Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2012, Article ID 736929, 9 pages
doi:10.1155/2012/736929

Research Article

Proper Splitting for the Generalized Inverse $A^{(2)}_{T,S}$ and Its Application on Banach Spaces

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Received 7 December 2011; Accepted 16 March 2012

Academic Editor: Jean Michel Combes

A possible type of the operator splitting is studied. Using this operator splitting, we introduce some properties and representations of generalized inverses as well as iterative method for computing various solutions of the restricted linear operator system $Ax = b$, $x \in T$, where $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ and T is an arbitrary but fixed subspace of $\mathbf{X}$.

1. Introduction

The subject of splitting was investigated by numerous authors. And there are several papers concerning the iterative methods of the general formas follows:

$$X_{i+1} = M^{(2)}_{T,K} X_i + M^{(2)}_{T,K}, \quad i = 0, 1, 2, \ldots, \tag{1.1}$$

where $A = M - N$ is a splitting of A, and K is an arbitrary closed subspaces of $\mathbf{Y}$ (see [1–4]). Particular results concerning the computation of the Drazin inverse and the Moore-Penrose inverse can be investigated in [5, 6].

In this paper, we will consider the linear operator system $Ax = b$, where $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$, $T \oplus L = \mathbf{X}$, and $AT \oplus S = \mathbf{Y}$. The concept of a operator splitting can be used in characterizations of the generalized inverse $A^{(2)}_{T,S}$ and in iterative method as follows:

$$X_{k+1} = M^{(2)}_{T,K} N X_k + M^{(2)}_{T,K} G b, \quad i = 0, 1, 2, \ldots \tag{1.2}$$

for solving linear operator system $Ax = b$. In particular, let $K = L$, the authors in [7] gave the iterative method

$$X_{i+1} = M_g N X_i + M_g G b, \quad i = 0, 1, 2, \ldots. \tag{1.3}$$

to solve linear system $Ax = b$.

Now, we introduce some notations and terminologies.

Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces and $\mathcal{L}(\mathbf{X}, \mathbf{Y})$ be the set of all bounded operators from $\mathbf{X}$ to $\mathbf{Y}$. We use $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\rho(A)$, respectively, to denote the range, the null space, and the spectral radius for an operator $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$. Suppose that there exists an operator $B \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$, such that

$$BAB = B, \qquad \mathcal{R}(B) = T, \qquad \mathcal{N}(B) = S. \tag{1.4}$$

Then B is usually denoted by $A^{(2)}_{T,S}$. Recall that the splitting $A = U - V$ is called a proper splitting of A if $\mathcal{R}(A) = \mathcal{R}(U)$ and $\mathcal{N}(A) = \mathcal{N}(U)$.

Let $T \in \mathcal{L}(X)$ if, for some nonnegative integer $k \geq 0$, there exists $S \in \mathcal{L}(X)$ such that

$$\left(1^k\right)\ TST^k = T^k, \qquad (2)\ STS = S, \qquad (5)\ TS = ST, \tag{1.5}$$

then S is called the Drazin inverse of T, and the smallest n is called the index of T and will be denoted by $\operatorname{ind}(T)$. If an operator $T \in \mathcal{L}(X)$ has a Drazin inverse, then it is unique and is denoted by T_d (see [8] for details). Particularly, when $\operatorname{ind}(T) \leq 1$, T_d is called the group inverse of T and is denoted by T_g.

The paper is organized as follows. In the remainder of this section, we will introduce some lemmas which are useful in the proofs. In Section 2, We express some properties and representations of generalized inverses based on the operator splitting. Moreover, by using these representations, we introduce iterative method for computing various solutions of linear operator system. In Section 3, we give a numerical example to demonstrate one of the results in Section 2.

Basic auxiliary results are summarized in the following lemmas.

