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Research Article

The Numerical Solution of the Bitsadze-Samarskii Nonlocal Boundary Value Problems with the Dirichlet-Neumann Condition

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We are interested in studying the stable difference schemes for the numerical solution of the nonlocal boundary value problem with the Dirichlet-Neumann condition for the multidimensional elliptic equation. The first and second orders of accuracy difference schemes are presented. A procedure of modified Gauss elimination method is used for solving these difference schemes for the two-dimensional elliptic differential equation. The method is illustrated by numerical examples.

1. Introduction

Methods of solution of the Bitsadze-Samarskii nonlocal boundary value problems for elliptic differential equations have been studied extensively by many researchers (see [1–22] and the references given therein).

Let Ω be the unit open cube in R^m $(x = (x_1, \ldots, x_m) : 0 < x_k < 1, 1 \leq k \leq m)$ with boundary S, $\overline{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the Bitsadze-Samarskii-type nonlocal boundary value problem for the multidimensional elliptic equation

$$-u_{tt} - \sum_{r=1}^{m} (a_r(x) u_{x_r})_{x_r} + \eta u = f(t,x), \quad 0 < t < 1, \; x = (x_1, \ldots, x_m) \in \Omega,$$

$$u(0,x) = \varphi(x), \quad u(1,x) = \sum_{j=1}^{J} \alpha_j u(\lambda_j, x) + \psi(x), \quad x \in \overline{\Omega},$$

$$\sum_{j=1}^{J}\left|\alpha_j\right| \leq 1, \quad 0<\lambda_1<\cdots<\lambda_J<1,$$
$$u(t,x)|_{x\in S^1}=0, \quad \left.\frac{\partial u(t,x)}{\partial \vec{n}}\right|_{x\in S^2}=0, \quad S^1\cup S^2=S \tag{1.1}$$

is considered. Here $a_r(x),(x\in\Omega),\psi(x),\varphi(x)\ (x\in\overline{\Omega})$, and $f(t,x)\ \ (t\in(0,1),\ x\in\Omega)$ are given smooth functions, $a_r(x)\geq a>0$, η is a positive number, and $\vec{\mathbf{n}}$ is the normal vector to Ω. We are interested in studying the stable difference schemes for the numerical solution of the nonlocal boundary value problem (1). The first and second orders of accuracy difference schemes are presented. The stability and almost coercive stability of these difference schemes are established. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of two-dimensional elliptic partial differential equations.

2. Difference Schemes: The Stability and Coercive Stability Estimates

The discretization of problem (1) is carried out in two steps. In the first step, let us define the grid sets

$$\begin{aligned}\widetilde{\Omega}_h=\{x=x_m=(h_1m_1,\ldots,h_mm_m),m=(m_1,\ldots,m_m),&\\ 0\leq m_r\leq N_r,h_rN_r=1,r=1,\ldots,m\},&\\ \Omega_h=\widetilde{\Omega}_h\cap\Omega,\quad S_h^r=\widetilde{\Omega}_h\cap S^r,\quad r=1,2.&\end{aligned}\tag{2.1}$$

We introduce the Hilbert space $L_{2h}=L_2(\widetilde{\Omega}_h)$ and $W_{2h}^2=W_2^2(\widetilde{\Omega}_h)$ of the grid functions $\varphi^h(x)=\{\varphi(h_1m_1,\ldots,h_mm_m)\}$ defined on $\widetilde{\Omega}_h$, equipped with the norms

$$\begin{aligned}\left\|\varphi^h\right\|_{W_{2h}^2}&=\left\|\varphi^h\right\|_{L_{2h}}+\left(\sum_{x\in\overline{\Omega_h}}\sum_{r=1}^{m}\left|\left(\varphi^h\right)_{x_r}\right|^2h_1\cdots h_m\right)^{1/2}\\&\quad+\left(\sum_{x\in\overline{\Omega_h}}\sum_{r=1}^{m}\left|\left(\varphi^h\right)_{x_r\overline{x_r},m_r}\right|^2h_1\cdots h_m\right)^{1/2},\\ \left\|\varphi^h\right\|_{L_2(\widetilde{\Omega}_h)}&=\left(\sum_{x\in\overline{\Omega_h}}\left|\varphi^h(x)\right|^2h_1\cdots h_m\right)^{1/2}.\end{aligned}\tag{2.2}$$

