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Research Article

Common Fixed Point Theorems for Commutating Mappings in Fuzzy Metric Spaces

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We generalize Jungck's theorem in Jungck (1976) to fuzzy metric spaces and prove common fixed point theorems for commutative mappings in fuzzy metric spaces.

1. Introduction

In 1965, Zadeh introduced initially the concept of fuzzy sets in [1]. Since then, many authors have expansively developed the theory of fuzzy sets. They applied the concept of fuzzy sets to topology and analysis theory and introduced the concept of fuzzy metric spaces in different ways. See [2–5].

In [5], Kramosil and Michálek provided a tool for developing a smoothing machinery in the field of fixed point theorems, in particular, for the study of contractive type maps. Many authors have studied the fixed point theory in fuzzy metric spaces. In [6], Grabiec followed Kramosil and Michálek [5] and obtained the fuzzy version of Banach contraction principle. Fang [7] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize, unify, and extend some main results in [8–11]. The most interesting references in this direction are [6, 7, 12, 13] and fuzzy mappings [14–17].

In this paper, we generalize Jungck's theorem in [18] to fuzzy metric spaces and prove common fixed point theorems for commutative mappings satisfying some conditions in fuzzy metric spaces in the sense of Kramosil and Michálek [5]. We also give an example to illustrate our main theorem.

To set up our results in the next section we recall some definitions and facts.

Definition 1.1 (see [19]). A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an Abelian topological monoid with unit 1 such that $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 1.2 (see [5]). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:

(FM-1) $M(x, y, 0) = 0$ for all $x, y \in X$;

(FM-2) for $x, y \in X$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $t, s > 0$;

(FM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous for all $x, y \in X$;

(FM-6) $\lim_{t \to \infty} M(x, y, t) = 1$ for all $x, y \in X$.

Definition 1.3 (see [6]). Let $(X, M, *)$ be a fuzzy metric space then we have the following:

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n \to \infty} x_n = x$) if $\lim_{n \to \infty} M(x_n, x, t) = 1$ for any $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a cauchy sequence if $\lim_{n \to \infty} M(x_{n+p}, x_n, t) = 1$ for any $t > 0$ and positive integer p.

(3) $(X, M, *)$ is said to be complete if every Cauchy sequence in X is convergent.

(4) A map $f : X \to X$ is called continuous at $x_0 \in X$ if $\{f(x_n)\}$ converges to $f(x_0)$ for each $\{x_n\}$ converging to x_0.

(5) A map $f : X \to X$ is called a continuous mapping on X if f is continuous at each point $x \in X$.

2. Main Results

Now, we begin with the following theorem, which is Jungck's generalization of the contraction principle for metric spaces.

Theorem 2.1 (see [18]). *Let f be a continuous mapping of a complete metric space (X, d) into itself and let $g : X \to X$ be a map. If*

(i) $g(X) \subseteq f(X)$,

(ii) *g commutes with f,*

(iii) $d(g(x), g(y)) \leq \alpha d(f(x), f(y))$ *for some $\alpha \in (0, 1)$ and all $x, y \in X$,*

then f and g have a unique common fixed point.

The above result can be generalized into the following theorem in fuzzy metric spaces.

Theorem 2.2. *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a continuous map and $g : X \to X$ a map. If*

(i) $g(X) \subseteq f(X)$,

(ii) *g commutes with f,*

(iii) $M(g(x), g(y), t) \geq M(f(x), f(y), \psi(t))$ *for all $x, y \in X$ and $t > 0$, where $\psi : [0, +\infty) \to (0, +\infty)$ is an increasing and left-continuous function with $\psi(t) > t$ for all $t > 0$,*

then f and g have a unique common fixed point.

Proof. Let φ^k be the kth iteration of φ, $k = 1, 2, \ldots$, that is, $\varphi^2(t) = \varphi(\varphi(t))$, and so forth.

We first prove that

$$\lim_{n\to\infty} \varphi^n(t) = +\infty \tag{2.1}$$

for any $t \geq 0$. Take any $t \geq 0$. It follows by the properties of φ in (iii) that the sequence $\{\varphi^n(t)\}$ is monotone increasing. Then $\lim_{n\to\infty}\varphi^n(t)$ exists or is $+\infty$. Suppose that $\lim_{n\to\infty}\varphi^n(t) = b \neq +\infty$. It is clear that $b > 0$. Since $\varphi(b) > b$ and $\varphi(t)$ is left-continuous, there exists $\delta > 0$ such that $\varphi(t) > b$ for any $t \in (b-\delta, b)$. And since $\lim_{n\to\infty}\varphi^n(t) = b$, there exists $N > 0$ such that $\varphi^n(t) > b - \delta/2$ for any $n > N$, hence by the increasing property of φ, it follows that $\varphi^{n+1}(t) = \varphi(\varphi^n(t)) > \varphi(b-\delta/2) > b$. Since the sequence $\{\varphi^n(t)\}$ is monotone increasing, $\varphi^{n+p}(t) > b$ for any positive integer p, which is contrary to $\lim_{n\to\infty}\varphi^n(t) = b$.

