Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 738472, 19 pages
doi:10.1155/2012/738472

Research Article

Perturbations of Half-Linear Euler Differential Equation and Transformations of Modified Riccati Equation

Ondřej Došlý and Hana Funková

Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

Correspondence should be addressed to Ondřej Došlý, dosly@math.muni.cz

Received 10 July 2012; Accepted 17 September 2012

Academic Editor: Allan Peterson

We investigate transformations of the modified Riccati differential equation and the obtained results we apply in the investigation of oscillatory properties of perturbed half-linear Euler differential equation. A perturbation is also allowed in the differential term.

1. Introduction

The half-linear Euler differential equation

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \qquad \Phi(x) := |x|^{p-2}x, \quad p > 1, \tag{1.1}$$

with the so-called *oscillation constant* $\gamma_p := ((p-1)/p)^{p-1}$ plays an important role in the oscillation theory of the half-linear differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{1.2}$$

with the continuous functions r, c, and $r(t) > 0$. The reason is that (1.1) represents a kind of borderline between oscillation and nonoscillation in the half-linear oscillation theory. More precisely, if $r(t) = 1$ in (1.2), then this equation is oscillatory provided

$$\liminf_{t\to\infty} t^p c(t) > \gamma_p, \tag{1.3}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^p c(t) < \gamma_p, \tag{1.4}$$

see, for example, [1]. Formulas (1.3), (1.4) show what "borderline" means. The potential $c(t) = \gamma_p/t^p$ "separates" potentials c in (1.2) with $r(t) \equiv 1$ for which this equation is oscillatory or nonoscillatory. Criteria (1.3), (1.4) can be extended to the general case $r(t) \not\equiv 1$. In this general setting, the Kneser type criterion is formulated in terms of the lower and upper limit of the expression

$$r^{q-1}(t)\left(\int^t r^{1-q}(s)ds\right)^p c(t), \tag{1.5}$$

if $\int^\infty r^{1-q}(t)dt = \infty$, $q = p/(p-1)$ being the conjugate exponent of p, and of the expression

$$r^{q-1}(t)\left(\int_t^\infty r^{1-q}(s)ds\right)^p c(t), \tag{1.6}$$

if $\int^\infty r^{1-q}(t)dt < \infty$. The constant γ_p in this criterion remains the same. In the linear case $p = 2$, (1.3) and (1.4) are the classical Kneser (non)oscillation criteria, see [2].

Our investigation is mainly motivated by the papers [3–5]. In [4], perturbations of (1.1) of the form,

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^n \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}\right]\Phi(x) = 0, \tag{1.7}$$

were investigated. Here, the notation

$$\operatorname{Log}_k t = \prod_{j=1}^k \log_j t, \qquad \log_k t = \log_{k-1}(\log t), \qquad \log_1 t = \log t \tag{1.8}$$

is used. It was shown that the crucial role in (1.7) plays the constant $\mu_p := (1/2)((p-1)/p)^{p-1}$. In particular, if $n = 1$ in (1.7), that is, this equation reduces to the so-called Riemann-Weber half-linear differential equation, then this equation is oscillatory if $\beta_1 > \mu_p$ and nonoscillatory in the opposite case. In general, if $\beta_j = \mu_p$ for $j = 1, \dots, n-1$, then (1.7) is oscillatory if and only if $\beta_n > \mu_p$.

In [5], the perturbations of the *linear* Euler differential equation were investigated and a perturbation was also allowed in the term-involving derivative. More precisely, the differential equation

$$\left[\left(1 + \sum_{j=1}^n \frac{\alpha_j}{\operatorname{Log}_j^2 t}\right)x'\right]' + \left[\frac{1}{4t^2} + \sum_{j=1}^n \frac{\beta_j}{t^2 \operatorname{Log}_j^2 t}\right]x = 0 \tag{1.9}$$

was considered. It was shown that if there exists $k \in \{1,\ldots,n\}$ such that $\beta_j - \alpha_j/4 = 1/4$ for $j = 1,\ldots,k-1$, and $\beta_k - \alpha_k \neq 1/4$, then (1.9) is oscillatory if and only if $\beta_k - \alpha_k/4 > 1/4$. If $\beta_j - \alpha_j = 1/4$ for all $j = 1,\ldots,n$, then (1.11) is nonoscillatory. This result was partially extended to half-linear equations in [3]. There,

$$\left[\left(1+\frac{\alpha}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\beta}{t^p \log^2 t}\right]\Phi(x) = 0 \tag{1.10}$$

was investigated and it was shown that (1.10) is oscillatory if and only $\beta - \alpha\gamma_p > \mu_p$. For some related results see also [6].

In this paper we deal with perturbations of the Euler half-liner differential equation in full generality. We consider

$$\left[\left(1+\sum_{j=1}^{n}\frac{\alpha_j}{\mathrm{Log}_j^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p} + \sum_{j=1}^{n}\frac{\beta_j}{t^p \mathrm{Log}_j^2 t}\right]\Phi(x) = 0, \tag{1.11}$$

and we find an explicit formula for the relationship between constants α_j, β_j in (1.11) which implies (non)oscillation of this equation. In the last section of the paper we explain why perturbations are just in the above considered form. Our result is based on a new method which consists in transformations of the modified Riccati equations associated with (1.2). The main result along this line is established in Section 3, while its application to the perturbed Euler equation is presented in Section 4. In the last section we present some remarks and comments concerning the results of our paper. In the next section we recall some essentials of the half-linear oscillation theory.

2. Preliminaries

It is a well-known fact that many of the results of the linear oscillation theory can be directly extended to half-linear equation (1.2), even if, in contrast to the case $p = 2$ (then (1.2) is a linear equation), the additivity of the solution space is lost and only homogeneity remains. In particular, the so-called Riccati technique, consisting in the relationship between (1.2) and its associated Riccati type equation (related to (1.2) by the substitution $w = r\Phi(x'/x)$)

$$R[w](t) := w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \qquad q := \frac{p}{p-1} \tag{2.1}$$

extends almost literally to (1.2). More precisely, the following statement holds (see [1, Theorem 2.2.1]).

Proposition 2.1. *Equation* (1.2) *is nonoscillatory if and only if there exists a differentiable function* w *such that* $R[w](t) = 0$ *for large* t.