To the differential operator A generated by problem (1), we assign the difference operator A_h^x by the formula

$$A_h^xu^h=-\sum_{r=1}^{m}\left(a_r(x)u_{\overline{x_r}}^h\right)_{x_r,j_r}+\eta u^h(x), \tag{2.3}$$

acting in the space of grid functions $u^h(x)$, satisfying the conditions $u^h(x) = 0$ for all $x \in S_h^1$ and $D_h u^h(x) = 0$ for all $x \in S_h^2$. Here, $D_h u^h(x)$ is an approximation to $\partial u/\partial \vec{\mathbf{n}}$. It is known that A_h^x is a self-adjoint positive definite operator in $L_2(\widetilde{\Omega}_h)$. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$
\begin{gathered}
-\frac{d^2 u^h(t,x)}{dt^2} + A_h^x u^h(t,x) = f^h(t,x), \quad 0 < t < 1, \ x \in \Omega_h, \\
u^h(0,x) = \varphi^h(x), \quad u^h(1,x) = \sum_{j=1}^{J} \alpha_j u^h(\lambda_j, x) + \psi^h(x), \quad x \in \widetilde{\Omega}_h, \\
\sum_{j=1}^{J} |\alpha_j| \le 1, \quad 0 < \lambda_1 < \cdots < \lambda_J < 1,
\end{gathered}
\tag{2.4}
$$

for an infinite system of ordinary differential equations. In the second step, we replace problem (2.4) by the first and second orders of accuracy difference schemes

$$
\begin{gathered}
-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), \\
f_k^h(x) = f^h(t_k, x), \quad t_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = 1, \ x \in \Omega_h, \\
u_0^h(x) = \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
u_N^h(x) = \sum_{j=1}^{J} \alpha_j u_{[\lambda_j/\tau]}^h(x) + \psi^h(x), \quad x \in \widetilde{\Omega}_h,
\end{gathered}
\tag{2.5}
$$

$$
\begin{gathered}
-\frac{u_{k+1}^h(x) - 2u_k^h(x) + u_{k-1}^h(x)}{\tau^2} + A_h^x u_k^h(x) = f_k^h(x), \qquad f_k^h(x) = f^h(t_k, x), \\
t_k = k\tau, \quad 1 \le k \le N-1, \quad N\tau = 1, \quad x \in \Omega_h, \\
u_0^h(x) = \varphi^h(x), \quad x \in \widetilde{\Omega}_h, \\
u_N^h(x) = \sum_{j=1}^{J} \alpha_j \left(u_{[\lambda_j/\tau]}^h(x) + \left(u_{[\lambda_j/\tau]+1}^h(x) - u_{[\lambda_j/\tau]}^h(x) \right) \left(\frac{\lambda_j}{\tau} - \left[\frac{\lambda_j}{\tau} \right] \right) \right) + \psi^h(x), \quad x \in \widetilde{\Omega}_h.
\end{gathered}
\tag{2.6}
$$

To formulate our result on well-posedness, we will give definition of $C_{01}^{\alpha}([0,1]_\tau, H)$ and $C([0,1]_\tau, H)$. Let $F([0,1]_\tau, H)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_1^{N-1}$ with values in the Hilbert space H. We denote $C([0,1]_\tau, H)$ normed space with the norm

$$
\|\varphi^\tau\|_{C([0,1]_\tau, H)} = \max_{1 \le k \le N-1} \|\varphi_k\|_H,
\tag{2.7}
$$

and $C_{01}^{\alpha}([0,1]_\tau, H)$ normed space with the norm

$$\|\varphi^\tau\|_{C_{01}^{\alpha}([0,1]_\tau, H)} = \|\varphi^\tau\|_{C([0,1]_\tau, H)} + \sup_{1\le k\le k+r\le N-1} \frac{((N-k)\tau)^\alpha((k+r)\tau)^\alpha}{(r\tau)^\alpha}\|\varphi_{k+r}-\varphi_k\|_H. \quad (2.8)$$