Lemma 1.1 (see [9, page 43]). *Let $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ and S, T be closed subspaces of $\mathbf{X}$ and $\mathbf{Y}$ such that $T \oplus L = \mathbf{X}$ and $AT \oplus S = \mathbf{Y}$. Then A has the following matrix form:*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} : \begin{bmatrix} T \\ L \end{bmatrix} \longrightarrow \begin{bmatrix} AT \\ S \end{bmatrix}, \tag{1.6}$$

where A_1 is invertible. Moreover,

$$A^{(2)}_{T,S} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} AT \\ S \end{bmatrix} \longrightarrow \begin{bmatrix} T \\ L \end{bmatrix}. \tag{1.7}$$

Lemma 1.2 (see [9, lemma 3.5.2]). *Let $B \in \mathcal{L}(\mathbf{X})$. L and M are closed subspaces of X such that $\mathbf{X} = L \oplus M$ and $P_{L,M}$ is a projection from $\mathbf{X}$ onto $\mathbf{Y}$ parallel to M. Then, one has*

(i) $P_{L,M}B = B$ *if and only if* $\mathcal{R}(B) \subseteq L$,

(ii) $BP_{L,M} = B$ *if and only if* $\mathcal{N}(B) \supseteq M$.

Lemma 1.3 (see [6]). *Let* $A \in \mathbb{C}_r^{m\times n}$ *be factorized in the following form:*

$$A = U\begin{bmatrix} I \\ C \end{bmatrix} A_{11} \begin{bmatrix} I & B \end{bmatrix} V, \tag{1.8}$$

where A_{11} *is an* $r \times r$ *nonsingular matrix, and* U *and* V *are permutation matrices. Then*

$$\mathcal{N}(M) = \mathcal{N}(A), \qquad \mathcal{R}(M) = \mathcal{R}(A) \tag{1.9}$$

(i.e., $A = M - N$ *is a proper splitting) if and only if*

$$M = U\begin{bmatrix} I \\ C \end{bmatrix} M_{11} \begin{bmatrix} I & B \end{bmatrix} V, \tag{1.10}$$

where M_{11} *is a nonsingular matrix of order* r.

Lemma 1.4 (see [10]). *Let* $A \in \mathbb{C}_r^{m\times n}$ *with* $\operatorname{ind}(A) = 1$ *and* A *be partitioned as*

$$A = U\begin{bmatrix} I \\ C \end{bmatrix} A_{11} \begin{bmatrix} I & B \end{bmatrix} V, \tag{1.11}$$

where U, V *and* A_{11} *are the same as Lemma 1.3. Then the group inverse* A_g *exists if and only if*

$$\begin{gathered} VU + BVUC \ \text{ is nonsingular}, \\ A_g = U\begin{bmatrix} I \\ C \end{bmatrix} (VU + BVUC)^{-1} A_{11}^{-1} (VU + BVUC)^{-1} \begin{bmatrix} I & B \end{bmatrix} V. \end{gathered} \tag{1.12}$$

2. Main Results

In this section, the representation of the generalized inverse $A_{T,S}^{(2)}$ of a linear operator is studied. Moreover, we consider the fundamental problem of solving a general linear operator equation of the type $Ax = b$, where $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$.

Now we are ready to present the representation theorem.

Theorem 2.1 (representation theorem). *Let* $A \in \mathcal{L}(\mathbf{X}, \mathbf{Y})$ *be given and* T *and* S*, respectively, be closed subspaces of* $\mathbf{X}$ *and* $\mathbf{Y}$ *such that there exists the generalized inverse* $A_{T,S}^{(2)}$. *And let* $G \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$ *with* $\mathcal{R}(G) = T$, $\mathcal{N}(G) = S$, $GA = M - N$ *be a proper splitting of* GA*, that is,* $\mathcal{N}(M) = \mathcal{N}(GA)$, $\mathcal{R}(M) = \mathcal{R}(GA)$. *Then, one has*

$$A_{T,S}^{(2)} = \left(I - M_{T,K}^{(2)} N\right)^{-1} M_{T,K}^{(2)} G, \tag{2.1}$$

where K *is a subspace of* $\mathbf{X}$.

Proof. By Lemma 1.2, it is easy to verify that

$$\begin{aligned}\left(I - M_{T,K}^{(2)}N\right)A_{T,S}^{(2)} &= A_{T,S}^{(2)} - M_{T,K}^{(2)}NA_{T,S}^{(2)} \\ &= A_{T,S}^{(2)} - M_{T,K}^{(2)}(M - GA)A_{T,S}^{(2)} \\ &= A_{T,S}^{(2)} - M_{T,K}^{(2)}MA_{T,S}^{(2)} + M_{T,K}^{(2)}GAA_{T,S}^{(2)} \\ &= M_{T,K}^{(2)}G.\end{aligned} \tag{2.2}$$

Now, we will show that $I - M_{T,K}^{(2)}N$ is invertible. Notice that

$$I - M_{T,K}^{(2)}N = I - M_{T,K}^{(2)}M + M_{T,K}^{(2)}GA. \tag{2.3}$$

Since $I - M_{T,K}^{(2)}M$ is a projection from Y onto L parallel to T and $M_{T,K}^{(2)}GA$ is an invertible operator from T to T, we get that $I - M_{T,K}^{(2)}M + M_{T,K}^{(2)}GA$ is invertible. Thus, the proof is completed. □

Now, we are in position to state the main result of this section.