The *modified Riccati equation* associated with (1.2) is introduced explicitly in [7], but it can be found implicitly already in some earlier papers, for example, [8–10]. Suppose that

(1.2) is nonoscillatory (i.e., every its nontrivial solution is eventually positive or negative) and let h be a positive differentiable function. Consider the substitution

$$v = h^p(t)w - G(t), \qquad G(t) := r(t)h(t)\Phi(h'(t)), \tag{2.2}$$

where w is a solution of (2.1). Then v is a solution of the *modified Riccati equation*

$$v' + \tilde{c}(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v, G(t)) = 0, \tag{2.3}$$

with

$$H(v, G) := |v+G|^q - q\Phi^{-1}(G)v - |G|^q, \tag{2.4}$$

$\Phi^{-1}(s) = |s|^{q-2}s$ being the inverse function of Φ, and

$$\tilde{c}(t) = h(t)\Big[(r(t)\Phi(h'(t)))' + c(t)\Phi(h(t))\Big]. \tag{2.5}$$

Note that the function $H(v, G)$ satisfies $H(v, G) \geq 0$ for every $v, G \in \mathbb{R}$ and $H(v, G) = 0 = H_v(v, G)$ if and only if $v = 0$. Observe also that Riccati equation (2.1) is a special case of (2.3) with $h(t) \equiv 1$, that is, $G(t) \equiv 0$.

In the investigation of perturbations of the half-linear Euler equation we will need the following criteria for (non)existence of a *proper* solution of (2.3). Recall that a solution v of (2.3) is called *proper* if it exists on some interval $[T, \infty)$. Nonexistence of a proper solution of (2.3) is equivalent to oscillation of (1.2) since it eliminates (via the transformation $w = h^{-p}(v+G)$) proper solutions of (2.1). For more details concerning this method, as well as the proof of the next two propositions, we refer to [11].

For the sake of the later application, we will write (2.3) in the form

$$v' + C(t) + (p-1)R^{-1}(t)H(v, G(t)) = 0, \tag{2.6}$$

with continuous functions C, R, and $R(t) > 0$.

Proposition 2.2. (i) *If* $C(t) \leq 0$ *for large* t, *then* (2.6) *possesses a (nonnegative) proper solution.*

In the remaining part of the proposition suppose that

$$\liminf_{t\to\infty} |G(t)| > 0, \qquad C(t) \geq 0, \quad \text{for large } t. \tag{2.7}$$

Denote

$$\mathcal{R}(t) = R^{-1}(t)|G(t)|^{q-2}, \tag{2.8}$$

and suppose that

$$\int^{\infty} \mathcal{R}(t)dt = \infty, \qquad \int^{\infty} C(t)dt < \infty. \tag{2.9}$$

(ii) *If*

$$\limsup_{t\to\infty}\left(\int^t \mathcal{R}(s)ds\right)\left(\int_t^\infty C(s)ds\right) < \frac{1}{2q}, \tag{2.10}$$

then (2.6) *has a proper solution.*

(iii) *If*

$$\liminf_{t\to\infty}\left(\int^t \mathcal{R}(s)ds\right)\left(\int_t^\infty C(s)ds\right) > \frac{1}{2q}, \tag{2.11}$$

then (2.6) *possesses no proper solution.*

Proposition 2.3. *Together with* (2.6) *consider the equation of the same form*

$$v' + D(t) + (p-1)R^{-1}(t)H(v,G(t)) = 0, \tag{2.12}$$

with the function D satisfying $D(t) \geq C(t)$ for large t. If the (majorant) equation (2.12) *has a proper solution, then* (2.6) *has a proper solution as well.*

Next, we recall basic properties of solutions of the "critical" half-linear Euler and Riemann-Weber differential equations as presented, for example, in [4]. Consider the half-linear Euler differential equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0. \tag{2.13}$$

This equation is nonoscillatory if and only if $\gamma \leq \gamma_p = ((p-1)/p)^p$. In the critical case $\gamma = \gamma_p$, (2.13) has the solution $h(t) = t^{(p-1)/p}$, and every linearly independent solution is asymptotically equivalent (up to a multiplicative factor) to the function $x(t) = t^{(p-1)/p}\log^{2/p} t$. The Riemann-Weber half-linear differential equation

$$(\Phi(x'))' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p\log^2 t}\right]\Phi(x) = 0 \tag{2.14}$$

is nonoscillatory if and only if $\mu \leq \mu_p = (1/2)((p-1)/p)^{p-1}$. In the critical case $\mu = \mu_p$, (2.14) has the (so-called principal) solution which is asymptotically equivalent (up to a multiplicative factor) to the function $h(t) = t^{(p-1)/p}\log^{1/p} t$, and every linearly independent solution is asymptotically equivalent to the function $x(t) = t^{(p-1)/p}\log^{1/p} t\log^{2/p}(\log t)$, see [4].

Finally, we recall the transformation method of the investigation of (1.9) which we extend in a modified form to half-linear equations. The Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0 \tag{2.15}$$

is the special case $p = 2$ in (1.2). The transformation $x = f(t)y$ gives the identity (suppressing the argument t)

$$f\left[(rx')' + cx\right] = \left(rf^2y'\right)' + f\left[(rf')' + cf\right]y. \tag{2.16}$$

In particular, if $f(t) \neq 0$, then x is a solution of (2.15) if and only if y is a solution of the equation

$$\left(r(t)f^2(t)y'\right)' + f(t)\left[(r(t)f'(t))' + c(t)f(t)\right]y = 0. \tag{2.17}$$

Let us emphasize at this moment that we have in disposal *no half-linear version* of transformation identity (2.16).

Let us denote

$$r(t) = 1 + \sum_{j=1}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}, \qquad c(t) = \frac{1}{4t^2} + \sum_{j=1}^{n} \frac{\beta_j}{t^2 \operatorname{Log}_j^2 t}. \tag{2.18}$$

First we apply the transformation $x = \sqrt{t}y$ to (1.9). Using (2.16) and the fact that $f(t) = \sqrt{t}$ is a solution of the critical Euler linear equation $x'' + (1/4t^2)x = 0$, we find that y is a solution of the equation

$$(tr(t)y')' + \left[\sum_{j=1}^{n} \frac{\beta_j - \alpha_j/4}{t \operatorname{Log}_j^2 t}\right] y = 0. \tag{2.19}$$

Now, we change the independent variable $t \mapsto e^t$, the resulting equation is

$$(r(e^t)y')' + \left[\sum_{j=1}^{n} \frac{\beta_j - \alpha_j/4}{t^2 \operatorname{Log}_{j-1}^2 t}\right] y = 0. \tag{2.20}$$

Here we take $\operatorname{Log}_0 t = 1$. Equation (2.20) is oscillatory by Kneser oscillation criterion if $\beta_1 - \alpha_1/4 > 1/4$ and nonoscillatory if $\beta_1 - \alpha_1/4 < 1/4$. Indeed, since $(r(e^t) \sim 1$ as $t \to \infty$, we have in (1.5) with $p = 2$

$$r(e^t)\left(\int^t r^{-1}(e^s)ds\right)^2 \sim t^2, \tag{2.21}$$

as $t \to \infty$, and hence

$$\lim_{t\to\infty} r(e^t)\left(\int^t r^{-1}(e^s)ds\right)^2 \left[\sum_{j=1}^{\infty} \frac{\beta_j - \alpha_j/4}{t^2 \operatorname{Log}_{j-1}^2 t}\right] = \frac{\beta_1 - \alpha_1}{4}. \tag{2.22}$$