Theorem 2.1. *Let τ and $|h|$ be sufficiently small positive numbers. Then, the solutions of difference schemes* (2.5) *and* (2.6) *satisfy the following stability and almost coercive stability estimates*

$$\begin{aligned} &\left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C([0,1]_\tau, L_{2h})} \le M_1\left[\left\|\varphi^h\right\|_{L_{2h}} + \left\|\psi^h\right\|_{L_{2h}} + \left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C([0,1]_\tau, L_{2h})}\right], \\ &\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\}_1^{N-1}\right\|_{C([0,1]_\tau, L_{2h})} + \left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C([0,1]_\tau,\ W_{2h}^2)} \\ &\quad\le M_2\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2} + \ln\frac{1}{\tau+|h|}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C([0,\,1]_\tau, L_{2h})}\right]. \end{aligned} \quad (2.9)$$

Here, M_1 and M_2 do not depend on τ, h, $\psi^h(x)$, $\varphi^h(x)$, and $f_k^h(x)$, $1 \le k \le N-1$.

Theorem 2.2. *Let τ and $|h|$ be sufficiently small positive numbers. Then, the solution of difference schemes* (2.5) *and* (2.6) *satisfies the following coercive stability estimate:*

$$\begin{aligned} &\left\|\left\{\frac{u_{k+1}^h - 2u_k^h + u_{k-1}^h}{\tau^2}\right\}_1^{N-1}\right\|_{C_{01}^{\alpha}([0,\,1]_\tau,\ L_{2h})} + \left\|\left\{u_k^h\right\}_1^{N-1}\right\|_{C_{01}^{\alpha}([0,\,1]_\tau,\ W_{2h}^2)} \\ &\quad\le M_3\left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|\psi^h\right\|_{W_{2h}^2} + \frac{1}{\alpha(1-\alpha)}\left\|\left\{f_k^h\right\}_1^{N-1}\right\|_{C_{01}^{\alpha}([0,\,1]_\tau,\ L_{2h})}\right]. \end{aligned} \quad (2.10)$$

M_3 is independent of τ, h, $\psi^h(x)$, $\varphi^h(x)$, and $f_k^h(x)$, $1 \le k \le N-1$.

Proofs of Theorems 2.1 and 2.2 are based on the symmetry properties of operator A_h^x defined by formula (2.3) and on the following formulas:

$$
\begin{aligned}
u_k^h(x) = {} & \left(I - R^{2N}\right)^{-1} \\
& \times \Bigg\{ \left(R^k - R^{2N-k}\right)\varphi^h(x) + \left(R^{N-k} - R^{N+k}\right)u_N^h(x) - \left(R^{N-k} - R^{N+k}\right) \\
& \qquad \times (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i^h(x)\tau \Bigg\} \\
& + (I+\tau B)(2I+\tau B)^{-1}B^{-1}\sum_{i=1}^{N-1}\left(R^{|k-i|} - R^{k+i}\right)f_i^h(x)\tau, \\
u_N^h(x) = {} & D^{-1}\Bigg(\sum_{k=1}^{J}\alpha_k\left(I - R^{2N}\right)^{-1} \\
& \times \Bigg\{\left(R^{[\lambda_k/\tau]} - R^{2N-[\lambda_k/\tau]}\right)\varphi^h(x) - \left(R^{N-[\lambda_k/\tau]} - R^{N+[\lambda_k/\tau]}\right)(I+\tau B)(2I+\tau B)^{-1}B^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i^h(x)\tau \Bigg\} + (I+\tau B)(2I+\tau B)^{-1}B^{-1} \\
& \times \Bigg(\sum_{i=1}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-i}f_i^h(x)\tau + \sum_{i=[\lambda_k/\tau]+1}^{N-1} R^{i-[\lambda_k/\tau]}f_i^h(x)\tau \\
& - \sum_{i=1}^{N-1} R^{[\lambda_k/\tau]+i}f_i^h(x)\tau\Bigg) + \psi^h(x)\Bigg),
\end{aligned}
\tag{2.11}
$$

