_l, m,n \in \mathbb{Z},\ l = 0,\dots,r-1\}$. A similar proof shows that if the lower condition in (3.11) is violated, then A cannot be a lower frame bound for $\{E_{mb_l}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l=0,\dots,r-1\}$. □

4. Parseval Multigenerater Gabor Frame

In applications of frames, it is inconvenient that the frame decomposition, stated in [12, Theorem 5.1.7], requires inversion of the frame operator. As we have seen in the discussion of general frame theory, one way of avoiding the problem is to consider tight frames. We will characterize Parseval multigenerater Gabor frames in this section. Noting that $L^2(\mathbb{S}) \subset L^2(\mathbb{R})$, we obtain from [11, Lemma 8.4.3] or [12, Lemma 9.1.4] the following lemma, which will be used in the rest of the section.

Lemma 4.1. *Let f be a bounded measure function with compact support and $g \in L^2(\mathbb{S})$. Then*

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}}\left|\left\langle f,E_{mb}T_{na}g\right\rangle\right|^2 &= \frac{1}{b}\int_{\mathbb{R}}|f(x)|^2\sum_{n\in\mathbb{Z}}|g(x-na)|^2dx\\ &\quad+\frac{1}{b}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\overline{f(x)}f\left(x-\frac{k}{b}\right)\sum_{n\neq 0}g(x-na)\overline{g\left(x-na-\frac{k}{b}\right)}dx\end{aligned}\tag{4.1}$$

for given $a, b > 0$.

Theorem 4.2. *Let $\varphi_0,\dots,\varphi_{r-1} \in L^2(\mathbb{S})$ and $a, b_0,\dots,b_{r-1} > 0$. Moreover, assume that $\{E_{mb_l}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l=0,\dots,r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A = 1$. Then,*

$$\sum_{l=0}^{r-1}\frac{1}{b_l}\left(\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) = \chi_{\mathbb{S}}(x), \quad a.e.\ \mathbb{R}.\tag{4.2}$$

Moreover, if $b_0 = \cdots = b_{r-1}$ and denote b_0 by b, then

$$\sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) = b\chi_{\mathbb{S}}(x),\tag{4.3}$$

$$\sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)}\right) = 0, \quad for\ k\neq 0\tag{4.4}$$

hold a.e. in $\mathbb{R}$.

Proof. Define

$$\frac{1}{b^0} := \min\left\{\frac{1}{b_l}, l = 0, \dots, r-1\right\}. \tag{4.5}$$

Consider

$$CL^2(\mathbb{S}) := \left\{f : f \in L^2(\mathbb{S}) \text{ and } \operatorname{supp} f \subset \left[0, \frac{1}{b^0}\right) \cap \mathbb{S}\right\}. \tag{4.6}$$

Note that $\{E_{mb_l}T_{na}\varphi_l, m, n \in \mathbb{Z},\ l = 0, \dots, r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A = 1$. Then

$$\sum_{l=0}^{r-1} \sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb_l}T_{na}\varphi_l\rangle\right|^2 = \|f\|^2, \quad \forall f \in CL^2(\mathbb{S}). \tag{4.7}$$

Again, we obtain from Lemma 4.1 that

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \left|\langle f, E_{mb_l}T_{na}\varphi_l\rangle\right|^2 &= \sum_{m\in\mathbb{Z}} \left|\int_{\mathbb{R}} f(x)\overline{\varphi_l(x-na)}e^{-2\pi m b_l x}dx\right|^2 \\
&= \frac{1}{b_l}\int_{\mathbb{R}} \left|f(x)\overline{\varphi_l(x-na)}\right|^2 dx, \\
&= \frac{1}{b_l}\int_0^{1/b^0} \left|f(x)\varphi_l(x-na)\right|^2 dx, \quad \forall f \in CL^2(\mathbb{S})
\end{aligned} \tag{4.8}$$

for fixed l and n. Thus,

$$\int_0^{1/b^0} |f(x)|^2 dx = \int_0^{1/b^0} |f(x)|^2 \left(\sum_{l=0}^{r-1} \frac{1}{b_l} \sum_{n\in\mathbb{Z}} |\varphi_l(x-na)|^2\right) dx \tag{4.9}$$

for any $f \in CL^2(\mathbb{S})$. This implies that

$$\sum_{l=0}^{r-1}\left(\frac{1}{b_l}\sum_{n\in\mathbb{Z}} |\varphi_l(x-na)|^2\right) = 1, \quad \text{a.e. } \left[0, 1/b^0\right) \cap \mathbb{S}. \tag{4.10}$$