If $\beta_1 - \alpha_1/4 = 1/4$, we can repeat the previous transformations and we obtain

$$\left(r(e_2(t))y'\right)' + \left[\frac{\beta_2 - \alpha_2/4}{t^2} + \cdots + \frac{\beta_n - \alpha_n/4}{t^2 \operatorname{Log}_{n-2}^2 t}\right] y = 0, \tag{2.23}$$

here $e_2(t) := e^{e^t}$. Now it should be clear how one can obtain the result of [5] concerning oscillation of (1.9). We repeat the transformation of dependent variable $y \mapsto \sqrt{t}y$ followed by the change of independent variable $t \mapsto e^t$ as long as the condition $\beta_j - \alpha_j/4 = 1/4$ is satisfied.

As we have emphasized above, we have no half-linear version of the linear transformation identity (2.16). Consequently, the above procedure cannot be applied directly to (1.2). However, as observed, for example, in [6, 11], the modified Riccati equation in the linear case $p = 2$ is

$$v' + h\left[(rh')' + ch\right] + \frac{v^2}{rh^2} = 0, \tag{2.24}$$

which is just the Riccati equation associated with differential equation (2.17). Hence, modified Riccati equation can be regarded, in a certain sense, as a half-linear substitution for the linear transformation identity (2.16). This is just the idea which we develop in the next section and apply it in the investigation of the perturbed Euler equation.

3. Transformation of Modified Riccati Equation

As a starting point of this section we consider the modified Riccati equation in the form

$$v' + C(t) + (p-1)R^{-1}(t)H(v, G(t)) = 0, \tag{3.1}$$

where the function H is given by (2.4), the functions R, C are supposed to be continuous and $R(t) > 0$. In this equation, we call the function C the *absolute term* (since this term does not contain the unknown function v).

We consider the transformation

$$z = f^p(t)v - U(t), \tag{3.2}$$

with a positive differentiable function f and with a function U which we determine as follows. We have (again suppressing the argument t, this argument we will suppress also now and then in the next parts of the paper) the following:

$$\begin{aligned} z' = {} & p\frac{f'}{f}(z+U) \\ & + f^p\Big\{-C - (p-1)R^{-1}\Big[f^{-pq}|z + U + f^pG|^q \\ & \qquad -q\Phi^{-1}(G)f^{-p}(z+U) - |G|^q\Big]\Big\} - U' \end{aligned}$$

$$\begin{aligned} &= -(p-1)R^{-1}f^{-q}\left|z+U+f^pG\right|^q + p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]z \\ &\quad + p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]U - (p-1)f^pR^{-1}|G|^q - U'. \end{aligned} \tag{3.3}$$

Next we determine the function U in such a way that the differential equation for z is again an equation of the form (3.1) (in which $H(0,G) = 0 = H_v(0,G)$). Denote $\Omega := U + f^pG$. The terms on the fourth line of the previous computation

$$-(p-1)R^{-1}f^{-q}|z+\Omega|^q + p\left[\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right]z \tag{3.4}$$

we will take as the first two terms in the function of the same form as H in (3.1). Differentiating (3.4) with respect to z, substituting $z = 0$, and setting the obtained expression equal to zero, we obtain

$$R^{-1}f^{-q}\Phi^{-1}(\Omega) = \frac{f'}{f} + R^{-1}\Phi^{-1}(G), \tag{3.5}$$

hence

$$\Omega = f\Phi\left(Rf' + f\Phi^{-1}(G)\right). \tag{3.6}$$

Consequently, we obtain the transformed modified Riccati equation

$$z' + \tilde{C} + (p-1)R^{-1}f^{-q}\left[|z+\Omega|^q - q\Phi^{-1}(\Omega)z - |\Omega|^q\right] = 0, \tag{3.7}$$

where

$$\begin{aligned} \tilde{C} = &-p\left(\frac{f'}{f}+R^{-1}\Phi^{-1}(G)\right)U + f^pC - (p-1)R^{-1}f^p|G|^q \\ &+ (p-1)R^{-1}f^{-q}|\Omega|^q + U', \end{aligned} \tag{3.8}$$

$$U = -f^pG + f\Phi\left(Rf' + f\Phi^{-1}(G)\right). \tag{3.9}$$

4. Perturbations of Euler Differential Equation

Now we apply the results of the previous section to the perturbed Euler half-linear differential equation

$$\left[\left(\sum_{j=0}^{n}\frac{\alpha_j}{\mathrm{Log}_j^2 t}\right)\Phi(x')\right]' + \left(\sum_{j=0}^{n}\frac{\beta_j}{t^p\,\mathrm{Log}_j^2 t}\right)\Phi(x) = 0, \tag{4.1}$$

where $\alpha_0 = 1$, $\beta_0 = \gamma_p := ((p-1)/p)^p$ and

$$\operatorname{Log}_j t = \prod_{i=1}^{j} \log_i t, \qquad \log_j t = \log\left(\log_{j-1} t\right), \tag{4.2}$$

with $\operatorname{Log}_0 t = 1$.

To simplify the next computations, we denote

$$r(t) = \sum_{j=0}^{n} \frac{\alpha_j}{\operatorname{Log}_j^2 t}, \qquad c(t) = \sum_{j=0}^{n} \frac{\beta_j}{t^p \operatorname{Log}_j^2 t}. \tag{4.3}$$

The Riccati equation associated with (4.1) is

$$w' + c(t) + (p-1) r^{1-q}(t) |w|^q = 0. \tag{4.4}$$

In order to better understand the next transformation procedure, we recommend the reader to compare it with the linear transformation idea described at the end of the previous section. The transformation

$$v_1 = t^{p-1} w - U_1, \tag{4.5}$$

with U_1 specified later, transforms (4.4) into

$$v' + \frac{\widetilde{c}_1(t)}{t} + \frac{p-1}{t} r^{1-q}(t) H(v, \Omega_1(t)) = 0, \tag{4.6}$$

with $\widetilde{c}_1/t$ given by (3.8), that is, $\widetilde{c}_1(t)/t = \widetilde{C}(t)$ with $f(t) = t^{(p-1)/p}$, $R = r^{q-1}$, and $G = 0$. This means that $\widetilde{c}_1/t = X_1 + Y_1 + Z_1 + U_1' + f^p c$, where

$$\begin{aligned}
U_1 &= -f^p G + f\Phi\left(Rf' + f\Phi^{-1}(G)\right) = r\Gamma_p, \qquad \Gamma_p := \left(\frac{p-1}{p}\right)^{p-1}, \\
X_1 &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}(G)\right) U_1 = -(p-1) t^{-1} r \Gamma_p, \\
Y_1 &= -(p-1) R^{-1} f^p |G|^q = 0, \\
\Omega_1 &= f\Phi\left(Rf' + f\Phi^{-1}(G)\right) = r\Gamma_p, \\
Z_1 &= (p-1) R^{-1} f^{-q} |\Omega_1|^q = (p-1) r^{1-q} t^{-1} r^q \gamma_p = (p-1) r t^{-1} \gamma_p.
\end{aligned} \tag{4.7}$$