for difference scheme (2.5), and

$$
\begin{aligned}
u_N^h(x) = {} & D^{-1}\Bigg(\sum_{k=1}^{J}\alpha_k\left(I - R^{2N}\right)^{-1} \\
& \times \Bigg\{\left(R^{[\lambda_k/\tau]} - R^{2N-[\lambda_k/\tau]}\right)\varphi^h(x) - \left(R^{N-[\lambda_k/\tau]} - R^{N+[\lambda_k/\tau]}\right)(I+\tau B)(2I+\tau B)^{-1} \\
& \times B^{-1}\sum_{i=1}^{N-1}\left(R^{N-i} - R^{N+i}\right)f_i^h(x)\tau \Bigg\} + (I+\tau B)(2I+\tau B)^{-1}B^{-1} \\
& \times \Bigg(\sum_{i=1}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-i}f_i^h(x)\tau + \sum_{i=[\lambda_k/\tau]+1}^{N-1} R^{i-[\lambda_k/\tau]}f_i^h(x)\tau - \sum_{i=1}^{N-1} R^{[\lambda_k/\tau]+i}f_i^h(x)\tau\Bigg) \\
& + \left(\frac{\lambda_k}{\tau} - \left[\frac{\lambda_k}{\tau}\right]\right)\left(I - R^{2N}\right)^{-1}
\end{aligned}
$$

$$\times \left\{ \tau B\left(R^{[\lambda_k/\tau]+1} - R^{2N-[\lambda_k/\tau]} \right)\varphi^h(x) - \left(R^{N-[\lambda_k/\tau]-1} - R^{N+[\lambda_k/\tau]} \right)(2I+\tau B)^{-1} \right.$$
$$\left. \times \sum_{i=1}^{N-1} \left(R^{N-i} - R^{N+i} \right) f_i^h(x)\tau^2 \right\}$$
$$+ (2I+\tau B)^{-1}\left(\left(\sum_{i=1}^{[\lambda_k/\tau]} R^{[\lambda_k/\tau]-i} f_i^h(x)\tau^2 + \sum_{i=[\lambda_k/\tau]+1}^{N-1} R^{i-[\lambda_k/\tau]-1} f_i^h(x)\tau^2 \right.\right.$$
$$\left.\left. - \sum_{i=1}^{N-1} R^{[\lambda_k/\tau]+i} f_i^h(x)\tau^2 \right) + \psi^h(x) \right), \tag{2.12}$$

for difference scheme (2.6). Here,

$$R = (I+\tau B)^{-1},$$
$$B = \frac{\tau A}{2} + \sqrt{\frac{\tau^2 A^2}{4} + A}, \quad A = A_h^x,$$
$$D = I - R^{2N} - \sum_{k=1}^{J} \alpha_k \left(R^{N-[\lambda_k/\tau]} - R^{N+[\lambda_k/\tau]} \right) \text{ for (2.5)}, \tag{2.13}$$
$$D = I - R^{2N} - \sum_{k=1}^{J} \alpha_k \left(R^{N-[\lambda_k/\tau]} - R^{N+[\lambda_k/\tau]} - \frac{1}{\tau}\left(\lambda_k - \left[\frac{\lambda_k}{\tau}\right]\tau \right) \right.$$
$$\left. \times B\left(R^{N-[\lambda_k/\tau]} - R^{N+[\lambda_k/\tau]+1} \right) \right) \text{ for (2.6)},$$

and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 2.3 (see [22]). *For the solution of the elliptic difference problem*

$$A_h^x u^h(x) = \omega^h(x), \quad x \in \widetilde{\Omega}_h,$$
$$u^h(x)|_{x\in S_h^1} = 0, \qquad D_h u^h(x)|_{x\in S_h^2} = 0, \quad S_h^1 \cup S_h^2 = S_h, \tag{2.14}$$

the following coercivity inequality holds:

$$\sum_{r=1}^{m} \left\| \left(u^h\right)_{\overline{x}_r \overline{x}_r,\, j_r} \right\|_{L_{2h}} \le M_4 \left\| \omega^h \right\|_{L_{2h}}, \tag{2.15}$$

where M_4 does not depend on h and $\omega^h(x)$.