Note that

$$\mathbb{S} = \bigcup_{k\in\mathbb{Z}} \left(\left[\frac{k}{b^0}, \frac{k+1}{b^0}\right) \cap \mathbb{S}\right), \tag{4.11}$$

and we obtain the desired result (4.2) and its special case (4.3).

Next, we prove (4.4). For fixed $l = 0, \ldots, r-1$, we obtain from Lemma 4.1 that

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}\varphi_l\rangle\right|^2 &= \frac{1}{b}\int_{\mathbb{R}} |f(x)|^2 \sum_{n\in\mathbb{Z}} |\varphi_l(x-na)|^2 dx \\ &\quad + \frac{1}{b}\sum_{k\neq 0}\int_{\mathbb{R}} \overline{f(x)} f\left(x-\frac{k}{b}\right)\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)}dx.\end{aligned} \tag{4.12}$$

Then,

$$\begin{aligned}\sum_{l=0}^{r-1}\sum_{m,n\in\mathbb{Z}} \left|\langle f, E_{mb}T_{na}\varphi_l\rangle\right|^2 &= \int_{\mathbb{R}} |f(x)|^2 \sum_{l=0}^{r-1}\frac{1}{b}\sum_{n\in\mathbb{Z}} |\varphi_l(x-na)|^2 dx \\ &\quad + \frac{1}{b}\sum_{k\neq 0}\int_{\mathbb{R}} \left(\overline{f(x)} f\left(x-\frac{k}{b}\right)\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\left(\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)}\right)\right)dx.\end{aligned} \tag{4.13}$$

This, together with (4.3), follows that

$$\sum_{k\neq 0}\int_{\mathbb{R}} \left(\overline{f(x)} f\left(x-\frac{k}{b}\right)\sum_{n\in\mathbb{Z}}\left(\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)}\right)\right)dx = 0. \tag{4.14}$$

A change of variable shows that the contribution in the above sum arising from any value of k is the complex conjugate of the contribution form the value $-k$. Therefore,

$$\sum_{k=1}^{\infty}\operatorname{Re}\left(\int_{\mathbb{R}} \overline{f(x)} f\left(x-\frac{k}{b}\right)G_k(x)dx\right) = 0, \tag{4.15}$$

where $G_k(x) := \sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l(x-na-k/b)}$ for $k \in \mathbb{N}$. Now we divide three cases to draw the result.

Case 1. $x \notin \mathbb{S}$. Note that $\mathbb{S}$ is a $a\mathbb{Z}$-periodic set. Then $x - na \notin \mathbb{S}$ for all $n \in \mathbb{Z}$. Therefore,

$$\varphi_l(x-na) = 0, \quad \forall n \in \mathbb{Z}. \tag{4.16}$$

Thus,

$$\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)} = 0, \quad \forall k \neq 0. \tag{4.17}$$

Case 2. $x - k/b \notin \mathbb{S}$ for fixed $k \in \mathbb{Z}\setminus\{0\}$. Then $x - k/b - na \notin \mathbb{S}$ for all $n \in \mathbb{Z}$. Therefore,

$$\sum_{l=0}^{r-1}\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)} = 0. \tag{4.18}$$

Case 3. $x \in \mathbb{S}$ and $x - k/b \in \mathbb{S}$ for fixed $k \in \mathbb{Z} \setminus \{0\}$. Consider $k \geqslant 1$ and let I be any interval in $\mathbb{R}$ of length at most $1/b$. Denote $I \cap \mathbb{S}$ by Δ^0 and $(I - k/b) \cap \mathbb{S} + k/b$ by Δ'. If $\mu(\Delta^0 \cap \Delta') = 0$, then $x \notin \Delta^0$ a.e. or $x \notin \Delta' - k/b$ a.e., thus

$$\sum_{l=0}^{r-1} \sum_{n\in\mathbb{Z}} \varphi_l(x - na)\overline{\varphi_l\left(x - na - \frac{k}{b}\right)} = 0. \tag{4.19}$$