Hence, by a direct computation we obtain

$$\frac{\widetilde{c}_1(t)}{t} = \sum_{j=1}^{n} \frac{B_j}{t \operatorname{Log}_j^2 t} + O\left(t^{-1} \log^{-3} t\right), \tag{4.8}$$

where

$$B_j = \beta_j - \alpha_j \gamma_p. \tag{4.9}$$

In (4.6), with the above given $\widetilde{c}_1(t)/t$, we change the independent variable $t \mapsto e^t$ and the resulting equation is

$$v_1' + c_1(t) + (p-1)r^{1-q}(e^t)\Big[\left|v_1 + \Omega_1(e^t)\right|^q - q\Phi^{-1}(\Omega_1(e^t))v - \Omega_1^q(e^t)\Big] = 0, \tag{4.10}$$

with

$$c_1(t) := \widetilde{c}_1(e^t) = \frac{B_1}{t^2} + \frac{B_2}{t^2 \log^2 t} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-1}^2 t} + O\Big(t^{-3}\Big). \tag{4.11}$$

As the next step, we consider the modified Riccati equation

$$v_1' + \frac{B_1}{t^2} + \frac{B_2}{t^2 \log^2 t} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-1}^2 t} + O\Big(t^{-3}\Big) + (p-1)\widetilde{r}_1^{1-q}(t) H\Big(v, \widetilde{\Omega}_1(t)\Big) = 0, \tag{4.12}$$

where now

$$\begin{aligned} \widetilde{r}_1(t) &:= \ r(e^t) = 1 + \frac{\alpha_1}{t^2} + \frac{\alpha_2}{t^2 \log^2 t} + \cdots + \frac{\alpha_n}{t^2 \operatorname{Log}_{n-1}^2 t}, \\ \widetilde{\Omega}_1(t) &:= \ \Omega_1(e^t) = r(e^t)\Gamma_p = \widetilde{r}_1(t)\Gamma_p. \end{aligned} \tag{4.13}$$

We apply the transformation $v_2 = t v_1 - U_2$, the quantity U_2 is again determined in such a way that we obtain a modified Riccati equation containing H type function for v_2. Hence, using the results from formula (3.8), with $f(t) = t^{1/p}$, $G(t) = \widetilde{\Omega}_1(t) = \widetilde{r}_1(t)\Gamma_p$, and $R^{-1}(t) = \widetilde{r}_1^{1-q}(t)$, we have

$$\begin{aligned} \Omega_2 &= f\Phi\Big(Rf' + f\Phi^{-1}\Big(\widetilde{\Omega}_1\Big)\Big) = t^{1/p}\Phi\bigg(\frac{1}{p}\widetilde{r}_1^{q-1} t^{(1/p)-1} + t^{1/p}\Phi^{-1}\big(\widetilde{r}_1\Gamma_p\big)\bigg) \\ &= \widetilde{r}_1 t \Gamma_p \left(1 + \frac{1}{(p-1)t}\right)^{p-1}, \end{aligned} \tag{4.14}$$

and using the binomial expansion

$$\begin{aligned} U_2 &= -f^p \widetilde{\Omega}_1 + \Omega_2 = -t\widetilde{r}_1\Gamma_p + \widetilde{r}_1 t \Gamma_p \left(1 + \frac{1}{(p-1)t}\right)^{p-1} \\ &= \widetilde{r}_1\Gamma_p\left[1 + \frac{p-2}{2(p-1)t} + O\Big(t^{-2}\Big)\right]. \end{aligned} \tag{4.15}$$

Futher,

$$
\begin{aligned}
X_2 &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}\left(\widetilde{\Omega}_1\right)\right)U_2 \\
&= -p\left(\frac{1}{p}t^{1/p-1}t^{1/p} + \widetilde{r}_1^{1-q}\Phi^{-1}(\widetilde{r}_1\Gamma)\right)\widetilde{r}_1\Gamma\left(1 + \frac{p-2}{2(p-1)t} + O\left(t^{-2}\right)\right) \\
&= \widetilde{r}_1\Gamma\left(-\frac{1}{t} - \frac{p-2}{2(p-1)t^2} - (p-1) - \frac{p-2}{2t} + O\left(t^{-2}\right)\right), \\
Y_2 &= -(p-1)R^{-1}f^p\left|\widetilde{\Omega}_1\right|^q = -(p-1)\widetilde{r}_1^{1-q}t\widetilde{r}_1^q\gamma = -t(p-1)\gamma\widetilde{r}_1, \\
Z_2 &= (p-1)R^{-1}f^{-q}|\Omega_2|^q = (p-1)\widetilde{r}_1^{1-q}t^{-q/p}\widetilde{r}_1^q\gamma t^q\left(1 + \frac{1}{(p-1)t}\right)^p \\
&= (p-1)\gamma\widetilde{r}_1 t + p\gamma\widetilde{r}_1 + \frac{p}{2t}\gamma\widetilde{r}_1 + O\left(t^{-2}\right).
\end{aligned}
\tag{4.16}
$$

Hence, the absolute term in the resulting modified Riccati equation is

$$
\begin{aligned}
\frac{\widetilde{c}_2(t)}{t} &:= X_2 + Y_2 + Z_2 + U_2' + t\left(\sum_{j=1}^{n}\frac{B_j}{t^2\operatorname{Log}_{j-1}^2}\right) \\
&= \widetilde{r}_1\left\{t[-(p-1)\gamma + (p-1)\gamma] + [-(p-1)\Gamma + p\gamma]\right. \\
&\qquad \left. + \frac{1}{t}\left[-\Gamma - \frac{p-2}{2}\Gamma + \frac{p}{2}\gamma\right] + O\left(t^{-2}\right)\right\} + \sum_{j=1}^{n}\frac{B_j}{t\operatorname{Log}_{j-1}^2 t} \\
&= \widetilde{r}_1\left[-\frac{\mu_p}{t} + O\left(t^{-2}\right)\right] + \frac{B_1}{t} + \cdots + \frac{B_n}{t\operatorname{Log}_{n-1}^2 t} \\
&= \frac{1}{t}(-\mu_p + B_1) + \frac{B_2}{t\log^2 t} + \cdots + \frac{B_n}{t\operatorname{Log}_{n-1}^2 t} + O\left(t^{-2}\right).
\end{aligned}
\tag{4.17}
$$