Note that we have not been able to obtain sharp estimate for the constants figuring in the stability estimates. Hence, in the following section, we study difference schemes (2.5) and (2.6) by numerical experiments.

3. Numerical Results

For the numerical result, we consider the nonlocal boundary value problem

$$-\frac{\partial^2 u(t,x)}{\partial t^2}-\frac{\partial^2 u(t,x)}{\partial x^2}+u=2\exp(-t)\left(x-\frac{1}{2}x^2+\frac{t}{2}-1\right),$$
$$0<t<1,\quad 0<x<1,$$
$$u(0,x)=x^2-2x, \tag{3.1}$$
$$u(1,x)=u\left(\frac{1}{2},x\right)+\left(\frac{x^2}{2}-x\right)\exp(-1)-\left(\frac{3x^2}{4}-\frac{3x}{2}\right)\exp\left(-\frac{1}{2}\right),\quad 0\le x\le 1,$$
$$u(t,0)=u_x(t,1)=0,\quad 0\le t\le 1,$$

for the elliptic equation. The exact solution of (3.1) is

$$u(t,x)=\left(tx-\frac{tx^2}{2}+x^2-2x\right)\exp(-t). \tag{3.2}$$

For the approximate solution of the nonlocal boundary Bitsadze-Samarskii problem (3.1), we consider the set $[0,1]_\tau\times[0,1]_h$ of a family of grid points depending on the small parameters τ and h

$$[0,1]_\tau\times[0,1]_h=\{(t_k,x_n):t_k=k\tau,1\le k\le N-1,N_\tau=1$$
$$x_n=nh,1\le n\le M-1,Mh=1\}. \tag{3.3}$$

Firstly, applying difference scheme (2.5), we present the first order of accuracy difference scheme for the approximate solution of problem (3.1) is

$$-\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k=f(t_k,x_n),$$
$$1\le k\le N-1,\quad 1\le n\le M-1,$$
$$u_n^0=\varphi(x_n),\quad 0\le n\le M,$$
$$u_n^N=u_n^{[N/2]}+\left(\frac{x_n^2}{2}-x_n\right)\exp(-1)$$
$$-\left(\frac{3x_n^2}{4}-\frac{3x_n}{2}\right)\exp\left(-\frac{1}{2}\right),\quad 0\le n\le M, \tag{3.4}$$
$$u_0^k=\frac{u_M^k-u_{M-1}^k}{h}=0,\quad 0\le k\le N,$$
$$f(t_k,x_n)=2\exp(-t_k)\left(x_n-\frac{x_n^2}{2}+\frac{t_k}{2}-1\right),$$
$$\varphi(x_n)=x_n^2-2x_n.$$

Then, we have an $(N+1)\times(M+1)$ system of linear equations and we will write them in the matrix form

$$AU_{n+1}+BU_n+CU_{n-1}=D\varphi_n,\quad 1\le n\le M-1,$$
$$U_0=\tilde{0},\quad U_M-U_{M-1}=\tilde{0}, \tag{3.5}$$

where

$$A=\begin{bmatrix}0&0&0&\cdot&0&\cdot&0&0&0\\0&a&0&\cdot&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&0&\cdot&0&a&0\\0&0&0&\cdot&0&\cdot&0&0&0\end{bmatrix}_{(N+1)\times(N+1)},$$
$$B=\begin{bmatrix}1&0&0&\cdot&0&\cdot&0&0&0\\c&b&c&\cdot&0&\cdot&0&0&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&0&0&\cdot&0&\cdot&c&b&c\\0&0&0&\cdot&-1&\cdot&0&0&1\end{bmatrix}_{(N+1)\times(N+1)}, \tag{3.6}$$