Theorem 2.2. *Under the hypotheses of Theorem 2.1, we have that*

$$X_{k+1} = M_{T,K}^{(2)}NX_k + M_{T,K}^{(2)}G \tag{2.4}$$

converges to $A_{T,S}^{(2)}$ for every $X_0 \in \mathbf{X}$ if and only if $\rho(M_{T,K}^{(2)}N) < 1$. Then we get that X_k has the error estimation

$$\|X_{k+1} - X_k\| \le \left\|M_{T,K}^{(2)}N\right\|^k \|X_1 - X_0\|. \tag{2.5}$$

Moreover, one has

$$\left\|X_k - A_{T,S}^{(2)}\right\| \le \left\|M_{T,K}^{(2)}N\right\|^{k+1} \left\|X_0 - A_{T,S}^{(2)}\right\|, \tag{2.6}$$

where K is a subspace of $\mathbf{X}$.

Proof. "$\Leftarrow$" Suppose that $\rho(M_{T,K}^{(2)}N) < 1$. According to Representation theorem, we obtain that

$$\begin{aligned}
X_{k+1} - A_{T,S}^{(2)} &= M_{T,K}^{(2)}NX_k + M_{T,K}^{(2)}G - \left(I - M_{T,K}^{(2)}N\right)^{-1}M_{T,K}^{(2)}G \\
&= M_{T,K}^{(2)}NX_k + M_{T,K}^{(2)}G - \sum_{k=0}^{\infty}\left(M_{T,K}^{(2)}N\right)^k M_{T,K}^{(2)}G \\
&= M_{T,K}^{(2)}NX_k - \sum_{k=1}^{\infty}\left(M_{T,K}^{(2)}N\right)^k M_{T,K}^{(2)}G \\
&= M_{T,K}^{(2)}N\left(X_k - \sum_{k=0}^{\infty}\left(M_{T,K}^{(2)}N\right)^k M_{T,K}^{(2)}G\right) \\
&= M_{T,K}^{(2)}N\left(X_k - A_{T,S}^{(2)}\right) \\
&= \left(M_{T,K}^{(2)}N\right)^{k+1}\left(X_0 - A_{T,S}^{(2)}\right).
\end{aligned} \tag{2.7}$$

By the hypothesis and (2.7), it is to see that X_k converges to $A_{T,S}^{(2)}$ for every $X_0 \in \mathbf{X}$.

"$\Rightarrow$" A simple computation shows that

$$\begin{aligned}
X_{k+1} - X_k &= M_{T,K}^{(2)}NX_k - M_{T,K}^{(2)}NX_{k-1} \\
&= M_{T,K}^{(2)}N(X_k - X_{k-1}) \\
&= \cdots \\
&= \left(M_{T,K}^{(2)}N\right)^k(X_1 - X_0),
\end{aligned} \tag{2.8}$$

it follows that $(M_{T,K}^{(2)}N)^k \to 0$ as $k \to \infty$, which implies that $\rho(M_{T,K}^{(2)}N) < 1$. Now the results follow immediately. □

From Theorem 2.2, we can immediately obtain the following theorem.

Theorem 2.3. *Under the hypotheses of Theorem 2.1, we have that*

$$X_{k+1} = M_{T,K}^{(2)}NX_k + M_{T,K}^{(2)}Gb \tag{2.9}$$

converges to $A_{T,S}^{(2)}b$ (the unique solution of linear operator system $Ax = b$.) for every $X_0 \in \mathbf{X}$ if and only if $\rho(M_{T,K}^{(2)}N) < 1$. Then we get that X_k has the error bounded

$$\|X_{k+1} - X_k\| \le \left\|M_{T,K}^{(2)}N\right\|^k \|X_1 - X_0\|. \tag{2.10}$$

Moreover, one has

$$\left\|X_k - A_{T,S}^{(2)}b\right\| \leq \left\|M_{T,K}^{(2)}N\right\|^{k+1}\left\|X_0 - A_{T,S}^{(2)}b\right\|, \tag{2.11}$$

where K is a subspace of $\mathbf{X}$.