Observe that the O term in U_2 and later in other U_j can be differentiated because of its special form. Hence, if $B_1 = \mu_p$, we obtain

$$
v_2' + \frac{B_2}{t\log^2 t} + \cdots + \frac{B_n}{t\operatorname{Log}_{n-1}^2 t} + O\left(t^{-2}\right) + (p-1)\widetilde{r}_1^{q-1}(t)t^{1-q}H(v_2,\Omega_2) = 0. \tag{4.18}
$$

In this equation we apply again the change of independent variable $t \mapsto e^t$ and the resulting equation is

$$
v_2' + c_2(t) + (p-1)\widetilde{r}_1^{1-q}(e^t)e^{(2-q)t}H\left(v_2,\widetilde{\Omega}_2\right) = 0 \tag{4.19}
$$

with $c_2(t) = \widetilde{c}_1(e^t)$, $\widetilde{\Omega}_2(t) = \Omega_2(e^t)$, and

$$c_2(t) = \frac{B_2}{t^2} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-2}^2 t} + O(e^{-t}). \tag{4.20}$$

We use the notation

$$\widetilde{r}_2(t) = \widetilde{r}_1(e^t)e^t, \ldots, \widetilde{r}_k(t) = \widetilde{r}_{k-1}(e^t)e^t, \tag{4.21}$$

in the next computations. With this notation, we have

$$v_2' + \frac{B_2}{t^2} + \cdots \frac{B_n}{t^2 \operatorname{Log}_{n-2}^2 t} + O(e^{-t}) + (p-1)\widetilde{r}_2^{\,1-q} e^t H\left(v_2, \widetilde{\Omega}_2\right) = 0. \tag{4.22}$$

We apply the transformation $v_3 = tv_2 - U_3$ to (4.22). We obtain

$$v_3' + \frac{\widetilde{c}_3(t)}{t} + (p-1)\widetilde{r}_2^{\,1-q} e^t t^{1-q} H(v_3, \Omega_3) = 0, \tag{4.23}$$

where, with $f(t) = t^{1/p}$ and $R^{-1}(t) = \widetilde{r}_2^{\,1-q} e^t$,

$$\Omega_3 = f\Phi\left(Rf' + \Phi^{-1}\left(\widetilde{\Omega}_2\right)\right) = \widetilde{r}_2 t \Gamma_p \left(1 + \frac{1}{(p-1)e^t} + \frac{1}{(p-1)te^t}\right)^{p-1}, \tag{4.24}$$

$$\frac{\widetilde{c}_3(t)}{t} := X_3 + Y_3 + Z_3 + U_3' + tc_2(t), \tag{4.25}$$

with

$$\begin{aligned}
U_3 = -t\widetilde{\Omega}_2 + \Omega_3 &= t\widetilde{r}_2\Gamma_p\left\{-\left(1 + \frac{1}{(p-1)e^t}\right)^{p-1} + \left(1 + \frac{1}{(p-1)e^t} + \frac{1}{(p-1)te^t}\right)^{p-1}\right\} \\
&= t\widetilde{r}_2\Gamma_p\left\{-\left(1 + \frac{1}{e^t} + \frac{p-2}{2(p-1)e^{2t}} + O\left(e^{-3t}\right)\right)\right. \\
&\qquad \left. +1 + \frac{1}{e^t} + \frac{1}{te^t} + \frac{p-2}{2(p-1)e^{2t}}\left(1 + \frac{1}{t}\right)^2 + O\left(e^{-3t}\right)\right\} \\
&= \widetilde{r}_2\Gamma_p\left\{\frac{1}{e^t} + \frac{p-2}{(p-1)e^{2t}} + \frac{p-2}{2(p-1)te^{2t}} + O\left(te^{-3t}\right)\right\},
\end{aligned}$$

$$
\begin{aligned}
X_3 &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}\left(\widetilde{\Omega}_2\right)\right)U_3 = -\left[\frac{1}{t} + (p-1)e^t\left(1 + \frac{1}{(p-1)e^t}\right)\right]\widetilde{r}_2\Gamma_p \\
&\quad \times \left[\frac{1}{e^t} + \frac{p-2}{(p-1)e^{2t}} + \frac{p-2}{2(p-1)te^{2t}} + O\left(te^{-3t}\right)\right] \\
&= -\widetilde{r}_2\Gamma_p\left[\frac{1}{te^t} + \frac{p-2}{(p-1)te^{2t}} + \frac{p-2}{2(p-1)t^2e^{2t}} + (p-1) + \frac{p-2}{e^t}\right. \\
&\qquad\qquad \left. + \frac{p-2}{2te^t} + \frac{1}{e^t} + \frac{p-2}{(p-1)e^{2t}} + \frac{p-2}{2(p-1)te^{2t}} + O\left(te^{-2t}\right)\right] \\
&= -\widetilde{r}_2\Gamma_p\left[(p-1) + \frac{p-1}{e^t} + \frac{p}{2te^t} + O\left(te^{-2t}\right)\right], \\
Y_3 &= -(p-1)\widetilde{r}_2^{1-q}e^t t^{1-q}\widetilde{\Omega}_2^q \\
&= -(p-1)\widetilde{r}_2^{1-q}e^t t^{1-q}\widetilde{r}_2^q\Gamma_p^q t^q\left[1 + \frac{1}{(p-1)e^t}\right]^p \\
&= -(p-1)\gamma_p\widetilde{r}_2 e^t t\left[1 + \frac{p}{(p-1)e^t} + \frac{p}{2(p-1)e^{2t}} + O\left(e^{-2t}\right)\right] \\
&= \widetilde{r}_2\left[-(p-1)\gamma_p te^t - p\gamma_p t - \frac{p}{2}\gamma_p\frac{t}{e^t} + O\left(te^{-2t}\right)\right], \\
Z_3 &= (p-1)\widetilde{r}_2^{1-q}e^t t^{1-q}\Omega_3^q \\
&= (p-1)\widetilde{r}_2^{1-q}e^t t^{1-q}t^q\widetilde{r}_2^q\gamma_p\left[1 + \frac{1}{(p-1)e^t} + \frac{1}{(p-1)te^t}\right]^p \\
&= (p-1)t\widetilde{r}_2\gamma_p e^t\left[1 + \frac{p}{(p-1)e^t} + \frac{p}{(p-1)te^t} + \frac{p}{2(p-1)e^{2t}}\left(1 + \frac{1}{t}\right)^2 + O\left(e^{-3t}\right)\right] \\
&= \widetilde{r}_2\left[(p-1)\gamma_p e^t t + p\gamma_p t + p\gamma_p + \frac{pt}{2e^t}\gamma_p + p\gamma_p\frac{1}{e^t} + \frac{p}{2}\gamma_p\frac{1}{te^t} + O\left(te^{-2t}\right)\right].
\end{aligned}
\tag{4.26}
$$