and $C=A$, D is an $(N+1)\times(N+1)$ identity matrix and

$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \tag{3.7}$$

where $s = n-1,\ n,\ n+1$,

$$\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}. \tag{3.8}$$

Here,

$$\begin{gathered} a = -\frac{1}{h^2}, \qquad b = \frac{2}{\tau^2} + \frac{2}{h^2} + 1, \qquad c = -\frac{1}{\tau^2}, \\ \varphi_n^k = \begin{cases} \left(x_n^2 - 2x_n\right), & k = 0, \\ f(t_k, x_n), & 1 \le k \le N-1, \\ \left(\dfrac{x_n^2}{2} - x_n\right)\exp(-1) - \left(\dfrac{3x_n^2}{4} - \dfrac{3x_n}{2}\right)\exp\left(-\dfrac{1}{2}\right), & k = N. \end{cases} \end{gathered} \tag{3.9}$$

So, we have a second-order difference equation with respect to n matrix coefficients. To solve this difference equation, we have applied a procedure of modified Gauss elimination method for difference equation with respect to n matrix coefficients. Hence, we seek a solution of the matrix equation in the following form:

$$\begin{gathered} U_j = \alpha_{j+1}U_{j+1} + \beta_{j+1}, \quad j = M-1,\dots,1, \\ U_M = \left(I - \alpha_M\right)^{-1}\beta_M, \\ \alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1}A, \\ \beta_{j+1} = \left(B + C\alpha_j\right)^{-1}\left(D_{\varphi_j} - C\beta_j\right), \quad j = 1,\dots,M-1, \end{gathered} \tag{3.10}$$

where α_j $(j = 1,\dots,M)$ are $(N+1)\times(N+1)$ square matrix and β_j $(j = 1,\dots,M)$ are $(N+1)\times 1$ column matrix and α_1 is the $(N+1)\times(N+1)$ zero matrix and β_j is the $(N+1)\times 1$ zero matrix. Secondly, applying difference scheme (2.6), we present the following second order of accuracy difference scheme for the approximate solutions of problem (3.1):

$$
\begin{gathered}
-\frac{u_n^{k+1}-2u_n^k+u_n^{k-1}}{\tau^2}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+u_n^k=f(t_k,x_n),\\
1\le k\le N-1,\quad 1\le n\le M-1,\\
u_n^0=\varphi(x_n),\quad 0\le n\le M,\\
u_n^N=u_n^{[N/2]}+\left(u_n^{[N/2]+1}-u_n^{[N/2]}\right)\left(\frac{N}{2}-\left[\frac{N}{2}\right]\right)+\left(\frac{x_n^2}{2}-x_n\right)\exp(-1)\\
-\left(\frac{3x_n^2}{4}-\frac{3x_n}{2}\right)\exp\left(-\frac{1}{2}\right),\quad 0\le n\le M,\\
u_0^k=0,\quad u_{M-2}^k-4u_{M-1}^k+3u_M^k=0,\quad 0\le k\le N,\\
f(t_k,x_n)=2\exp(-t_k)\left(x_n-\frac{x_n^2}{2}+\frac{t_k}{2}-1\right),\\
\varphi(x_n)=x_n^2-2x_n.
\end{gathered}
\tag{3.11}
$$

So, we have again an $(N+1)\times(M+1)$ system of linear equations and we will write in the matrix form

$$
\begin{gathered}
AU_{n+1}+BU_n+CU_{n-1}=R\varphi_n,\quad 1\le n\le M-1,\\
U_0=\tilde{0},\quad U_{M-2}-4U_{M-1}+3U_M=\tilde{0},
\end{gathered}
\tag{3.12}
$$

where

$$
A=\begin{bmatrix}
0&0&0&\cdot&0&0&\cdot&0&0&0\\
0&a&0&\cdot&0&0&\cdot&0&0&0\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
0&0&0&\cdot&0&0&\cdot&0&a&0\\
0&0&0&\cdot&0&0&\cdot&0&0&0
\end{bmatrix}_{(N+1)\times(N+1)},
$$