Proof. The proof is similar to that of Theorem 2.2. □

Similar to [7, Theorems 3.1 and 3.3]. In the case when $K = L$, the previous theorems reduce to the following corollary.

Corollary 2.4. *Let A, T and S be the same as Theorem 2.1 and let $G \in \mathcal{L}(\mathbf{X},\mathbf{Y})$, $\mathcal{R}(G) = T$ and $N(G) = S$. Let $GA = M - N$ be a proper splitting of GA, that is,*

$$\mathcal{N}(M) = \mathcal{N}(GA), \qquad \mathcal{R}(M) = \mathcal{R}(GA). \tag{2.12}$$

Then, one has

(i) $\operatorname{ind}(M) = 1$,

(ii) $A_{T,S}^{(2)} = (I - M_g N)^{-1} M_g G$, *and*

(iii) *the iteration $X_{i+1} = M_g N X_i + M_g G$ converges to $A_{T,S}^{(2)}$ for each $X_0 \in \mathbf{X}$ if and only if $\rho(M_g N) < 1$.*

Now, we will consider the proper splitting of AG.

Theorem 2.5. *Let $A \in \mathcal{L}(\mathbf{X},\mathbf{Y})$ be given and T and S, respectively, be closed subspaces of $\mathbf{X}$ and $\mathbf{Y}$, such that there exists the generalized inverse $A_{T,S}^{(2)}$. And let $G \in \mathcal{L}(\mathbf{Y},\mathbf{X})$ with $\mathcal{R}(G) = T$, $\mathcal{N}(G) = S$, $AG = M - N$ be a proper splitting of AG, that is, $\mathcal{N}(M) = \mathcal{N}(AG)$, $\mathcal{R}(M) = \mathcal{R}(AG)$. Then, one has*

$$A_{T,S}^{(2)} = GM_{K,S}^{(2)}\left(I - NM_{K,S}^{(2)}\right)^{-1}, \tag{2.13}$$

where K is a subspace of $\mathbf{Y}$.

Theorem 2.6. *Under the hypotheses of Theorem 2.5, we have that*

$$X_{k+1} = NM_{K,S}^{(2)}X_k + GM_{K,S}^{(2)} \tag{2.14}$$

converges to $A_{T,S}^{(2)}$ for every $X_0 \in \mathbf{X}$ if and only if $\rho(NM_{K,S}^{(2)}) < 1$. Then we get that X_k has the error estimation as follows:

$$\left\|X_k - A_{T,S}^{(2)}\right\| \leq \left\|NM_{K,S}^{(2)}\right\|^{k+1}\left\|X_0 - A_{T,S}^{(2)}\right\|. \tag{2.15}$$

Moreover, consider that

$$\|X_{k+1} - X_k\| \leq \left\| N M_{K,S}^{(2)} \right\|^k \|X_1 - X_0\|, \tag{2.16}$$

where K is a subspace of $\mathbf{Y}$.

It is well known that $A_d = A^{(2)}_{R(A^k),N(A^k)}$. Let $G = A^k$; thus, the following result follows from Corollary 2.4.

Corollary 2.7. *Let $A \in \mathcal{L}(\mathbf{X},\mathbf{Y})$, $G \in \mathcal{L}(\mathbf{Y},\mathbf{X})$, T, S be closed subspaces of X and Y such that $A^{(2)}_{T,S}$ exists with* $\operatorname{Ind}(A) = k$. *Suppose that $R(A^k) = T, N(A^k) = S$ and $A^{k+1} = M - N$ be a proper splitting of GA, that is,*

$$\mathcal{N}(M) = \mathcal{N}\left(A^{k+1}\right), \qquad \mathcal{R}(M) = \mathcal{R}\left(A^{k+1}\right). \tag{2.17}$$

Then the iteration

$$x_{i+1} = M_g N x_i + M_g A^k \tag{2.18}$$

or

$$x_{i+1} = M_g N x_i + M_g A^k b \tag{2.19}$$

converges to A_d or $A_d b$ for every $X_0 \in \mathbf{X}$ if and only if $\rho(M_g N) < 1$.