Substituting into (4.25) the above computed quantities, we have

$$
\frac{\widetilde{c}_3(t)}{t} = \frac{B_2 - \mu_p}{t} + \frac{B_3}{t\log^2 t} + \cdots + \frac{B_n}{t\,\mathrm{Log}_{n-2}^2 t} + O\left(te^{-t}\right). \tag{4.27}
$$

Consequently, if $B_2 = \mu_p = (1/2)((p-1)/p)^{p-1}$, we obtain

$$
v_3' + \sum_{j=3}^{n}\frac{B_j}{t\,\mathrm{Log}_{j-2}^2 t} + O\left(te^{-t}\right) + (p-1)\widetilde{r}_2^{1-q}(t)e^t t^{1-q}H(v_3, \Omega_3(t)) = 0. \tag{4.28}
$$

In this equation, the change of independent variable $t \mapsto e^t$ results

$$v_3' + c_3(t) + (p-1)\tilde{r}_3^{1-q}(t)E_2(t)H\left(v_3, \tilde{\Omega}_3(t)\right) = 0, \qquad c_3(t) := \tilde{c}_3(e^t). \tag{4.29}$$

Here, and also in the sequel, we use the notation

$$e_1(t) := e^t, \dots, e_n(t) := e_{n-1}(e^t), \qquad E_n(t) := e_n(t) \cdots e_1(t), \tag{4.30}$$

where n is the integer in (4.1).

Now we are already in a position to make the induction step in transformations of modified Riccati equations. We suppose that $B_j = \mu_p$ for $j = 1, \dots, k-2$ for some $k \in \{3, \dots, n\}$, so we have

$$v_{k-1}' + c_{k-1}(t) + (p-1)\tilde{r}_{k-1}^{1-q}(t)E_{k-2}(t)H\left(v_{k-1}, \tilde{\Omega}_{k-1}(t)\right) = 0, \tag{4.31}$$

with

$$\tilde{\Omega}_{k-1}(t) = \Gamma_p \tilde{r}_{k-1}(t)\left(1 + \frac{1 + E_1(t) + \cdots + E_{k-3}(t)}{E_{k-2}(t)}\right)^{p-1}, \tag{4.32}$$

$$c_{k-1}(t) = \frac{B_{k-1}}{t^2} + \cdots + \frac{B_n}{t^2 \operatorname{Log}_{n-k+1}^2 t} + O\left(\frac{tE_{k-3}^3(t)}{E_{k-2}(t)}\right), \tag{4.33}$$

where $\tilde{r}_k$ is given by (4.21). We will also use the notation

$$r_k(t) := r_{k-1}(e^t), \qquad r_1(t) := \tilde{r}_1(t) = r(e^t). \tag{4.34}$$

Then $\tilde{r}_k(t) = r_k(t)E_{k-1}(t)$ and $r_k(t) = r(e_k(t))$ with r given by (4.3).

We put $v_k = tv_{k-1} - U_k$. We have

$$\begin{aligned} U_k &= -t\tilde{\Omega}_{k-1} + \Omega_k \\ &= t\tilde{r}_{k-1}\Gamma_p \left\{ -\left[1 + \frac{1 + \cdots + E_{k-3}}{(p-1)E_{k-2}}\right]^{p-1} + \left[1 + \frac{1 + \cdots + E_{k-3}}{(p-1)E_{k-2}} + \frac{1}{(p-1)tE_{k-2}}\right]^{p-1} \right\} \\ &= r_{k-1}\Gamma_p \left[1 + \frac{(p-2)(1 + \cdots + E_{k-3})}{(p-1)E_{k-2}} + \frac{p-2}{2(p-1)tE_{k-2}} + O\left(\frac{tE_{k-3}^3}{E_{k-2}^2}\right)\right], \end{aligned} \tag{4.35}$$

and with $f(t) = t^{1/p}$, $R^{-1} = \tilde{r}_{k-1}^{1-q} E_{k-2}$ and $\tilde{\Omega}_{k-1}$ given by (4.32)

$$\begin{aligned}
X_k &= -p\left(\frac{f'}{f} + R^{-1}\Phi^{-1}\left(\tilde{\Omega}_{k-1}\right)\right)U_k \\
&= -\tilde{r}_{k-1}\Gamma_p\left[\frac{1}{t} + (p-1)E_{k-2} + (1+\cdots+E_{k-3})\right] \\
&\quad\times\left[\frac{1}{E_{k-2}} + \frac{(p-2)(1+\cdots+E_{k-3})}{(p-1)E_{k-2}^2} + \frac{p-2}{2(p-1)tE_{k-2}^2} + O\left(\frac{tE_{k-3}^3}{E_{k-2}^2}\right)\right], \\
&= -r_{k-1}\Gamma_p\left[(p-1)E_{k-2} + (p-1)(1+\cdots+E_{k-3}) + \frac{p}{2t} + \frac{(p-2)(1+\cdots+E_{k-3})^2}{(p-1)E_{k-2}} + O\left(\frac{tE_{k-3}^3}{E_{k-2}}\right)\right], \\
Y_k &= -(p-1)R^{-1}f^p\tilde{\Omega}_{k-1}^q = -(p-1)\tilde{r}_{k-1}^{1-q}E_{k-2}t\tilde{r}_{k-1}^q\gamma_p\left(1 + \frac{1+\cdots+E_{k-3}}{(p-1)E_{k-2}}\right)^p \\
&= -r_{k-1}\left[(p-1)\gamma_p tE_{k-2}^2 - p\gamma_p tE_{k-2}(1+\cdots+E_{k-3}) + \frac{p\gamma_p}{2}t(1+\cdots+E_{k-3})^2 + O\left(\frac{tE_{k-3}^3}{E_{k-2}}\right)\right], \\
Z_k &= (p-1)R^{-1}f^{-q}\tilde{\Omega}_{k-1}^q = (p-1)\tilde{r}_{k-1}E_{k-2}t\gamma_p\left[1 + \frac{(1+\cdots+E_{k-3})}{(p-1)E_{k-2}} + \frac{1}{(p-1)tE_{k-2}}\right]^p \\
&= r_{k-1}\Bigg[(p-1)\gamma_p tE_{k-2}^2 + p\gamma_p tE_{k-2}(1+\cdots+E_{k-3}) + p\gamma_p E_{k-2} \\
&\qquad + \frac{p\gamma_p}{2}t(1+\cdots+E_{k-3})^2 + p\gamma_p(1+\cdots+E_{k-3}) + \frac{p\gamma_p}{2t} + O\left(\frac{tE_{k-3}^3}{E_{k-2}}\right)\Bigg].
\end{aligned} \tag{4.36}$$