$$
B=\begin{bmatrix}
1&0&0&\cdot&0&0&\cdot&0&0&0\\
c&b&c&\cdot&0&0&\cdot&0&0&0\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
0&0&0&\cdot&0&0&\cdot&c&b&c\\
0&0&0&\cdot&d&e&\cdot&0&0&1
\end{bmatrix}_{(N+1)\times(N+1)},
$$

$$C = A, \quad R = D,$$
$$U_s = \begin{bmatrix} u_s^0 \\ u_s^1 \\ \cdot \\ u_s^{N-1} \\ u_s^N \end{bmatrix}_{(N+1)\times 1}, \tag{3.13}$$

where $s = n-1,\ n,\ n+1$ and $\varphi_n = \begin{bmatrix} \varphi_n^0 \\ \varphi_n^1 \\ \cdot \\ \varphi_n^{N-1} \\ \varphi_n^N \end{bmatrix}_{(N+1)\times 1}$.

Here,

$$a = -\frac{1}{h^2}, \qquad b = \frac{2}{h^2} + \frac{2}{\tau^2} + 1, \qquad c = -\frac{1}{\tau^2}, \qquad d = \left[\frac{N}{2}\right] - \frac{N}{2}, \qquad e = -1 - d,$$
$$\varphi_n^k = \begin{cases} (x_n^2 - 2x_n), & k = 0, \\ f(t_k, x_n), & 1 \le k \le N-1, \\ \left(\dfrac{x_n^2}{2} - x_n\right)\exp(-1) - \left(\dfrac{3x_n^2}{4} - \dfrac{3x_n}{2}\right)\exp\left(-\dfrac{1}{2}\right), & k = N. \end{cases} \tag{3.14}$$

Thus, we have a second-order difference equation with respect to n matrix coefficients. To solve this difference equation, we have applied the same procedure of modified Gauss elimination method (3.10) for difference equation with respect to n matrix coefficients with

$$U_M = (3I + \alpha_M\alpha_{M-1} - 4\alpha_M)^{-1}(-\beta_M\alpha_{M-1} - \beta_{M-1} + 4\beta_M). \tag{3.15}$$

Now, we will give the results of the numerical analysis. The errors computed by

$$E_M^N = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k, x_n) - u_n^k\right|^2 h\right)^{1/2} \tag{3.16}$$

of the numerical solutions for different values of M and N, where $u(t_k, x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k, x_n). Table 1 gives the error analysis between the exact solution and solutions derived by difference schemes for $N = M = 20$, 40, and 60, respectively.

4. Conclusion

In this work, the first and second orders of accuracy difference schemes for the approximate solution of the Bitsadze-Samarskii nonlocal boundary value problem for elliptic equations are presented. Theorems on the stability estimates, almost coercive stability estimates, and coercive stability estimates for the solution of difference schemes for elliptic equations are

Table 1: Error analysis.

Difference schemes	$N = M = 20$	$N = M = 40$	$N = M = 60$
Difference scheme (2.5)	0.0049	0.0025	0.0012
Difference scheme (2.6)	$3.7155e - 005$	$9.4107e - 006$	$2.3679e - 006$

proved. The theoretical statements for the solution of these difference schemes are supported by the results of numerical examples. The second order of accuracy difference scheme is more accurate comparing with the first order of accuracy difference scheme. As a future work, high orders of accuracy difference schemes for the approximate solutions of this problem could be established. Theorems on the stability estimates, almost coercive stability estimates, and coercive stability estimates for the solution of difference schemes for elliptic equations could be proved.

References

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, D. Van Nostrand, London, UK, 1965.

[2] S. G. Krein, *Linear Differential Equations in Banach Space*, Nauka, Moscow, Russia, 1966.

[3] A. L. Skubachevskii, *Elliptic Functional-Differential Equations and Applications*, vol. 91 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 1997.