Remark 2.8. In [11], the author present $\{T,S\}$ splitting for computing the generalized inverse $A^{(2)}_{T,S}$, particular result concerning the computation Drazin inverse. Let $T = \mathcal{R}(A^k)$, $S = \mathcal{N}(A^k)$, $k = \operatorname{ind}(A)$, and $A = U - V$ be a $\{T,S\}$ splitting of A. Then the iteration

$$X_{i+1} = U_d V X_i + U_d b \tag{2.20}$$

converges to $A_d b$ for every $X_0 \in \mathbf{X}$ if and only if $\rho(U_d V) < 1$. Moreover, the iteration (2.20) can be reduced to

$$\mathcal{R}\left(U^k\right) = \mathcal{R}\left(A^k\right), \qquad \mathcal{N}\left(U^k\right) = \mathcal{N}\left(A^k\right). \tag{2.21}$$

The splitting (2.17) is more practical than (2.21). Since M in (2.17) can easily be calculated by Lemmas 1.3 and 1.4, while U in (2.21) is more difficult to be getten.

Table 1: Convergence of (2.6) with any choice of initial x_0.

x_0^T	k	r_k	R_k	X_k^T
[10; 2; −5; 9]	98	0.0016	0.0014	[5.0001, 5.0001, −2.0000, −2.0000]
[1; 20; 84; 9]	98	0.0039	0.0027	[5.0008, 5.0009, −2.0003, −2.0003]
[55; 6; 5; 2; 4]	98	0.0065	0.0013	[4.9990, 4.9993, −1.9997, −1.9997]
[0; 45; 600; 87]	98	0.0222	0.0088	[5.0058, 5.0058, −2.0023, −2.0023]
[66; 22; 1; −9]	98	0	$3.1765e-004$	[4.9998, 4.9998, −1.9999, −1.9999]
[−1; −99; 8; −7]	98	0.0023	$6.3226e-004$	[4.9997, 4.9995, −1.9998, −1.9998]
[−555; 0; 99; 34]	98	0.0004	0.0019	[4.9984, 4.9993, −1.9995, −1.9995]

3. Illustration Example

The following matrix A is from [11]. Let

$$A = \begin{bmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{bmatrix} \in \mathcal{R}^{4\times 4}, \tag{3.1}$$

where $\operatorname{ind}(A) = 2$, $\operatorname{rank}(A^2) = 2$.

We will use Corollary 2.7 to compute the unique solution $A_d b$ of the restricted linear equation $Ax = b,\quad x \in \mathcal{R}(A^2) = T$ and $b = (8, 7, -3, -3)^T \in \mathcal{R}(A^2)$. Take

$$N = \begin{bmatrix} 0 & -8 & -8 & -8 \\ -2 & 0 & -2 & -2 \\ 0.4 & 1.6 & 2.0 & 2.0 \\ 0.4 & 1.6 & 2.0 & 2.0 \end{bmatrix}, \qquad M = \begin{bmatrix} 3 & 0 & 3 & 3 \\ 0 & 7 & 7 & 7 \\ -0.6 & -1.4 & -2.0 & -2.0 \\ -0.6 & -1.4 & -2.0 & -2.0 \end{bmatrix} \tag{3.2}$$

which satisfies the conditions of Corollary 2.7. Therefore, by Lemmas 1.3 and 1.4,

$$M_d = \begin{bmatrix} \frac{25}{7} & \frac{20}{7} & \frac{45}{7} & \frac{45}{7} \\ \frac{20}{7} & \frac{55}{21} & \frac{115}{21} & \frac{115}{21} \\ -\frac{9}{7} & -\frac{23}{21} & -\frac{50}{21} & -\frac{50}{21} \\ -\frac{9}{7} & -\frac{23}{21} & -\frac{50}{21} & -\frac{50}{21} \end{bmatrix},$$
$$A_d = \begin{bmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{bmatrix}. \tag{3.3}$$

Denote that $r_k = \|x_{k+1} - x_k\|$ and $R_k = \|x_k - A_D b\|$. We have Table 1 for the norm $\|\cdot\|_2$, and from which we conclude that X_k is an approximation of the exact solution x of $Ax = b$.

Acknowledgments

The authors would like to thank the referees for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (11061005, 61165015), the Ministry of Education Science and Technology Key Project under Grant (210164), Grants HCIC201103 of Guangxi Key Laboratory of Hybrid Computational and IC Design Analysis Open Fund, Key issues for Department of Education of Guangxi (201202ZD031).

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