Then, using that $(p-1)\Gamma_p = p\gamma_p$ and

$$\frac{p}{2}(\gamma_p - \Gamma_p) = -\frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1} = -\mu_p, \tag{4.37}$$

we have

$$X_k + Y_k + Z_k = -\frac{\mu_p}{t} + O\left(\frac{tE_{k-3}^3}{E_{k-2}}\right). \tag{4.38}$$

The last formula is the result of a direct computation where one needs to show that all terms with the faster growth than t^{-1} vanish. Further, again by a direct computation

$$\begin{aligned}
U_k' &= \left\{r_{k-1}\Gamma_p\left[1 + \frac{(p-2)(1+\cdots+E_{k-3})}{(p-1)E_{k-2}} + \frac{p-2}{2(p-1)tE_{k-2}} + O\left(\frac{tE_{k-3}^3}{E_{k-2}^2}\right)\right]\right\}' \\
&= O\left(\frac{tE_{k-3}^2}{E_{k-2}}\right).
\end{aligned} \tag{4.39}$$

Consequently, in the resulting modified Riccati equation for v_k

$$v_k' + \frac{\widetilde{c}_k(t)}{t} + (p-1)R^{-1}(t)H(v_k, \Omega_k(t)) = 0, \tag{4.40}$$

with $R^{-1}(t) = \widetilde{r}_{k-1}^{-1}(t)E_{k-2}(t)t^{1-q}$ we have

$$\frac{\widetilde{c}_k(t)}{t} := X_k + Y_k + Z_k + U_k' + tc_{k-1} = \frac{B_{k-1} - \mu_p}{t} + \sum_{j=k}^{n} \frac{B_j}{t \operatorname{Log}_{j+1-k}^2} + O\left(\frac{tE_{k-3}^3}{E_{k-2}}\right), \tag{4.41}$$

as $t \to \infty$.

Now we can summarize the previous computations as follows.

Theorem 4.1. *Suppose that there exists $k \in \{2, \dots, n\}$ such that*

$$\beta_j - \gamma_p \alpha_j = \mu_p, \quad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}, \quad j = 1, \dots, k-1, \tag{4.42}$$

and $\beta_k - \gamma_p \alpha_k \neq 0$. Then (1.11) *is oscillatory if $\beta_k - \gamma_p \alpha_k > \mu_p$ and nonoscillatory if $\beta_k - \gamma_p \alpha_k < \mu_p$. If* (4.42) *holds for all $j = 1, \dots, n$,* (1.11) *is nonoscillatory.*

Proof. We apply Proposition 2.2 to the modified Riccati equation (4.40) for v_k. In this equation, with the notation from Proposition 2.2,

$$\begin{aligned} \mathcal{R} &= R^{-1}|\Omega_k|^{q-2} \sim t^{q-2}\widetilde{r}_{k-1}^{q-2}\Gamma_p^{q-2}\widetilde{r}_{k-1}^{1-q}E_{k-2}t^{1-q} \\ &= r_{k-1}t^{-1}q^{p-2} \sim q^{p-2}t^{-1}, \end{aligned} \tag{4.43}$$

hence $\int^t \mathcal{R}(s)ds \sim q^{p-2}\log t$ and

$$\int_t^\infty c_k(s)ds \sim \frac{B_k}{\log t}, \qquad B_k = \beta_k - \alpha_k \gamma_p. \tag{4.44}$$

Here (and also earlier), $f \sim g$ for a pair of functions f, g means $\lim_{t\to\infty}(f(t)/g(t)) = 1$. Consequently, if $B_k q^{p-2} > 1/(2q)$, what happens if and only if $B_k > \mu_p$, modified Riccati equation (4.40) for v_k has no proper solution in view of Proposition 2.2 (iii). Now, via the "back" transformations

$$v_{j-1} = -U_j + \frac{v_j}{t}, \quad j = 2, \dots, k, \qquad w = -U_1 + t^{(1-p)/p}v_1, \tag{4.45}$$

the same holds for the Riccati equation associated with (1.11) and hence this equation is oscillatory by Proposition 2.1.

If $B_k < \mu_p$, nonoscillation of (1.11) follows from parts (i) (when $B_k < 0$) and (ii) (when $0 \leq B_k < \mu_p$) of Proposition 2.2 since the existence of a proper solution of the modified

Riccati equation for v_{k+1} implies the existence of a proper solution for the Riccati equation (4.4) associated with (1.11), hence this equation is nonoscillatory by Proposition 2.1.

Finally, if (4.42) holds for all $j = 1, \dots, n$, then the absolute term in the modified Riccati equation for v_{n+1} is $d(t) := \tilde{c}_{n+1}(t)/t = O(tE_{n-2}^3(t)/E_{n-1}(t))$ and replacing d by its nonnegative part $d^+ = \max\{0, d\}$, we get a majorant of the modified Riccati equation for v_{n+1} (in the sense of Proposition 2.3). The function d^+ satisfies the same asymptotic estimate as d. To estimate the integral $\int_t^\infty d^+(s)ds$ we proceed as follows. We have, via the substitution $e_{n-2}(s) = u$, $E_{n-2}(s)\ ds = du$, using the inequality $\log_j u \le u$, and followed by integration by parts,

$$
\begin{aligned}
\int_t^\infty \frac{sE_{n-2}^3(s)}{E_{n-1}(s)}ds &= \int_t^\infty \frac{s(e_1(s)\cdots e_{n-2}(s))^3}{e_{n-1}(s)\cdots e_1(t)}ds \\
&= \int_{e_{n-2}(t)}^\infty \frac{\log_{n-2} u \log_{n-3} u \cdots \log u \cdot u}{e^u}du \\
&\le \int_{e_{n-2}(u)}^\infty \frac{u^{n-1}}{e^u}du \\
&\sim -u^{n-1}e^{-u}\Big|_{e_{n-2}(u)}^\infty = \frac{e_{n-2}^{n-1}(u)}{e_{n-1}(u)}.
\end{aligned}
\tag{4.46}
$$

Consequently,

$$
\lim_{t\to\infty} \log t \int_t^\infty d^+(s)ds = 0, \tag{4.47}
$$

hence the modified Riccati equation with for v_{n+1} with d^+ instead of $\tilde{c}_{n+1}(t)/t$ possesses a proper solution by Proposition 2.2 and, by Proposition 2.3, the Riccati equation for v_{n+1} has the same property. This implies that (1.11) is nonoscillatory using the same argument as in the previous part of the proof. □