Let $x_0 \in X$. By (i) we can find x_1 such that $f(x_1) = g(x_0)$. By induction, we can find a sequence $\{x_n\}$ in X such that $f(x_n) = g(x_{n-1})$. Take any $t > 0$. For any positive integers n and p, by induction again, we have

$$\begin{aligned} M(f(x_n), f(x_{n+p}), t) &= M(g(x_{n-1}), g(x_{n+p-1}), t) \\ &\geq M(f(x_{n-1}), f(x_{n+p-1}), \varphi(t)) \geq \cdots \geq M(f(x_0), f(x_p), \varphi^n(t)). \end{aligned} \tag{2.2}$$

Because $\lim_{n\to\infty}\varphi^n(t) = +\infty$, it follows by (FM-6) that $\lim_{n\to\infty}M(f(x_0), f(x_p), \varphi^n(t)) = 1$, hence $\lim_{n\to\infty}M(f(x_n), f(x_{n+p}), t) = 1$. Thus $\{f(x_n)\}$ is a Cauchy sequence and by the completeness of X, $\{f(x_n\}$ is a convergent sequence in X. Suppose that $\lim_{n\to\infty}f(x_n) = y$. So $\lim_{n\to\infty}g(x_n) = \lim_{n\to\infty}f(x_{n+1}) = y$.

It can be seen from (iii) that the continuity of f implies that of g. So $\lim_{n\to\infty}g(f(x_n)) = g(y)$. By the commutativity of f and g, it follows that $\lim_{n\to\infty}f(g(x_n)) = \lim_{n\to\infty}g(f(x_n)) = g(y)$. Because of the uniqueness of limits, we have $f(y) = g(y)$. So $f(f(y)) = f(g(y)) = g(f(y)) = g(g(y))$ by commutativity of f and g. So we have

$$\begin{aligned} M(g(y), g(g(y)), t) &\geq M(f(y), f(g(y)), \varphi(t)) \\ &\geq M(g(y), g(g(y)), \varphi(t)) \geq \cdots \geq M(g(y), g(g(y)), \varphi^n(t)). \end{aligned} \tag{2.3}$$

Since $\lim_{n\to\infty}\varphi^n(t) = +\infty$, $\lim_{n\to\infty}M(g(y), g(g(y)), \varphi^n(t)) = 1$ hence, $M(g(y), g(g(y)), t) = 1$. So by (FM-2), we have $g(y) = g(g(y))$. Thus $g(y) = g(g(y)) = f(g(y))$, that is, $g(y)$ is a common fixed point of f and g.

If x and z are two common fixed points of f and g, that is, $f(x) = g(x) = x$, $f(z) = g(z) = z$, then for any $t > 0$,

$$\begin{aligned} M(x, z, t) &= M(g(x), g(z), t) \geq M(f(x), f(z), \varphi(t)) \\ &= M(x, z, \varphi(t)) \geq \cdots \geq M(x, z, \varphi^n(t)). \end{aligned} \tag{2.4}$$

Since $\lim_{n\to\infty}\varphi^n(t) = +\infty$, by (FM-6), $\lim_{n\to\infty}M(x, z, \varphi^n(t)) = 1$ hence, $M(x, z, t) = 1$. So $x = z$ by (FM-2). $\square$

Remark 2.3. Our result is an extension of Grabiec's contraction principle in fuzzy metric spaces from [6].

Remark 2.4. Jungck introduced the notion of compatibility of a pair of mappings in [20]. We can weaken the commutability of f and g in Theorem 2.2 to compatibility.

Remark 2.5. We now give an example that illustrates Theorem 2.2. Let X be the subset $[0,+\infty)$ of the real number set. Define

$$M(x,y,t) = \begin{cases} 0, & t=0, \\ e^{-|x-y|/t}, & t>0, \end{cases} \quad x,y \in X. \tag{2.5}$$

Clearly $M(x,y,*)$ is a fuzzy metric on X, where $*$ is defined by $a*b = ab$. It is easy to see that $(X,M,*)$ is a complete fuzzy metric space. Define $g(x) = \ln(1+Ax)$ and $f(x) = A^{-1}(e^x - 1)$ on X, where $A \in (0,1)$ is a constant. It is evident that $g(X) \subseteq f(X)$. Also, for $t > 0$,

$$\begin{aligned} M(g(x),g(y),t) &= e^{-|\ln(1+Ax)-\ln(1+Ay)|/t} \geq e^{-A|x-y|/t} \\ &\geq e^{-|A^{-1}(e^x-1)-A^{-1}(e^y-1)|/A^{-2}t} = M\Big(f(x),f(y),A^{-2}t\Big). \end{aligned} \tag{2.6}$$

Take $\psi(t) = A^{-2}t > t$. Thus all the conditions of Theorem 2.2 are satisfied and f and g have the common fixed point 0.

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