Now consider $\mu(\Delta^0 \cap \Delta') > 0$. Define a function $f \in L^2(\mathbb{S})$ by

$$f(x) := \begin{cases} e^{-\arg G_{k_0}(x)}, & x \in \Delta^0 \cap \Delta' \\ 1, & x \in \Delta^0 \cap \Delta' - \dfrac{k}{b}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.20}$$

Then, by (4.15),

$$\begin{aligned} 0 &= \sum_{k=1}^{\infty} \operatorname{Re}\left(\int_{\mathbb{R}} \overline{f(x)} f\left(x - \frac{k}{b}\right) G_k(x)dx\right) \\ &= \operatorname{Re}\left(\int_{\mathbb{R}} \overline{f(x)} f\left(x - \frac{k_0}{b}\right) G_{k_0}(x)dx\right) = \int_{\Delta^0\cap\Delta'} |G_{k_0}(x)|dx. \end{aligned} \tag{4.21}$$

It follows that $G_{k_0}(x) = 0$, a.e. on $\Delta^0 \cap \Delta'$. Since I is an arbitrary interval of length at most $1/b$, we conclude that $G_{k_0}(x) = 0$, a.e. in $\mathbb{S}$. A direct computation shows that

$$G_{-k_0}(x) = G_{k_0}\left(x + \frac{k_0}{b}\right). \tag{4.22}$$

Thus, we obtain the desired results. □

To proceed further, we need use the following symbols. For $b_0, \ldots, b_{r-1} > 0$, define

$$\begin{aligned} \frac{1}{b^0} &:= \min\left\{\frac{1}{b_l},\ l = 0, \ldots, r-1\right\}, \\ \frac{1}{b^1} &:= \min\left\{\frac{1}{b_l} : b_l < b^0,\ l = 1, \ldots, r-1\right\}, \\ &\vdots \\ \frac{1}{b^q} &:= \min\left\{\frac{1}{b_l} : b_l < b^{q-1},\ l = 1, \ldots, r-1\right\}, \\ I_0 &:= \{l : b_l = b^0,\ l = 0, \ldots, r-1\}, \\ I_1 &:= \{l : b_l = b^1,\ l = 0, \ldots, r-1\}, \\ &\vdots \\ I_q &:= \{l : b_l = b^q,\ l = 0, \ldots, r-1\}. \end{aligned} \tag{4.23}$$

Then there exists a unique nonnegative integer q_0 such that

$$\begin{aligned} &I_q \neq \emptyset, \quad \text{for } q = 0,\dots,q_0, \\ &I_{q_1} \bigcap I_{q_2} = \emptyset, \quad \text{for } q_1 \neq q_2, \\ &\bigcup_{q=0}^{q_0} I_q = \{0,\dots,r-1\}. \end{aligned} \tag{4.24}$$

Theorem 4.3. *Let q_0 be the unique nonnegative integer satisfying* (4.24)*. Assume that $\varphi_0,\dots,\varphi_{r-1} \in L^2(\mathbb{S})$ and $a, b_0,\dots,b_{r-1} > 0$ satisfy*

$$\sum_{l=0}^{r-1} \frac{1}{b_l}\left(\sum_{n\in\mathbb{Z}} \left|\varphi_l(x-na)\right|^2\right) = \chi_{\mathbb{S}}(x), \tag{4.25}$$

$$\sum_{l\in I_q}\left(\sum_{n\in\mathbb{Z}} \varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b_l}\right)}\right) = 0, \quad \text{for } k \neq 0,\ q = 0,\dots,q_0 \tag{4.26}$$

a.e. in $\mathbb{R}$. Then $\{E_{mb_l}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l = 0,\dots,r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A = 1$.