5. Remarks and Comments

(i) In the previous section, we applied successively the transformation $v_k = tv_{k-1} - U_k$ to the modified Riccati equation, followed by the change of independent variable $t \mapsto e^t$. This change of the independent variable was motivated by the linear case and also by the fact that upon this transformation the modified Riccati equation simplifies. Without this change of independent variable, the transformation procedure can be "reformulated" as follows. As shown at the beginning of the previous section, the transformation (2.2), that is, $v = h^p(t)w - G(t)$, $G(t) = r(t)h(t)\Phi(h'(t))$, transforms the Riccati equation (2.1) associated with (1.2) into the modified Riccati equation (2.3). The transformation (3.2), that is, $z = f^p(t)v - U(t)$, transforms (2.3) into an equation of the same form, with the function $\tilde{C}$ given by a relatively complicated formula (3.8). The composition of these transformations gives

$$
z = \big(f(t)h(t)\big)^p w - \big(f^p(t)G(t) + U(t)\big), \tag{5.1}
$$

and by a direct computation, using (3.9), we have $f^pG + U = rfh\Phi((fh)')$. So, the resulting modified Riccati equation for z is just the modified Riccati equation resulting from (2.1) via (2.2) with h replaced by fh. In this equation, the function $\widetilde{c}$ is given by (2.5) with h replaced by fh, that is, $\widetilde{c} = fh[(r\Phi((fh)')' + c\Phi(fh)]$.

Now, consider the function

$$h(t) = t^{(p-1)/p}\left(\mathrm{Log}_n t\right)^{1/p} = t^{(p-1)/p}\log^{1/p}t\cdots\log_n^{1/p}t. \tag{5.2}$$

In view of the previous consideration, the application of transformation (2.2) with this h can be decomposed into the successive transformations $v_1 = t^{p-1}w - G$, $v_j = \log_{j-1}tv_j - U_j$, $j = 2,\dots,k$. Hence, the successive transformations treated in the previous section can be replaced by just one transformation, with the transformation function (5.2).

This idea has been used in [3] in the case that $n = 1$ in (1.11) and in (5.2). However, as shown in the computations of that paper (where also substantially the results of [12] have been used), this method is technically complicated even in this relatively simple case. This is also the reason why we developed the method of successive transformations of modified Riccati equation presented in the previous section.

(ii) The reason why the perturbation terms in (1.11) are just $\alpha_j/\mathrm{Log}_j^2 t$ in the differential term and $\beta_j/t^p\mathrm{Log}_j^2 t$ by $\Phi(x)$ is motivated by the fact that in this form they "match together". More precisely if we replace some of them by a term with a faster asymptotic growth, then this term "overrules" the remaining terms and the equation becomes (non)oscillatory for any positive value of the corresponding parameter α_j or β_j. On the other hand, functions with slower asymptotic growth have no influence on the oscillatory behavior. These considerations are closely related to concepts of strong (non)oscillation of half-linear equations as treated for example in [13].

(iii) In [6], and partially also in [11], we have considered

$$\left[(r(t) + \lambda\widetilde{r}(t))\Phi\left(x'\right)\right]' + \left[c(t) + \mu\widetilde{c}(t)\right]\Phi(x) = 0 \tag{5.3}$$

as a perturbation of (1.2). We found assumptions of the functions $r, \widetilde{r}, c, \widetilde{c}$ (which are satisfied in case of the perturbed Euler equation) which guarantee that there exists a constant γ such that (5.3) is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$. The limiting case $\mu - \lambda = \gamma$ remained undecided, mainly because of technical computational problems. In view of perturbations of Euler equation with $n = 1$ $r(t) = 1$, $\widetilde{r}(t) = \gamma_p\log^{-2}t$, $c(t) = t^{-p}$, and $\widetilde{c}(t) = t^{-p}\log^{-2}t$ (then $\gamma = \mu_p$) we hope to prove that (5.3) is nonoscillatory also in the limiting case $\mu - \lambda = \gamma$. We also hope that the method of transformations of modified Riccati equation elaborated in this paper can be applied to treat the "multiparametric" general case, not only for perturbations of Euler equation.

Acknowledgment

The authors thank the referee for his/her remarks which contributed to better presentation of the result of the paper. Research supported by the Grant 201/11/0768 of the Grant Agency of the Czech Republic and the Research Project MUNI/A/0964/2009 of Masaryk University.

References

[1] O. Došlý and P. Řehák, *Half-Linear Differential Equations*, vol. 202 of *North-Holland Mathematics Studies*, Elsevier Science, Amsterdam, The Netherlands, 2005.

[2] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1964.

[3] O. Došlý and H. Haladová, "Half-linear Euler differential equations in the critical case," *Tatra Mountains Mathematical Publications*, vol. 48, pp. 41–49, 2011.

[4] Á. Elbert and A. Schneider, "Perturbations of the half-linear euler differential equation," *Results in Mathematics*, vol. 37, no. 1-2, pp. 56–83, 2000.

[5] H. Krüger and G. Teschl, "Effective Prüfer angles and relative oscillation criteria," *Journal of Differential Equations*, vol. 245, no. 12, pp. 3823–3848, 2008.

[6] O. Došlý and S. Fišnarová, "Two-parametric conditionally oscillatory half-linear differential equations," *Abstract and Applied Analysis*, vol. 2011, Article ID 182827, 16 pages, 2011.

[7] O. Došlý and M. Ünal, "Half-linear differential equations: linearization technique and its application," *Journal of Mathematical Analysis and Applications*, vol. 335, no. 1, pp. 450–460, 2007.

[8] J. Jaroš, K. Takaŝi, and T. Tanigawa, "Nonoscillation theory for second order half-linear differential equations in the framework of regular variation," *Results in Mathematics*, vol. 43, no. 1-2, pp. 129–149, 2003.

[9] J. Jaroš, K. Takaŝi, and T. Tanigawa, "Nonoscillatory half-linear differential equations and generalized Karamata functions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 64, no. 4, pp. 762–787, 2006.

[10] J. Sugie and N. Yamaoka, "Comparison theorems for oscillation of second-order half-linear differential equations," *Acta Mathematica Hungarica*, vol. 111, no. 1-2, pp. 165–179, 2006.

[11] O. Došlý and S. Fišnarová, "Half-linear oscillation criteria: perturbation in term involving derivative," *Nonlinear Analysis*, vol. 73, no. 12, pp. 3756–3766, 2010.

[12] O. Došlý, "Perturbations of the half-linear Euler-Weber type differential equation," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 426–440, 2006.

[13] T. Kusano, Y. Naito, and A. Ogata, "Strong oscillation and nonoscillation of quasilinear differential equations of second order," *Differential Equations and Dynamical Systems*, vol. 2, no. 1, pp. 1–10, 1994.