[4] A. V. Bitsadze and A. A. Samarskiĭ, "Some elementary generalizations of linear elliptic boundary value problems," *Doklady Akademii Nauk SSSR*, vol. 185, pp. 739–740, 1969.

[5] P. E. Sobolevskii, "Well-posedness of difference elliptic equation," *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, vol. 28, no. suppl., pp. 337–382, 1996.

[6] A. Ashyralyev, "Well-posed solvability of the boundary value problem for difference equations of elliptic type," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 24, no. 2, pp. 251–256, 1995.

[7] A. Ashyralyev and P. E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, vol. 148 of *Operator Theory: Advances and Applications*, Birkhäuser, Basel, Switzerland, 2004.

[8] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions—II," *Communications on Pure and Applied Mathematics*, vol. 17, pp. 35–92, 1964.

[9] G. Berikelashvili, "On a nonlocal boundary-value problem for two-dimensional elliptic equation," *Computational Methods in Applied Mathematics*, vol. 3, no. 1, pp. 35–44, 2003.

[10] D. G. Gordeziani, "On a method of resolution of Bitsadze-Samarskii boundary value problem," *Abstracts of Reports of Institute of Applied Mathematics, Tbilisi State University*, vol. 2, pp. 38–40, 1970.

[11] D. V. Kapanadze, "On a nonlocal Bitsadze-Samarskiĭ boundary value problem," *Differentsial'nye Uravneniya*, vol. 23, no. 3, pp. 543–545, 1987.

[12] V. A. Il'in and E. I. Moiseev, "A two-dimensional nonlocal boundary value problem for the Poisson operator in the differential and the difference interpretation," *Matematicheskoe Modelirovanie*, vol. 2, no. 8, pp. 139–156, 1990.

[13] A. Ashyralyev, "A note on the Bitsadze-Samarskii type nonlocal boundary value problem in a Banach space," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 557–573, 2008.

[14] A. P. Soldatov, "A problem of Bitsadze-Samarskiĭ type for second-order elliptic systems on the plane," *Rossiĭskaya Akademiya Nauk. Doklady Akademii Nauk*, vol. 410, no. 5, pp. 607–611, 2006.

[15] R. P. Agarwal, D. O'Regan, and V. B. Shakhmurov, "*B*-separable boundary value problems in Banach-valued function spaces," *Applied Mathematics and Computation*, vol. 210, no. 1, pp. 48–63, 2009.

[16] V. Shakhmurov, "Linear and nonlinear abstract equations with parameters," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 73, no. 8, pp. 2383–2397, 2010.

[17] A. A. Dosiev and B. S. Ashirov, "On a numerical solution of a multipoint problem for a second-order ordinary differential equation with singular coefficients," in *Approximate Methods for Solving Operator Equations*, pp. 41–46, Bakinskii Gosudarstvennogo Universiteta, Baku, Azerbaijan, 1991.

[18] A. Ashyralyev, "Nonlocal boundary-value problems for abstract elliptic equations: well-posedness in Bochner spaces," *AIP Conference Proceedings*, vol. 1309, pp. 66–85, 2010.

[19] M. P. Sapagovas, “A difference method of increased order of accuracy for the Poisson equation with nonlocal conditions,” *Differentsial'nye Uravneniya*, vol. 44, no. 7, pp. 988–998, 2008.

[20] A. Ashyralyev, C. Cuevas, and S. Piskarev, “On well-posedness of difference schemes for abstract elliptic problems in $L_p([0,1,E])$ spaces,” *Numerical Functional Analysis and Optimization*, vol. 29, no. 1-2, pp. 43–65, 2008.

[21] A. Ashyralyev, “Well-posedness of the difference schemes for elliptic equations in $C_\tau^{\beta,\gamma}(E)$ spaces,” *Applied Mathematics Letters*, vol. 22, no. 3, pp. 390–395, 2009.

[22] P. E. Sobolevskii, *Difference Methods for the Approximate Solution of Differential Equations*, Voronezh State University Press, Voronezh, Russia, 1975.