Proof. For fixed $l = 0,\dots,r-1$, we obtain from Lemma 4.1 that

$$\begin{aligned} \sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_{mb_l}T_{na}\varphi_l\right\rangle\right|^2 = {}& \frac{1}{b_l}\int_{\mathbb{R}} \left|f(x)\right|^2 \sum_{n\in\mathbb{Z}} \left|\varphi_l(x-na)\right|^2 dx \\ &+ \frac{1}{b_l}\sum_{k\neq 0}\int_{\mathbb{R}} \overline{f(x)}f\left(x-\frac{k}{b_l}\right)\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b_l}\right)}dx. \end{aligned} \tag{4.27}$$

Then,

$$\sum_{l=0}^{r-1}\sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_{mb_l}T_{na}\varphi_l\right\rangle\right|^2 = \int_{\mathbb{R}} \left|f(x)\right|^2 \sum_{l=0}^{r-1}\frac{1}{b_l}\sum_{n\in\mathbb{Z}} \left|\varphi_l(x-na)\right|^2 dx + (*), \tag{4.28}$$

where

$$(*) := \int_{\mathbb{R}} \sum_{l=0}^{r-1}\left\{\frac{1}{b_l}\sum_{k\neq 0}\left[\overline{f(x)}f\left(x-\frac{k}{b_l}\right)\sum_{n\in\mathbb{Z}}\left(\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b_l}\right)}\right)\right]\right\}dx. \tag{4.29}$$

This, together with (4.25), follows that

$$\sum_{l=0}^{r-1}\sum_{m,n\in\mathbb{Z}} \left|\left\langle f, E_{mb_l}T_{na}\varphi_l\right\rangle\right|^2 = \int_{\mathbb{R}} \left|f(x)\right|^2 dx + (*). \tag{4.30}$$

Define

$$G_k^q(x) := \sum_{l\in I_q}\left(\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b_l}\right)}\right), \quad \text{for } q=0,\ldots,q_0. \tag{4.31}$$

Then, we obtain from (4.26) that

$$(*) = \sum_{k\neq 0}\left(\int_{\mathbb{R}}\sum_{q=0}^{q_0}\frac{1}{b^q}\overline{f(x)}f\left(x-\frac{k}{b_q}\right)G_k^q(x)dx\right) = 0. \tag{4.32}$$

This, together with (4.30), follows that

$$\sum_{l=0}^{r-1}\sum_{m,n\in\mathbb{Z}}\left|\langle f, E_{mb_l}T_{na}\varphi_l\rangle\right|^2 = \int_{\mathbb{R}}|f(x)|^2dx = \|f\|^2, \quad \forall f\in L^2(\mathbb{S}). \tag{4.33}$$

Therefore, $\{E_{mb_l}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l=0,\ldots,r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A=1$. □

Note that $q_0 = 0$ when $b_0 = b_1 = \cdots = b_{r-1} = b$. We obtain from Theorems 4.2 and 4.3 that following Theorem 4.4, which is a necessary and sufficient condition for multigenerater Gabor Parseval frame.

Theorem 4.4. *Let $\varphi_0,\ldots,\varphi_{r-1}\in L^2(\mathbb{S})$ and $a,b>0$. Then $\{E_{mb}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l=0,\ldots,r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A=1$ if and only if*

$$\begin{gathered}\sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) = b\chi_{\mathbb{S}}(x),\\ \sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l\left(x-na-\frac{k}{b}\right)}\right) = 0, \quad \text{for } k\neq 0\end{gathered} \tag{4.34}$$

hold a.e. in $\mathbb{R}$.

Remark 4.5. If $\mathbb{S} = \mathbb{R}$, then $\mathbb{S}$ is $a\mathbb{Z}$-periodic set for any given $a > 0$. In this case, $\{E_{m/a}T_{na}\varphi_l, m,n\in\mathbb{Z},\ l=0,\ldots,r-1\}$ is a tight frame for $L^2(\mathbb{S})$ with $A=1$ if and only if

$$\begin{gathered}\sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}|\varphi_l(x-na)|^2\right) = \frac{1}{a}\chi_{\mathbb{S}}(x),\\ \sum_{l=0}^{r-1}\left(\sum_{n\in\mathbb{Z}}\varphi_l(x-na)\overline{\varphi_l(x-na-ka)}\right) = 0, \quad \text{for } k\neq 0\end{gathered} \tag{4.35}$$

hold a.e. in $\mathbb{R}$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant no. 11071152) and the Natural Science Foundation of Guangdong Province (Grant nos. 10151503101000025 and S2011010004511); this research was also partially supported by the Opening Project of Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University (Grant no. 201206012).

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